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Free vibration analysis of a Timoshenko beam carrying multiple spring-mass systems with the effects of shear deformation and rotary inertia

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Abstract. Because of complexity, the literature regarding the free vibration analysis of a Timoshenko beam carrying "multiple" spring-mass systems is rare, particular that regarding the "exact" solutions. As to the "exact" solutions by further considering the joint terms of shear deformation and rotary inertia in the differential equation of motion of a Timoshenko beam carrying multiple concentrated attachments, the information concerned is not found yet. This is the reason why this paper aims at studying the natural frequencies and mode shapes of a uniform Timoshenko beam carrying multiple intermediate spring-mass systems using an exact as well as a numerical assembly method. Since the shear deformation and rotary inertia terms are dependent on the slenderness ratio of the beam, the shear coefficient of the cross-section, the total number of attachments and the support conditions of the beam, the individual and/or combined effects of these factors on the result are investigated in details. Numerical results reveal that the effect of the shear deformation and rotary inertia joint terms on the lowest five natural frequencies of the combined vibrating system is somehow complicated.

Keywords: Timoshenko beam; shear deformation; rotary inertia; spring-mass systems; shear coefficient; natural frequency.

1. Introduction

Since many practical vibrating systems can be modelled as a beam carrying one or more elastically mounted lumped masses, the literature concerned appears sufficiently adequate. Most of the existing literatures use the Euler-Bernoulli beam theory with the effects of shear deformations and rotary inertias neglected (Gurgoze 1998, Wu and Chou 1998, 1999, Cha 2001, Chen and Wu

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2002, Qiao *el al.* 2002), and the use of more advanced the Timoshenko beam theory is relatively scarce (Laura *et al.* 1977, Rossi *et al.* 1993, Wu and Chen 2001) because the formulation gives rise to some complications. The problem becomes much more difficult when the Timoshenko beam carries "multiple" spring-mass systems and an "exact" solution is required. To the author's knowledge, the work of Wu and Chen (2001) is the only one that has been reported in the literature. However, the coupling terms were neglected in the paper of Wu and Chen, it became necessary to include these coupling terms on the "exact" solutions for a more advanced and refined apply. The present research, which is partly motived by the above paper, will undertake this investigation.

Abramovich and Elishakoff (1990) have studied the influence of shear deformation and rotary inertia on the vibration frequencies of a Timoshenko beam without any attachments. Later, Abramovizh and Hamberger (1991) investigated the influence of shear deformation and rotary inertia on the natural frequencies of a cantilever Timoshenko beam with a tip mass having rotary inertia. The following year, they repeated the same study (1992) by adding the effects of a translational spring and a rotational spring at any arbitrary point on the beam. Besides, Rossi *et al.* (1993) have presented an exact solution for the natural frequencies and mode shapes of a Timoshenko beam carrying a "single" elastically mounted concentrated mass for three different types of boundary conditions. On the other hand, using the numerical assembly method (NAM), Wu and Chen (2001) have performed the free vibration analysis of a uniform Timoshenko beam carrying "multiple" spring-mass systems and achieved satisfactory results. As was the case in the working, Abramovich and Elishakoff (1990) and Abramovizh and Hamberger (1991, 1992), the "joint terms" of shear deformation and rotary inertia in the differential equation of Timoshenko beam were also omitted by Rossi *et al.* (1993) and Wu and Chen (2001).

For an "approximate" solution, it may be reasonable on some occasions to neglect the effect of the above-mentioned "joint terms" of shear deformation and rotary inertia. However, when seeking an "exact" solution such as the attempts made by Rossi *et al.* (1993), Wu and Chen (2001), the effects could be significant and must be considered in the analysis. Thus the key point of this paper is focused on the influence of the slenderness ratio of the beam, the shear coefficient of the cross-section, the total number of attachments and the boundary (supporting) conditions on the lowest five natural frequencies of the Timoshenko beam.

2. Formulation of the problem

By considering the effects of shear deformation and rotary inertia, the equation of motion for a freely vibrating uniform beam is given by (Abramovich and Elishakoff 1990, Thomson 1981, Meirovitch 1967)

$$EI\frac{\partial^2\varphi}{\partial x^2} + k'AG\left(\frac{\partial y}{\partial x} - \varphi\right) - \rho I\frac{\partial^2\varphi}{\partial t^2} = 0 \tag{1}$$

$$\rho A\frac{\partial^2 y}{\partial t^2} - k'AG\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial\varphi}{\partial x}\right) = 0 \tag{2}$$

where y is transverse deflection, φ is bending slope, E is Young's modulus, G is shear modulus, A is cross-sectional area, I is moment of inertia of the cross-sectional area A, ρ is density of the beam material, k' is shear coefficient (or shape factor) for the cross section, x is the spatial coordinate

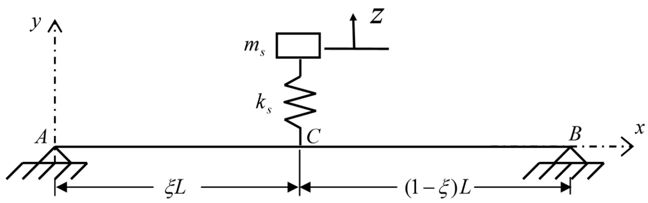

Fig. 1 A simply supported Timoshenko beam carrying a spring-mass system at the arbitrary point C with coordinate $x = \xi L$

along the beam length and t is time (see Fig. 1).

