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An approximate formula to calculate the fundamental period of a fixed-free mass-spring system with varying mass and stiffness

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Abstract. A formula to approximate the fundamental period of a fixed-free mass-spring system with varying mass and varying stiffness is formulated. The formula is derived mainly by taking the dominant parts from the general form of the characteristic polynomial, and adjusting the initial approximation by a coefficient derived from the exact solution of a uniform case. The formula is tested for a large number of randomly generated structures, and the results show that the approximated fundamental periods are within the error range of 4% with 90% of confidence. Also, the error is shown to be normally distributed with zero mean, and the width of the distribution (as measured by the standard deviation) tends to decrease as the total number of discretized elements in the system increases. Other possible extensions of the formula are discussed, including an extension to a continuous cantilever structure with distributed mass and stiffness. The suggested formula provides an efficient way to estimate the fundamental period of building structures and other systems that can be modeled as mass-spring systems.

Keywords: earthquake engineering; eigenvalue; eigensystem; fundamental period.

1. Introduction

Mass-spring systems are used as idealizations in many areas of science and engineering. Although eigenfrequency of complex systems are often calculated by using advanced methods such as matrix

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condensation (Li 2003) or a variation of Galerkin's method (Lee and Renshaw 2002), the amount of information obtained from such methods can be superfluous according to the purpose the analysis serves. Furthermore, the methods usually necessitate execution of a sophisticated computer simulation. If only an approximate bound of the frequency is needed, a method developed by Rajendran (2002) or Chen *et al.* (2004) can be used. Access to highly accurate description of complex dynamic systems through these methods may be suitable for analytical purposes, however engineers may opt for a simpler yet representative view of the structures. Hence, dynamic systems are often represented by a simple array of beams, springs, and attached masses. There are numerous studies that propose methods of calculating eigenfrequencies for such systems. For simple systems, Dunkerley's method (Jacobsen and Ayre 1958, Thomson 1981), or Rayleigh's method (Meirovitch 1986, Thomson 1981, Chopra 1995) can be used. However, utilization of the methods involves assuming a suitable deflection shape, and iteration may be required to increase accuracy of the initial calculation. Low's study (Low 2000) illustrates how concentrated mass can be treated in calculating approximate eigenfrequencies by virtue of a modified Dunkerley's method, illustrating another example of such type of research.

In more advanced cases, modern studies provide solutions. Gürgöze (1996, 2005), and Gürgöze and Zeren (2006) propose methods to calculate eigenfrequencies in various configurations of cantilevers with attached masses. For configurations where additional attachments are other than masses, study of Cha (2005) provides a framework of formulating the frequency equation. Especially, Gürgöze in a recent study (Gürgöze 2006) proposes a formula to calculate eigenfrequencies of torsional vibration on an array of elastic bars and masses. The author tackles the problem by directly investigating the related kernel matrices, which is similar to the method used in this study. Strong contribution of matrix trace to eigenfrequncies is used in the study, and the trend is originally reported by Braun (2003), and Strobach and Braun (2003). The contribution of matrix trace is also rediscovered in this study.

In special cases, exact solutions may be available. Chen (2006) gives a theoretical solution for a uniform circular shaft with concentrated elements such as rotary inertia or torsional springs, and Gökda and Kopmaz (2005) give a solution for combination of a beam and a rod. However, the range of application can be fairly limited, and computational cost may be considerable for using the exact formulas.

An example of the practical application is determining the fundamental period of building structures subjected to earthquake. In the field of practical engineering, a quick and easy-to-use formula to evaluate the fundamental period is often necessary. Uniform Building Code (Structural Engineers Association of California 1997) suggests a statistically based formula for the purpose. However, the formula leaves room for improvement, in that the formula does not account for structural details such as floor stiffness and mass. For future development of design codes, more advanced approximation method may be necessary. An analysis method proposed by Kim *et al.* (2006) is an effort to improve the code, and their research takes advantage of the formula suggested in this study.