Eliminating φ from Eq. (1) and y from Eq. (2), one obtains the following two complete differential equations in y and φ in similar form

$$EI\frac{\partial^4 y}{\partial x^4} + m\frac{\partial^2 y}{\partial t^2} - \left(J + \frac{mEI}{k'GA}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2} + \left(\frac{mJ}{k'AG}\right)\frac{\partial^4 y}{\partial t^4} = 0 \tag{3}$$

$$EI\frac{\partial^4 \varphi}{\partial x^4} + m\frac{\partial^2 \varphi}{\partial t^2} - \left(J + \frac{mEI}{k'GA}\right)\frac{\partial^4 \varphi}{\partial x^2 \partial t^2} + \left(\frac{mJ}{k'AG}\right)\frac{\partial^4 \varphi}{\partial t^4} = 0 \tag{4}$$

where $m = \rho A$ is the mass per unit length of beam and $J = \rho I$ is the rotary inertia per unit length of beam.

The two underlined terms in Eqs. (3) and (4) are sometimes called the "joint terms" of shear deformation and rotary inertia. In some of the existing literature these joint terms are neglected (Abramovich and Elishakoff 1990, Abramovizh and Hamberger 1991, 1992, Rossi *et al.* 1993, Wu and Chen 2001). However, they are essentially considered here, because one of the main purposes of this paper is to study their effect on the free vibration characteristics of a Timoshenko beam carrying one or more intermediate spring-mass systems. It is evident that these joint terms will render the mathematical expressions of this paper to be more complicated than the ones given in the fore-mentioned papers.

Free vibration of the beam takes the form

$$y(x, t) = \overline{Y}(x)e^{i\overline{\omega}t}, \qquad \varphi(x, t) = \overline{\Psi}(x)e^{i\overline{\omega}t} \tag{5}$$

where $\overline{\omega}$ is the natural frequency of the combined vibrating system shown in Fig. 1 and $i = \sqrt{-1}$, while $\overline{Y}(x)$ and $\overline{\Psi}(x)$ are the amplitudes of $y(x, t)$ and $\varphi(x, t)$, respectively.

Substituting Eq. (5) into Eqs. (3) and (4) one obtains

$$\frac{d^4\overline{Y}(x)}{dx^4} + \frac{\overline{\omega}^2}{EI}\left(\frac{mEI}{k'GA} + J\right)\frac{d^2\overline{Y}(x)}{dx^2} - \frac{m\overline{\omega}^2}{EI}\left(1 - \frac{J\overline{\omega}^2}{k'GA}\right)\overline{Y}(x) = 0 \tag{6}$$

$$\frac{d^4\overline{\Psi}(x)}{dx^4} + \frac{\overline{\omega}^2}{EI}\left(\frac{mEI}{k'GA} + J\right)\frac{d^2\overline{\Psi}(x)}{dx^2} - \frac{m\overline{\omega}^2}{EI}\left(1 - \frac{J\overline{\omega}^2}{k'GA}\right)\overline{\Psi}(x) = 0 \tag{7}$$

Introducing the following parameters

$$a' = \frac{m\overline{\omega}^2}{k'GA}, \quad b' = \frac{J\overline{\omega}^2}{k'GA}, \quad c' = \frac{m\overline{\omega}^2}{EI} \qquad \text{(8a)~(8c)}$$

Eq. (6) reduces to

$$\frac{d^4\overline{Y}(x)}{dx^4} + (a' + b')\frac{d^2\overline{Y}(x)}{dx^2} - (c' - a'b')\overline{Y}(x) = 0 \tag{9}$$

Set

$$\overline{Y}(x) = \overline{Y}_0 e^{\lambda x} \tag{10}$$

then the substitution of Eq. (10) into Eq. (9) yields the characteristic equation

$$\lambda^4 + (a' + b')\lambda^2 - (c' - a'b') = 0 \tag{11}$$

If the roots of Eq. (11) are denoted by $\pm\lambda_1$ and $\pm i\lambda_2$, respectively, then

$$\lambda_1^2 = \sqrt{c' + \frac{1}{4}(a' - b')^2} - \frac{1}{2}(a' + b') \tag{12a}$$

$$\lambda_2^2 = \sqrt{c' + \frac{1}{4}(a' - b')^2} + \frac{1}{2}(a' + b') \tag{12b}$$

Therefore, the solution of Eq. (9) or (6) is given by

$$\begin{aligned}\overline{Y}(x) &= A_1 e^{\lambda_1 x} + A_2 e^{-\lambda_1 x} + A_3 e^{i\lambda_2 x} + A_4 e^{-i\lambda_2 x} \\ &= C_1\cosh(\lambda_1 x) + C_2\sinh(\lambda_1 x) + C_3\cos(\lambda_2 x) + C_4\sin(\lambda_2 x)\end{aligned} \tag{13}$$

Similarity, the solution of Eq. (7) takes the form

$$\overline{\Psi}(x) = C_1'\cosh(\lambda_1 x) + C_2'\sinh(\lambda_1 x) + C_3'\cos(\lambda_2 x) + C_4'\sin(\lambda_2 x) \tag{14}$$