When the total number of masses is large, the mass-spring system with varying stiffness represents a discretization of a continuous cantilever beam or rod with varying section, and the formula for calculating the fundamental period of the discrete system can be extended to the continuous case. A number of such systems have been solved for exact eigenvalues, and for specific types of stiffness distribution. Distributions including linear, quadratic, exponential, and trigonometric were obtained by Kumar and Sujith (1997). More results about the non-uniform

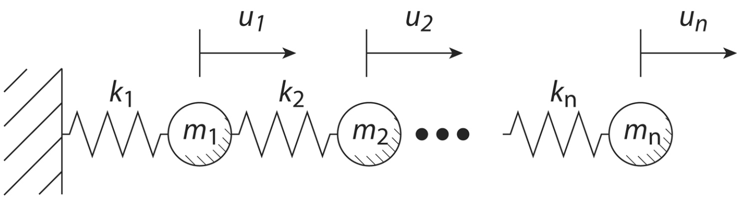

Fig. 1 Configuration of the varying mass-spring system

cantilever systems can be found in studies of Bapat (1995) and in Abrate (1986). The formula presented in this study can treat not only the cases of smooth distribution of stiffness, but may be also applied to the cases of discontinuous distributions where different materials are stacked with jumps. Such layered structures were studied by Li *et al.* (2002). An example is given in this study that extends the proposed discrete model to continuous limit, and accuracy is measured by comparing the approximate results with the exact solution provided by Kumar and Sujith (1997).

2. Formulation

Configuration of the varying mass-spring system is shown in Fig. 1, where the quantity u_i, measured from the equilibrium position of mass m_i, denotes the displacement relative to the fixed base. The governing equation of the system with no external force is

$$\mathbf{M\ddot{u}} + \mathbf{Ku} = \mathbf{0} \tag{1}$$

where mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ are in size of $n \times n$, and displacement vector $\mathbf{u}$ and zero vector $\mathbf{0}$ are in size of $n \times 1$. Elements M_{ij} of the diagonal matrix $\mathbf{M}$ with ranges of i and j in $1, \ldots, n$ can be represented by

$$M_{ij} = m_i \delta_{ij} \tag{2}$$

where δ_{ij} is the Kronecker delta with the property $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. Elements K_{ij} of the tri-diagonal matrix $\mathbf{K}$ can be expressed as

$$K_{ij} = (k_i + k_{i+1})\delta_{ii} - k_j \delta_{(i+1)j} - k_i \delta_{i(j+1)} \tag{3}$$

$\mathbf{K}$ can be decomposed by an upper triangular matrix $\mathbf{E}$ and a diagonal matrix $\mathbf{k}$, such that

$$\mathbf{K} = \mathbf{EkE}^T \tag{4}$$

where

$$\mathbf{k} = [k_i \delta_{ij}] \tag{5}$$

and

$$\mathbf{E} = [\delta_{ij} - \delta_{(i+1)j}] \tag{6}$$

Note that Eq. (4) forms a Cholesky decomposition in a uniform case, such that $\mathbf{K} = k\mathbf{E}\mathbf{E}^T$ where $\mathbf{k} = k\mathbf{I}$. Inserting Eq. (4) into Eq. (1) yields

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{E}\mathbf{k}\mathbf{E}^T\mathbf{u} = \mathbf{0} \tag{7}$$

The problem of approximating the fundamental period of this system is essentially equivalent to approximating the largest eigenvalue of a matrix $\mathbf{E}^{-T}\mathbf{k}^{-1}\mathbf{E}^{-1}\mathbf{M}$, or the smallest eigenvalue of $\mathbf{M}^{-1}\mathbf{E}\mathbf{k}\mathbf{E}^T$. Unfortunately, deriving an exact eigenvalue of the case is not feasible, and the complexity of the problem suggests that an exact solution, if any, may be impractical.

Artin (1991) suggested that the characteristic polynomial $p(\lambda)$ of a $n \times n$ matrix $\mathbf{A}$ has the form

$$p(\lambda) = \lambda^n - (tr\mathbf{A})\lambda^{n-1} + \text{L.O.D} + (-1)^n(\det \mathbf{A}) \tag{8}$$

where $tr\mathbf{A}$, the *trace* of $\mathbf{A}$, is the sum of the diagonal entries

$$tr\mathbf{A} = a_{11} + a_{22} + \ldots + a_{nn} \tag{9}$$

λ is an eigenvalue, det $\mathbf{A}$ is the determinant of the matrix $\mathbf{A}$, and *L.O.D* denotes lower order terms of λ.