In Eqs. (13) and (14), C_i and $C_i' (i = 1, 2, 3, 4)$ are constants determined by the boundary conditions. For a Timoshenko beam, its boundary conditions may be obtained from the following relationship (Meirovitch 1967)

$$\overline{\Psi}(x) = \frac{d\overline{Y}(x)}{dx} + \frac{EI}{k'GA}\frac{d^2\overline{\Psi}(x)}{dx^2} + \frac{J\varpi^2}{k'GA}\overline{\Psi}(x) \tag{15}$$

or

$$\frac{d\overline{Y}(x)}{dx} = \left(1 - \frac{J\varpi^2}{k'GA}\right)\overline{\Psi}(x) - \frac{EI}{k'GA}\frac{d^2\overline{\Psi}(x)}{dx^2} \tag{16}$$

The substitution of Eqs. (13) and (14) into Eq. (16) yields

$$\begin{aligned}&C_1\lambda_1\sinh(\lambda_1 x) + C_2\lambda_1\cosh(\lambda_1 x) - C_3\lambda_2\sin(\lambda_2 x) + C_4\lambda_2\cos(\lambda_2 x) \\ &= \left(1 - \frac{J\varpi^2}{k'GA}\right)[C_1'\cosh(\lambda_1 x) + C_2'\sinh(\lambda_1 x) + C_3'\cos(\lambda_2 x) + C_4'\sin(\lambda_2 x)] \\ &- \frac{EI}{k'GA}[C_1'\lambda_1^2\cosh(\lambda_1 x) + C_2'\lambda_1^2\sinh(\lambda_1 x) - C_3'\lambda_2^2\cos(\lambda_2 x) - C_4'\lambda_2^2\sin(\lambda_2 x)]\end{aligned} \tag{17}$$

From the last expression one obtains the following relationships between C_i and C_i' $(i = 1\sim4)$

$$C_1' = \delta_1' C_2, \quad C_2' = \delta_1' C_1, \quad C_3' = \delta_2' C_4, \quad C_4' = -\delta_2' C_3 \tag{18a)~(18d}$$

where

$$\delta_1' = \frac{\lambda_1}{\left(1 - \frac{J\overline{\omega}^2}{k'GA}\right) - \frac{EI}{k'GA}\lambda_1^2} \tag{19a}$$

$$\delta_2' = \frac{\lambda_2}{\left(1 - \frac{J\overline{\omega}^2}{k'GA}\right) + \frac{EI}{k'GA}\lambda_2^2} \tag{19b}$$

For the simply supported beam shown in Fig. 1, if the functions for the deflection and slope amplitudes are respectively denoted by $\overline{Y}_1(x)$ and $\overline{\Psi}_1(x)$ in the first region with $0 \le x \le \xi L$ and $\overline{Y}_2(x)$ and $\overline{\Psi}_2(x)$ in the second region with $\xi L \le x \le L$, then the boundary conditions of the problem are given by

$$\overline{Y}_1(x) = 0 \quad \text{and} \quad \overline{\Psi}_1'(x) = 0 \quad \text{at} \quad x = 0 \tag{20a}$$

$$\overline{Y}_2(x) = 0 \quad \text{and} \quad \overline{\Psi}_2'(x) = 0 \quad \text{at} \quad x = L \tag{20b}$$

The continuity of deformations at $x = \xi L$ requires that

$$\overline{Y}_1(x) = \overline{Y}_2(x) \tag{21a}$$

$$\overline{\Psi}_1(x) = \overline{\Psi}_2(x) \tag{21b}$$

$$\overline{\Psi}_1'(x) = \overline{\Psi}_2'(x) \tag{21c}$$

and the equilibrium of forces at $x = \xi L$ requires that

$$k'GA[\overline{\Psi}_1(x) - \overline{Y}_1'(x)] + F_s\overline{Y}_1(x) = k'GA[\overline{\Psi}_2(x) - \overline{Y}_2'(x)] \tag{21d}$$

where F_s is the interactive force between the beam and the attached spring-mass system given by (Laura *et al*. 1977)

$$F_s = \frac{m_s\overline{\omega}^2}{1 - (m_s\overline{\omega}^2/k_s)} \tag{22}$$

In the last expression, m_s and k_s are the lumped mass and spring constant of the spring-mass system, respectively.

The equation of motion for the attached spring-mass system is given by

$$m_s\ddot{z}(t) + k_s[z(t) - y_C(t)] = 0 \tag{23}$$

where $\ddot{z}(t)$ and $z(t)$ are the acceleration and displacement of the spring mass and $y_C(t)$ is the transverse deflection of the beam at the attaching point C (see Fig. 1).

It is similar to Eq. (5) that one has

$$z(t) = Ze^{i\bar{\omega}t} \tag{24a}$$

$$y_C(t) = \bar{Y}_C e^{i\bar{\omega}t} \tag{24b}$$

where Z is the amplitude of the lumped mass.

To insert Eqs. (24a) and (24b) into Eq. (23), one obtains

$$\bar{Y}_C + (\gamma_s^2 - 1)Z = 0 \tag{25}$$

where

$$\gamma_s^2 = \frac{\bar{\omega}^2}{\omega_s^2} \tag{26}$$

$$\omega_s = \sqrt{k_s/m_s} \tag{27}$$

In the last expressions, ω_s denotes the natural frequency of the attached spring-mass system itself.

Substituting the boundary conditions given by Eq. (20), the continuity requirements given by Eq. (21) and the force equilibrium condition given by Eq. (25) into Eqs. (13) and (14), one obtains nine homogeneous equations consisting of nine unknowns $\bar{C}_i$ (i = 1~8) and Z.

In matrix form, the homogeneous equations are given by

$$[B]\{\bar{C}\} = 0 \tag{28}$$

where

$$\{\bar{C}\} = [\bar{C}_1\ \ \bar{C}_2\ \ \ldots\ \ \bar{C}_8\ \ Z]^T \tag{29a}$$

$$[B] = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & 0 & 0 & 0 & 0 & 0 \\ B_{21} & B_{22} & B_{23} & B_{24} & 0 & 0 & 0 & 0 & 0 \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & B_{36} & B_{37} & B_{38} & 0 \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} & B_{46} & B_{47} & B_{48} & 0 \\ B_{51} & B_{52} & B_{53} & B_{54} & B_{55} & B_{56} & B_{57} & B_{58} & 0 \\ B_{61} & B_{62} & B_{63} & B_{64} & B_{65} & B_{66} & B_{67} & B_{68} & 0 \\ 0 & 0 & 0 & 0 & B_{75} & B_{76} & B_{77} & B_{78} & 0 \\ 0 & 0 & 0 & 0 & B_{85} & B_{86} & B_{87} & B_{88} & 0 \\ B_{91} & B_{92} & B_{93} & B_{94} & 0 & 0 & 0 & 0 & -1+\gamma_s^2 \end{bmatrix} \tag{29b}$$

The coefficients of matrix $[B]$ are as follows

$$B_{11} = 1,\ \ B_{12} = 0,\ \ B_{13} = 1,\ \ B_{14} = 0,\ \ B_{21} = \delta_1'\lambda_1,\ \ B_{22} = 0,\ \ B_{23} = -\delta_2'\lambda_2,\ \ B_{24} = 0$$

$$B_{31} = \cosh(\lambda_1 x),\ \ B_{32} = \sinh(\lambda_1 x),\ \ B_{33} = \cos(\lambda_2 x),\ \ B_{34} = \sin(\lambda_2 x)$$

$$B_{35} = -\cosh(\lambda_1 x),\ \ B_{36} = -\sinh(\lambda_1 x),\ \ B_{37} = -\cos(\lambda_2 x),\ \ B_{38} = -\sin(\lambda_2 x)$$

$$B_{41} = \delta_1' \sinh(\lambda_1 x), \quad B_{42} = \delta_1' \cosh(\lambda_1 x), \quad B_{43} = -\delta_2' \sin(\lambda_2 x), \quad B_{44} = \delta_2' \cos(\lambda_2 x)$$

$$B_{45} = -\delta_1' \sinh(\lambda_1 x), \quad B_{46} = -\delta_1' \cosh(\lambda_1 x), \quad B_{47} = \delta_2' \sin(\lambda_2 x), \quad B_{48} = -\delta_2' \cos(\lambda_2 x)$$

$$B_{51} = \delta_1' \lambda_1 \cosh(\lambda_1 x), \quad B_{52} = \delta_1' \lambda_1 \sinh(\lambda_1 x), \quad B_{53} = -\delta_2' \lambda_2 \cos(\lambda_2 x)$$

$$B_{54} = -\delta_2' \lambda_2 \sin(\lambda_2 x), \quad B_{55} = -\delta_1' \lambda_1 \cosh(\lambda_1 x), \quad B_{56} = -\delta_1' \lambda_1 \sinh(\lambda_1 x)$$

$$B_{57} = \delta_2' \lambda_2 \cos(\lambda_2 x), \quad B_{58} = \delta_2' \lambda_2 \sin(\lambda_2 x)$$

$$B_{61} = k'GA\delta_1' \sinh(\lambda_1 x) - k'GA\lambda_1 \sinh(\lambda_1 x) + F_s \cosh(\lambda_1 x)$$

$$B_{62} = k'GA\delta_1' \cosh(\lambda_1 x) - k'GA\lambda_1 \cosh(\lambda_1 x) + F_s \sinh(\lambda_1 x)$$

$$B_{63} = -k'GA\delta_2' \sin(\lambda_2 x) + k'GA\lambda_2 \sin(\lambda_2 x) + F_s \cos(\lambda_2 x)$$

$$B_{64} = k'GA\delta_2' \cos(\lambda_2 x) - k'GA\lambda_2 \cos(\lambda_2 x) + F_s \sin(\lambda_2 x)$$

$$B_{65} = -k'GA\delta_1' \sinh(\lambda_1 x) + k'GA\lambda_1 \sinh(\lambda_1 x)$$

$$B_{66} = -k'GA\delta_1' \cosh(\lambda_1 x) + k'GA\lambda_1 \cosh(\lambda_1 x)$$

$$B_{67} = k'GA\delta_2' \sin(\lambda_2 x) - k'GA\lambda_2 \sin(\lambda_2 x)$$

$$B_{68} = -k'GA\delta_2' \cos(\lambda_2 x) + k'GA\lambda_2 \cos(\lambda_2 x)$$

$$B_{75} = \cosh(\lambda_1 L), \quad B_{76} = \sinh(\lambda_1 L), \quad B_{77} = \cos(\lambda_2 L), \quad B_{78} = \sin(\lambda_2 L)$$

$$B_{85} = \delta_1' \lambda_1 \cosh(\lambda_1 L), \quad B_{86} = \delta_1' \lambda_1 \sinh(\lambda_1 L), \quad B_{87} = -\delta_2' \lambda_2 \cos(\lambda_2 L)$$

$$B_{88} = -\delta_2' \lambda_2 \sin(\lambda_2 L), \quad B_{91} = \cosh(\lambda_1 x), \quad B_{92} = \sinh(\lambda_1 x), \quad B_{93} = \cos(\lambda_2 x)$$

$$B_{94} = \sin(\lambda_2 x) \tag{30}$$

Non-trivial solution of Eq. (28) requires that its coefficient determinant is equal to zero, i.e.