In the application of the formula, the matrix $\mathbf{A}$ should be substituted by $\mathbf{E}^{-T}\mathbf{k}^{-1}\mathbf{E}^{-1}\mathbf{M}$. Note that the determinant of $\mathbf{M}^{-1}\mathbf{E}\mathbf{k}\mathbf{E}^T$ is

$$\det(\mathbf{M}^{-1}\mathbf{E}\mathbf{k}\mathbf{E}^T) = \frac{k_1}{m_1}\frac{k_2}{m_2}\ldots\frac{k_n}{m_n}\det\mathbf{E}\det\mathbf{E}^T \tag{10}$$

since $\det(\mathbf{M}^{-1}\mathbf{E}\mathbf{k}\mathbf{E}^T) = \det(\mathbf{M}^{-1})\det(\mathbf{E})\det(\mathbf{k})\det(\mathbf{E}^T)$, where $\det(\mathbf{k}) = k_1k_2\ldots k_n$, and $\det(M^{-1}) = m_1^{-1}m_2^{-1}\ldots m_n^{-1}$ (the determinant of a diagonal matrix is the product of the diagonal terms). det $\mathbf{E}$ and det $\mathbf{E}^T$ are 1, since both are triangular matrices with diagonal entries equal to 1. Eq. (10) can be written as

$$\det(\mathbf{M}^{-1}\mathbf{E}\mathbf{k}\mathbf{E}^T) = \frac{k_1}{m_1}\frac{k_2}{m_2}\ldots\frac{k_n}{m_n} \tag{11}$$

Inverse of Eq. (11) is the determinant of $\mathbf{E}^{-T}\mathbf{k}^{-1}\mathbf{E}^{-1}\mathbf{M}$, and is

$$\det(\mathbf{E}^{-T}\mathbf{k}^{-1}\mathbf{E}^{-1}\mathbf{M}) = \frac{m_1}{k_1}\frac{m_2}{k_2}\ldots\frac{m_n}{k_n} \tag{12}$$

The value of m_i/k_i can be related to the period of a local oscillator such that $T_i = 2\pi\sqrt{m_i/k_i} = 2\pi/\omega_i$ where ω_i is the local radial frequency. The whole system is a chain of such local oscillators and the fundamental period of the total structure tends to be larger than the fundamental periods of the small scale compartments. Note that the global radial frequency ω can be related to the largest eigenvalue of the system λ by $T = 2\pi/\omega = 2\pi\lambda$. Hence, it can be said $T_i < T$, and $m_i/k_i < \lambda$. Thus, $\Pi(m_i/k_i)$ can be neglected since the contribution of the product is small compared to the leading term λ^n which is the product of $\lambda > \lambda_i$. If λ is taken as the largest eigenvalue of the system, *L.O.D* also can be neglected, and Eq. (8) further simplifies to

$$p(\lambda) \simeq \lambda^n - (tr\mathbf{A})\lambda^{n-1} \tag{13}$$

where lower order terms and the determinant are neglected. The largest eigenvalue λ can be evaluated by setting $p(\lambda) = 0$. It is

$$\lambda - (tr\mathbf{A}) \approx 0 \tag{14}$$

Hence, λ can be approximated by

$$\lambda \approx tr\mathbf{A} \tag{15}$$

Matrix **A**, which is $\mathbf{E}^{-T}\mathbf{k}^{-1}\mathbf{E}^{-1}\mathbf{M}$, is obtained by inverting each matrix in the composition. The resulting matrix is of the form