$$|B| = 0 \tag{31}$$

Eq. (31) is the frequency equation, from which the natural frequencies $\bar{\omega}_j$ (j = 1, 2, …) can be obtain by using the half-interval technique (Faires and Burden 1993). To substitute each value of $\bar{\omega}_j$ into Eq. (28) one may determine the values of unknowns $\bar{C}_i$ (i = 1~8) and Z. Finally, the substitution of $\bar{C}_i$ (i = 1~8) into Eq. (13) (four for the first region and four for the second region) will define the corresponding mode shape $\bar{Y}_j(x)$ of the combined vibrating system shown in Fig. 1.

3. Numerical results and discussions

From Eqs. (3) and (4) one sees that the joint terms are proportional to $mJ/(k'GA)$. Thus, the influence of the joint terms is studied with two cases in this section. In the first case the slenderness ratio (L/r_g) and the shear coefficient (k') are kept constant and in the second case the last two factors are varied. Since the radius of gyration of a cross-section, r_g, is given by $r_g = \sqrt{I/A}$, one may change the slenderness ratio of a beam, L/r_g, by changing its length L and keeping its cross-section area to be constant.

3.1 Influence of joint terms with constant slenderness ratio and shear coefficient

To realize the influence of the joint terms of shear deformation and rotary inertia in the differential equation of a Timoshenko beam, the lowest five natural frequencies and the associated mode shapes of the simply supported Timoshenko beam carrying a spring-mass system as shown in Fig. 1 are studied. The dimensions and physical properties of the Timoshenko beam are: beam

Table 1 The lowest five frequency coefficients $\bar{\Omega}_j = \bar{\omega}_j\sqrt{\rho A L^4/EI}$ for a simply supported uniform Timoshenko beam carrying one spring-mass system located at $\xi = x/L = 0.5$

k_s^*	m_s^*	Methods	$\bar{\Omega}_1$	$\bar{\Omega}_2$	$\bar{\Omega}_3$	$\bar{\Omega}_4$	$\bar{\Omega}_5$
1.0	0.2	Present	2.20962	9.51998	33.54929	65.66024	101.38258
		Rossi *et al*. (1993)	2.20963	9.52000	33.54940	65.66070	101.38400
	0.5	Present	1.39803	9.51630	33.54929	65.66023	101.38258
		Rossi *et al*. (1993)	1.39803	9.51632	33.54940	65.66070	101.38400
	1.0	Present	0.98867	9.51513	33.54929	65.66023	101.38258
		Rossi *et al*. (1993)	0.98868	9.51515	33.54940	65.66070	101.38400
	2.0	Present	0.69914	9.51455	33.54929	65.66023	101.38258
		Rossi *et al*. (1993)	0.69914	9.51457	33.54940	65.66070	101.38400
	3.0	Present	0.57086	9.51436	33.54929	65.66023	101.38258
		Rossi *et al*. (1993)	0.57086	9.51438	33.54940	65.66070	101.38400
10.0	0.2	Present	6.01749	11.02088	33.54929	65.78824	101.38258
		Rossi *et al*. (1993)	6.01750	11.02090	33.54940	65.78870	101.38400
	0.5	Present	3.95992	10.59210	33.54929	65.78725	101.38258
		Rossi *et al*. (1993)	3.95993	10.59210	33.54940	65.78770	101.38400
	1.0	Present	2.82864	10.48522	33.54929	65.78693	101.38258
		Ross *et al*. (1993)	2.82865	10.48520	33.54940	65.78740	101.38400
	2.0	Present	2.00934	10.43730	33.54929	65.78676	101.38258
		Rossi *et al*. (1993)	2.00935	10.43730	33.54940	65.78720	101.38400
	3.0	Present	1.64302	10.42208	33.54929	65.78671	101.38258
		Rossi *et al*. (1993)	1.64302	10.42210	33.54940	65.78720	101.38400
100.0	0.2	Present	7.81406	25.97072	33.54929	67.23363	101.38258
		Rossi *et al*. (1993)	7.81406	25.97070	33.54940	67.23410	101.38400
	0.5	Present	6.28649	20.45214	33.54929	67.12322	101.38258
		Rossi *et al*. (1993)	6.28650	20.45220	33.54940	67.12370	101.38400
	1.0	Present	4.92971	18.45193	33.54929	67.08959	101.38258
		Rossi *et al*. (1993)	4.92972	18.45200	33.54940	67.09000	101.38400
	2.0	Present	3.68639	17.45253	33.54929	67.07333	101.38258
		Rossi *et al*. (1993)	3.69340	17.45260	33.54940	67.07380	101.38400
	3.0	Present	3.06797	17.12381	33.54929	67.06798	101.38258
		Rossi *et al*. (1993)	3.06798	17.12380	33.54940	67.06840	101.38400

Note: The joint terms are considered in the “Present” paper and neglected in Rossi *et al*. (1993).

length L = 40 *in*, Young's modulus $E = 3.0 \times 10^7$ *psi*, shear modulus $G = 1.154 \times 10^6$ *psi*, cross-sectional area A = 13.865 in^2, area moment of inertia I = 55.426 in^4, mass density of beam material $\rho = 0.283$ *lbm*, mass per unit length $m = \rho A = 3.921$ *lbm/in*, rotary inertia per unit length $J = 15.685$ $lbm - in$, shear coefficient (or shape factor) $k' = 5/6$, total mass of the beam $m_b = \rho AL = 156.855$ *lbm* and reference stiffness for the beam $k_b = EI/L^3 = 25980.760$ *lbf/in*. For convenience, two non-dimensional parameters for the spring-mass system are also introduced: $m_s^* = m_s/m_b$ and $k_s^* = k_s/k_b$.