$$\mathbf{ME}^{-1}\mathbf{k}^{-1}\mathbf{E}^{-T} = \begin{bmatrix} \frac{m_1}{k_1} & \frac{m_2}{k_1} & \frac{m_3}{k_1} & \cdots & \frac{m_n}{k_n} \\ \frac{m_1}{k_1} & \frac{m_2}{k_1}+\frac{m_2}{k_2} & \frac{m_3}{k_1}+\frac{m_3}{k_2} & \cdots & \frac{m_n}{k_1}+\frac{m_n}{k_2} \\ \frac{m_1}{k_1} & \frac{m_2}{k_1}+\frac{m_2}{k_2} & \frac{m_3}{k_1}+\frac{m_3}{k_2}+\frac{m_3}{k_3} & \cdots & \frac{m_n}{k_1}+\frac{m_n}{k_2}+\frac{m_n}{k_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{m_1}{k_1} & \frac{m_2}{k_1}+\frac{m_2}{k_2} & \frac{m_3}{k_1}+\frac{m_3}{k_2}+\frac{m_3}{k_3} & \cdots & \frac{m_n}{k_1}+\frac{m_n}{k_2}+\ldots+\frac{m_n}{k_n} \end{bmatrix} \tag{16}$$

Trace of the matrix in Eq. (16) is

$$tr\mathbf{A} = \frac{m_1+m_2+\ldots+m_n}{k_1}+\frac{m_2+m_3+\ldots+m_n}{k_2}+\ldots+\frac{m_n}{k_n} \tag{17}$$

The varying masses m_i can be consolidated by introducing an equivalent uniform mass-varying stiffness system. Eq. (17) can be transformed as

$$\begin{aligned} tr\mathbf{A} &= \frac{\sum_{j=1}^{n} m_j}{k_1}+\frac{\sum_{j=2}^{n} m_j}{k_2}+\ldots+\frac{m_n}{k_n} \\ &= \frac{nM_1}{\hat{k}_1}+\frac{(n-1)M_2}{\hat{k}_2}+\ldots+\frac{M_1}{\hat{k}_n} \end{aligned} \tag{18}$$

where $M_1 = \sum_{i=1}^{n} m_i/n$ is the average value of m_i, and the equivalent stiffness $\hat{k}_i$ can be solved by comparing each term in the summation

$$\begin{aligned} \frac{\sum_{j=i}^{n} m_j}{k_i} &= \frac{(n-i+1)}{n}\frac{\sum_{j=1}^{n} m_j}{\hat{k}_i} \\ \Leftrightarrow \hat{k}_i &= \frac{(n-i+1)}{\sum_{j=i}^{n} m_j}\frac{\sum_{j=1}^{n} m_j}{n}\cdot k_i = \frac{M_1}{M_i}\cdot k_i \end{aligned} \tag{19}$$

where M_i is the average value of m_j with j ranging from $j = i$ to $j = n$, such that $M_i = \sum_{j=i}^{n} m_j/(n-i+1)$. Thus, the fundamental period of this system can be calculated by substituting k_i with $\hat{k}_i$, and m_i with M_i. Let $m = M_1$, which is the average value of all m_i. Then, it can be said that

$$\lambda \approx tr\mathbf{A} = m\left[\frac{n}{\hat{k}_1} + \frac{n-1}{\hat{k}_2} + \ldots + \frac{1}{\hat{k}_n}\right] \tag{20}$$

Although Eq. (20) gives an initial approximation of the solution, accuracy is expected to be poor, especially by noting the fact that (Goldberg 1992)

$$tr\mathbf{A} = \lambda_1 + \lambda_2 + ,\ldots, + \lambda_n \tag{21}$$

In other words, the approximate largest eigenvalue is the totality of the eigenvalues, and eigenvalues for higher modes are negligible.

Fortunately, accuracy of Eq. (20) can be further improved by inspection. That is comparing the approximate solution to the exact eigenvalues for a special case. The special case can be defined by $\hat{k}_i = \hat{k}$, i.e., the uniform case. The exact solution of the eigenvalue for the uniform case can be derived by solving the Chebyshev polynomial of the first kind that satisfies the curious determinant equation. The solution is well known and has the form of (Elliot 1953, Gregory 1978)

$$\frac{1}{\lambda_{exact}} = \frac{2\hat{k}}{m}\left[1 - \cos\left(\frac{\pi}{2n+1}\right)\right] \tag{22}$$

The eigenvalue for the same case can be approximated by using Eq. (20), such that

$$\lambda_{approx} = \frac{m}{\hat{k}}(1 + 2 + \ldots + n) = \frac{m}{\hat{k}}\frac{n(n+1)}{2} \tag{23}$$

Hence, the correction factor Ψ, which is only dependent on the total number of masses, can be written as

$$\frac{1}{\Psi} = \frac{\lambda_{approx}}{\lambda_{exact}} = n(n+1)\left[1 - \cos\left(\frac{\pi}{2n+1}\right)\right] \tag{24}$$

λ in Eq. (20) should be corrected by the correction factor Ψ in Eq. (24) to give better accuracy. Thus, the corrected formula for the largest eigenvalue λ' becomes

$$\begin{aligned} \lambda' &= \lambda\Psi \\ &= m\left[\frac{n}{\hat{k}_1} + \frac{(n-1)}{\hat{k}_2} + \ldots + \frac{1}{\hat{k}_n}\right]\left\{n(n+1)\left[1 - \cos\left(\frac{\pi}{2n+1}\right)\right]\right\}^{-1} \end{aligned} \tag{25}$$

The corresponding fundamental period T is

$$\begin{aligned} T &= 2\pi\sqrt{\lambda'} = 2\pi\sqrt{\lambda\Psi} \\ &= 2\pi\sqrt{\frac{m}{n(n+1)}\left[\frac{n}{\hat{k}_1} + \frac{(n-1)}{\hat{k}_2} + \ldots + \frac{1}{\hat{k}_n}\right]\left[1 - \cos\left(\frac{\pi}{2n+1}\right)\right]^{-1}} \end{aligned} \tag{26}$$

3. Numerical test of accuracy

To verify the accuracy of the proposed formula, a Monte-Carlo style experiment is executed. When $\hat{k}_{\max} = \max[\hat{k}_i]$ is factored out from Eq. (26), it becomes

$$T = 2\pi\sqrt{\frac{m}{\hat{k}_{\max}}\frac{1}{n(n+1)}\left[\frac{n}{\eta_1}+\frac{(n-1)}{\eta_2}+\ldots+\frac{1}{\eta_n}\right]\left[1-\cos\left(\frac{\pi}{2n+1}\right)\right]^{-1}} \tag{27}$$

where $\eta_i = \hat{k}_i/\hat{k}_{\max} \le 1$. Hence, the fundamental period of the structure can be expressed as

$$T = 2\alpha\pi\sqrt{\frac{m}{\hat{k}_{\max}}} \tag{28}$$

where

$$\alpha = \sqrt{\frac{1}{n(n+1)}\left[\frac{n}{\eta_1}+\frac{(n-1)}{\eta_2}+\ldots+\frac{1}{\eta_n}\right]\left[1-\cos\left(\frac{\pi}{2n+1}\right)\right]^{-1}} \tag{29}$$

The exact value of α can be calculated by evaluating the eigenvalues of $\mathbf{K}/\hat{k}_{\max}$. The relative error of the approximation in Eq. (26), then, can be calculated as

$$\varepsilon = \frac{T_{approx}}{T_{exact}} - 1 = \frac{\alpha_{approx}}{\alpha_{exact}} - 1 \tag{30}$$

For the numerical experiment, 3000 mass-spring systems were generated for a fixed n, with randomly generated η_i. Note that by using the average value m, and the equivalent stiffness $\hat{k}_i$, the simulated cases can cover both varying mass and stiffness. The range of η_i was set to 0.1~1 and uniform probabilistic distribution was used, so that the maximum value of $\hat{k}_i$ should not exceed 10 times the minimum value of $\hat{k}_i$. The exact value of α was calculated by using a numerical routine