Table 2 The key is the same as Table 1 except that $\xi = x/L = 2/3$

k_s^*	m_s^*	Methods	$\bar{\Omega}_1$	$\bar{\Omega}_2$	$\bar{\Omega}_3$	$\bar{\Omega}_4$	$\bar{\Omega}_5$
1.0	0.2	Present	2.21493	9.49269	33.57049	65.64620	101.38933
		Rossi *et al*. (1993)	2.21494	9.42271	33.57060	65.64670	101.39000
	0.5	Present	1.40126	9.48991	33.57043	65.64620	101.38933
		Rossi *et al*. (1993)	1.40126	9.48993	33.57060	65.64670	101.39000
	1.0	Present	0.99093	9.48902	33.57042	65.64620	101.38933
		Rossi *et al*. (1993)	0.99093	9.48904	33.57060	65.64670	101.39000
	2.0	Present	0.70072	9.48859	33.57041	65.64620	101.38933
		Rossi *et al*. (1993)	0.70073	9.48861	33.57060	65.64670	101.39000
	3.0	Present	0.57215	9.48844	33.57040	65.64620	101.38933
		Rossi *et al*. (1993)	0.57215	9.48846	33.57050	65.64670	101.39000
10.0	0.2	Present	6.18205	10.67413	33.77157	65.64620	101.45039
		Rossi *et al*. (1993)	6.18205	10.67420	33.77170	65.64670	101.45100
	0.5	Present	4.04919	10.30872	33.76558	65.64620	101.45019
		Rossi *et al*. (1993)	4.04920	10.30880	33.76570	65.64670	101.45100
	1.0	Present	2.88762	10.22218	33.76365	65.64620	101.45012
		Rossi *et al*. (1993)	2.88762	10.22220	33.76380	65.64670	101.45100
	2.0	Present	2.04958	10.18392	33.76270	65.64620	101.45009
		Rossi *et al*. (1993)	2.04959	10.18390	33.76280	65.64670	101.45100
	3.0	Present	1.67548	10.17184	33.76238	65.64620	101.45008
		Rossi *et al*. (1993)	1.67548	10.17190	33.76250	65.64670	101.45100
100.0	0.2	Present	8.10813	23.03747	37.11827	65.64620	102.08729
		Rossi *et al*. (1993)	8.10814	23.03760	37.11830	65.64670	102.08800
	0.5	Present	6.66860	18.16903	36.20167	65.64620	102.06676
		Rossi *et al*. (1993)	6.66860	18.16910	36.20170	65.64670	102.06800
	1.0	Present	5.28137	16.32501	35.97650	65.64620	102.06017
		Rossi *et al*. (1993)	5.28137	16.32510	35.97660	65.64670	102.06100
	2.0	Present	3.96568	15.41724	35.87559	65.64620	102.05692
		Rossi *et al*. (1993)	3.96568	15.41730	35.87570	65.64670	102.05800
	3.0	Present	3.30383	15.12359	35.84352	65.64620	102.05585
		Rossi *et al*. (1993)	3.30383	15.12370	35.84360	65.64670	102.05700

Note: The joint terms are considered in the "Present" paper and neglected in Rossi *et al*. (1993).

For the values of non-dimensional point mass $m_s^* = m_s/m_b = 0.2$, 0.5, 1.0, 2.0 and 3.0, and those of non-dimensional spring constant $k_s^* = k_s/k_b = 1.0$, 10.0 and 100.0, the influence of the joint terms on the lowest five frequency coefficients $\overline{\Omega}_j = \overline{\omega}_j\sqrt{\rho A L^4/EI}$ (j = 1~5) are shown in Table 1 and Table 2 for the cases of spring-mass system located at $\xi = x/L = 0.5$ and 2/3, respectively. In Tables 1 and 2, the results of "Present" paper are obtained with the joint terms considered, but those of Rossi *et al.* (1993) are with the joint terms neglected. From the two tables one sees that the values of $\overline{\Omega}_j$ (j = 1~5) obtained from the present paper are smaller than the corresponding ones obtained from Rossi *et al.* (1993) and the divergence increases with the increase of vibration order (j). In other words, the consideration of effect of the joint terms of shear deformation and rotary inertia will reduce the natural frequencies of the Timoshenko beam. Although the reducing percentage is not very large, but their effect on the accuracy of the "exact" solutions, such as those of Rossi *et al.* (1993) and this paper, will be significant and cannot be neglected.

The influence of the joint terms on the lowest five mode shapes of the simply supported Timoshenko beam carrying a spring-mass system located at $\xi = x/L = 0.5$ with non-dimensional point mass $m_s^* = 0.2$ and spring constant $k_s^* = 1.0$ are shown in Fig. 2. In which the dashed curves with stars (---★---) denote the mode shapes obtained with joint terms neglected; while the solid curves (——) denote those with joint terms considered. Because the associated natural frequencies shown in Table 1 are very close to each other, so are the corresponding mode shapes shown in Fig. 2.