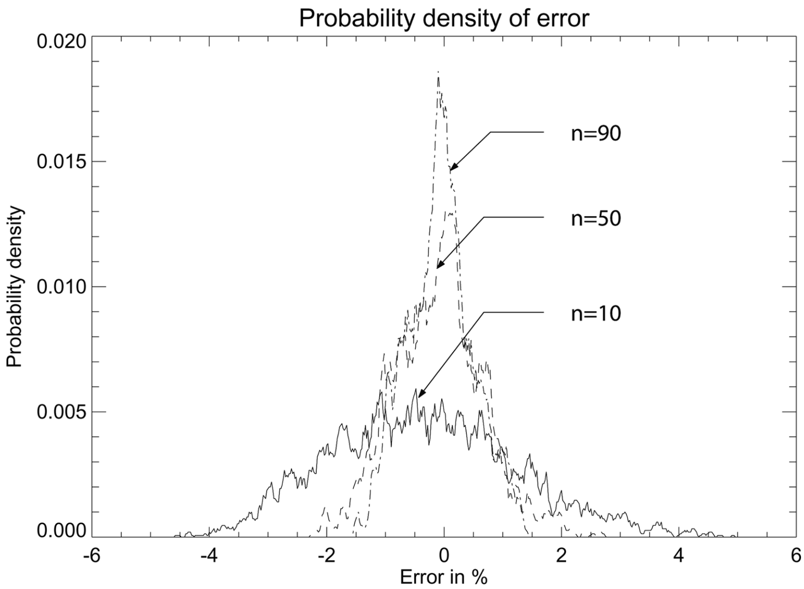

Fig. 2 Probability density of error for n = 10, 50, and 90

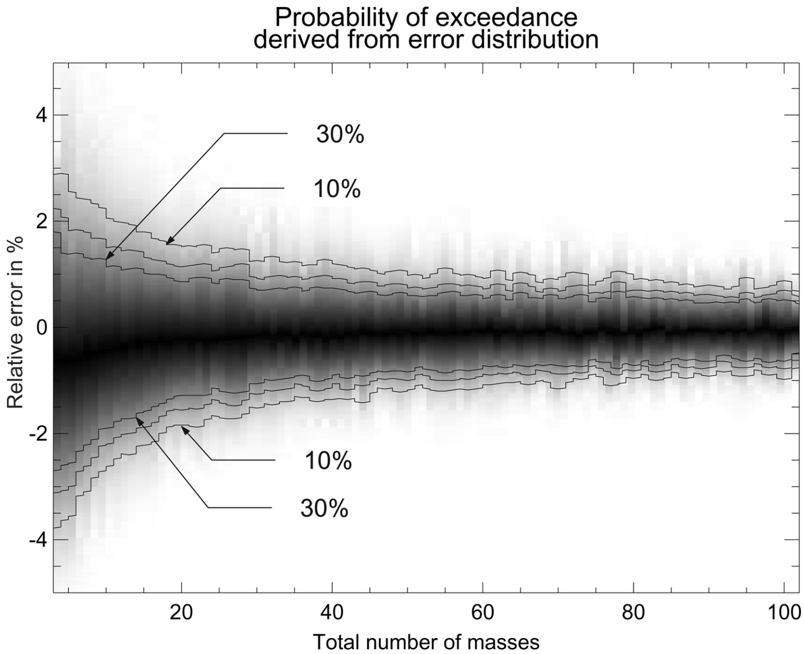

Fig. 3 Probability of exceedance derived from error probability distribution

for tri-diagonal matrices (Press *et al.* 1992). Then, the error in Eq. (30) was calculated for each case in the 3000 sample structures. A relative frequency histogram, using a bin size of 0.02%, was constructed using the error results. The process was repeated for n = 3~103. Note that the case of $n = 2$ is omitted since the case is trivial and the fundamental period can be calculated directly from the characteristic polynomial without approximation. The histograms, which are estimations of the probability density functions, are shown in Fig. 2 for the cases of n = 10, 50, and 90. It is shown that the probability density curves resemble the shape of normal distribution, and the mean values are located near 0%. Furthermore, the width of the distributions become smaller as n increases, implying that the error range becomes smaller and the accuracy of the estimation increases for larger n.