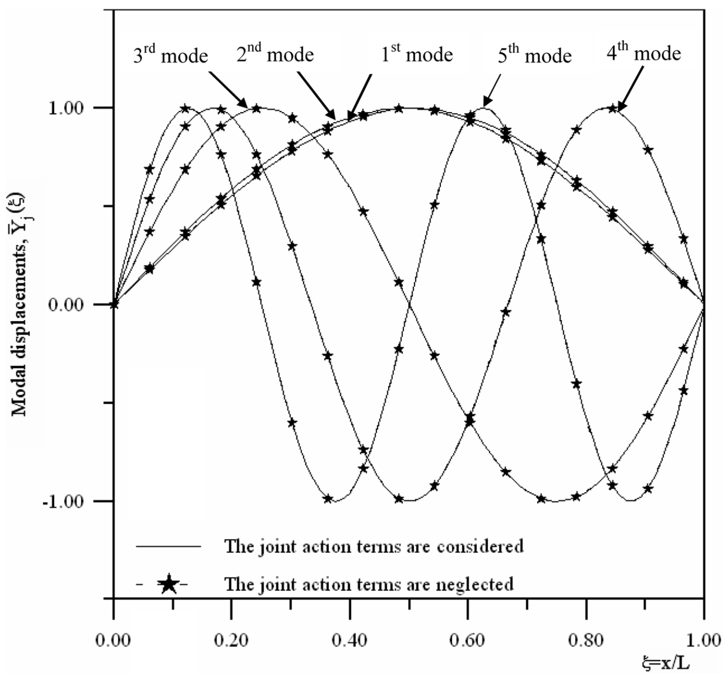

Fig. 2 The lowest five mode shapes for a simply supported Timoshenko beam carrying a spring-mass system located at $\xi = x/L = 0.5$ with non-dimensional point mass $m_s^* = 0.2$ and spring constant $k_s^* = 1.0$: (a) first mode, (b) second mode, (c) third mode, (d) fourth mode and (e) fifth mode. Key: ---★--- with joint terms neglected; —— with joint terms considered

3.2 Influence of joint terms with varying slenderness ratio and shear coefficient

This subsection studies the influence of beam length L (or slenderness ratio L/r_g) and shear coefficient k' on the lowest five natural frequencies of the simply supported uniform Timoshenko beam carrying "one" spring-mass system (cf. Fig. 1). The location of the spring-mass system is at $\xi = x/L = 0.5$ and the non-dimensional spring constant and lumped mass are given by

Table 3 Influence of shear coefficient k' and beam length L on the lowest five natural frequencies of the simply supported uniform Timoshenko beam carrying one spring-mass system (located at $\xi = x/L = 0.5$ with $k_s^* = k_s/k_b = 1.0$ and $m_s^* = m_s/m_b = 0.2$)

Shear coeff. k'	Beam length L (*in*)	Methods	$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$
2/3	40	Present A	28.43040	121.51134	420.84653	810.25640	1234.51057
		Present B	28.43044	121.51154	420.84864	810.26298	1234.52363
	50	Present A	18.20404	79.24979	285.46890	569.93133	893.18167
		Present B	18.20410	79.24988	285.46994	569.93493	893.18939
	60	Present A	12.64492	55.62513	205.39451	420.92041	674.60254
		Present B	12.64499	55.62528	205.39507	420.92251	674.60736
	70	Present A	9.29152	41.13765	154.41200	322.52217	526.13179
		Present B	9.29163	41.13772	154.41233	322.52347	526.13492
	80	Present A	7.11452	31.63306	120.08921	254.39455	420.84653
		Present B	7.11462	31.63308	120.08942	254.39539	420.84864
	90	Present A	5.62171	25.06926	95.94635	205.42501	3436.4203
		Present B	5.62181	25.06928	95.94648	205.42557	343.64349
	100	Present A	4.55389	20.35012	78.35285	169.13309	285.46890
		Present B	4.55390	20.35013	78.35294	169.13348	285.46994
5/6	40	Present A	28.43785	122.52192	431.77823	845.04518	1304.79043
		Present B	28.43786	122.52208	431.78003	845.05104	1304.80247
	50	Present A	18.20713	79.68977	290.81128	588.77308	934.27360
		Present B	18.20714	79.68984	290.81216	588.77622	934.28057
	60	Present A	12.64644	55.84488	208.26713	431.84981	699.96647
		Present B	12.64645	55.84492	208.26760	431.85162	699.97074
	70	Present A	9.29242	41.25882	156.07690	329.22345	542.48335
		Present B	9.29243	41.25884	156.07718	329.22456	542.48608
	80	Present A	7.11508	31.70507	121.11391	258.69717	431.77823
		Present B	7.11509	31.70508	121.11408	258.69788	431.78004
	90	Present A	5.62210	25.11466	96.60866	208.29716	351.18141
		Present B	5.62211	25.11467	96.60877	208.29763	351.18266
	100	Present A	4.55400	20.36294	78.54569	169.99752	287.82380
		Present B	4.55408	20.38012	78.79863	171.11510	290.81128

Note: The joint terms are considered in the "Present A" and neglected in "Present B".

$k_s^* = k_s/k_b = 1.0$ and $m_s^* = m_s/m_b = 0.2$, respectively. The dimensions and physical properties of the Timoshenko beam are exactly the same as those for the last subsection except that the beam length L is changed from 40 *in* to 100 *in* (with increment 10 *in*) and the shear coefficient k' is changed from 2/3 to 5/6. The results are shown in Table 3, where “Present A” refers to the natural frequencies of the combined system obtained from the present paper with the joint terms considered and “Present B” refers to those with the joint terms neglected. It is seen that the natural frequencies associated with “Present A” are smaller than the corresponding ones associated with “Present B”. In other words, consideration of the joint terms will reduce the natural frequencies of the combined vibrating system. This phenomenon agrees with the conclusion of the last subsection.

Besides, the effect of the joint terms increases with increasing the beam length L (or slenderness ratio) if the shear coefficient k' is kept unchanged. On the other hand, the last effect decreases with increasing the shear coefficient k' if the beam length L (or slenderness ratio) is kept unchanged.

3.3 Influences of total number of attachments and boundary conditions

In this subsection the same Timoshenko beam as the previous examples is studied, but the total number of attached spring-mass systems is three (rather than one) and the boundary conditions include CF, CS and CC (in addition to SS). In which, C = clamped, F = free and S = simply supported. The locations and magnitudes of the three spring-mass systems are shown in Table 4. It is seen that the locations of the “three” spring-mass systems are: $\xi_i = x_i/L =$ 0.1, 0.5 and 0.9 ($i = 1$ to 3); the non-dimensional spring constants are: $k_{si}^* = k_{si}/k_b = 3.0$, 4.5 and 6.0; the non-dimensional

Table 4 The locations and magnitudes of the “three” spring-mass systems on the uniform Timoshenko beam

Locations of spring-mass systems $\xi_i = x_i/L$			Magnitudes of non-dimensional spring constants $k_{si}^* = k_{si}/k_b$			Magnitudes of non-dimensional point masses $m_{si}^* = m_{si}/m_b$		
ξ_1	ξ_2	ξ_3	k_{s1}^*	k_{s2}^*	k_{s3}^*	m_{s1}^*	m_{s2}^*	m_{s3}^*
0.1	0.5	0.9	3.0	4.5	6.0	0.2	0.5	1.0

Table 5 Influence of boundary conditions on the lowest five natural frequencies for a uniform Timoshenko beam carrying “three” spring-mass systems with parameters shown in Table 4

Boundary conditions	Methods	Natural frequencies (rad/sec)				
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
CF	Present A	18.42326	37.26291	49.76759	76.70573	250.82646
	Present B	18.42328	37.26292	49.76760	76.70575	250.82737
CS	Present A	31.26508	37.64204	49.76929	183.57115	504.22925
	Present B	31.26511	37.64206	49.76930	183.57174	504.23251
CC	Present A	31.43121	37.98989	49.77189	246.30786	571.02117
	Present B	31.43127	37.98990	49.77190	246.30916	571.02608
SS	Present A	31.14518	36.67430	49.60288	128.74461	432.90807
	Present B	31.14519	36.67431	49.60289	128.74477	432.90829

Note: The joint terms are considered in the “Present A” and neglected in “Present B”.

point masses are: $m_{si}^{*} = m_{si}/m_b = 0.2$, 0.5 and 1.0. The influence of boundary conditions on the lowest five natural frequencies of the uniform Timoshenko beam is shown in Table 5. From Tables 3 and 5, one finds that the influence of the joint terms on the lowest five natural frequencies of the SS Timoshenko beam carrying "one" spring-mass system is greater than that carrying "three" spring-mass systems. It is believed that this is a reasonable result, because the influence on the dynamic characteristics of a SS beam due to a "concentrated" attachment is greater than that due to "distributed" attachments. From Table 5 one also finds that, among the four boundary conditions (CF, CS, CC and SS), the effect of the joint terms is smallest for the CF beam and is largest for the CC beam. This is also a reasonable result, because, for the same order, the "wave length" of the mode shape of a CF beam is longer than the corresponding one of a CC beam.

4. Conclusions

1. In most of the existing literature, the joint terms of shear deformation and rotary inertia in the differential equation of motion for the Timoshenko beam are neglected for simplicity. However, the reports regarding the influence of these joint terms on the free vibration characteristics of the Timoshenko beam are not found yet. Thus, the information presented in this paper will be helpful for clarifying the reasonability of neglecting these joint terms.
2. Numerical results of this paper reveal that, if only the "approximate" natural frequencies of the combined vibrating systems are required, then the effect of the joint terms may be neglected. However, if the "exact" solutions are required, then the effect of the joint terms should be considered, because it affects the accuracy of the results to some degree, particularly for the higher-mode natural frequencies.
3. Consideration of the joint terms will reduce the natural frequencies of the combined vibrating system. The effect of the joint terms increases with increasing the beam length L (or slenderness ratio) if the shear coefficient k' is kept unchanged. On the contrary, the last effect decreases with increasing the shear coefficient k' if the beam length L (or slenderness ratio) is kept unchanged.
4. The effect of the joint terms on the lowest five natural frequencies of a Timoshenko beam carrying a "concentrated" spring-mass system is greater than that carrying multiple (or "distributed") spring-mass systems.
5. The effect of the joint terms on the lowest five natural frequencies of a Timoshenko beam is closely related to the boundary (supporting) conditions. For the four boundary conditions (CF, CS, CC and SS) studied, the effect of the joint terms is smallest for the CF beam and is largest for the CC beam. In which, C = clamped, F = free and S = simply supported.

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