Probability of exceedance $P(\mu, |\varepsilon|)$ about the mean error μ can be calculated by integrating the density function as

$$P(\mu, |\varepsilon|) = 1 - \int_{\mu - |\varepsilon|}^{\mu + |\varepsilon|} p(\varepsilon) d\varepsilon \tag{31}$$

where $p(\varepsilon)$ is the probability density function of the error. Fig. 3 shows the probability of exceedance for all n (n = 3~103). Contour lines are added to help identify the 10% separation of the probability of exceedance. Note that the mean value μ at each n is not exactly zero, and hence Fig. 3 is not symmetric about zero axis (The chart is symmetric about μ at each n, instead). In fact the chart shows that the location of the mean value (dark area in Fig. 3) tends to curve down to the negative side as n decreases. The chart clearly shows a pattern of increased accuracy at larger n. With 90% confidence (10% probability of exceedance), the error range is −3.5%~2.8% at n = 3, and −0.8% ~0.8% at n = 103. The result indicates that the formula in Eq. (26) may be adequately accurate when the minimum value of $\hat{k}_i$ is bounded at a reasonable level (0.1 $\hat{k}_{\max}$ for the presented test).

4. Example 1 : Calculation of the fundamental period of shear buildings with rigid floor beams

Fundamental periods of pre-existing buildings at Los Angeles are approximated by using the proposed formula, to give a practical example. The sample buildings were presented in the works of Gupta (Gupta and Krawinkler 2000). Basic configurations of the buildings are composed of 3, 9, and 20 stories, and floor beams are assumed rigid for convenience. When the rigid floor beam is assumed, story stiffness k_i at an ith floor can be calculated as $k_i = \sum[12(EI)_i/h_i^3]$, where h_i denotes the ith story height, and the summation is over all columns. Table 1 summarizes the story stiffnesses, and story masses of the sample buildings.

Approximated fundamental periods T_{Kim} for the buildings are tabulated in Table 2, along with the exact values calculated by using a numerical routine for tri-diagonal matrices (Press *et al.* 1992). For comparison with conventional approximation method, an additional column of $T_{Rayleigh}$ is provided. $T_{Rayleigh} = 2\pi/\lambda$ is calculated such that

$$\lambda = \frac{\Phi^T \mathbf{k} \Phi}{\Phi^T \mathbf{m} \Phi} \tag{32}$$

where Φ is the eigenvector of the first mode. Using Rayleigh's quotient requires the first mode shape Φ be known before evaluation of Eq. (32). Thus, a test Φ is often assumed and the method

Table 1 Story stiffnesses and masses of sample buildings

Story stiffness k_i in 100 kN/m, and story mass m_i in ton.											
3 story building			9 story building			20 story building					
Story	k_i	m_i	Story	k_i	m_i	Story	k_i	m_i	Story	k_i	m_i
1	2482	4.79	1	2315	5.03	1	3101	2.82	11	4781	2.76
2	2482	4.79	2	5812	5.03	2	8130	2.76	12	4346	2.76
3	2482	5.18	3	5490	5.03	3	8039	2.76	13	4346	2.76
			4	4771	5.03	4	8039	2.76	14	3544	2.76
			5	4110	5.03	5	6554	2.76	15	2833	2.76
			6	3539	5.03	6	5239	2.76	16	2833	2.76
			7	3018	5.03	7	5239	2.76	17	2676	2.76
			8	2839	5.03	8	5239	2.76	18	2524	2.76
			9	2666	5.34	9	5239	2.76	19	2086	2.76
						10	5239	2.76	20	1694	2.92

Table 2 Approximated and exact fundamental periods of sample buildings by using the method suggested in this study T_{Kim}, and by using Rayleigh's quotient $T_{Rayleigh}$

T_{exact}, T_{Kim}, $T_{Rayleigh}$, and the relative error $T_{aprroximated}/T_{exact} - 1$					
Sample building	T_{exact} (sec)	T_{Kim} (sec)	Error (%)	$T_{Rayleigh}$ (sec)	Error (%)
3 story	0.633	0.633	0.00	0.633	−0.96
9 story	1.40	1.42	1.43	1.31	−6.43
20 story	1.92	1.98	3.13	1.81	−5.24

becomes iterative (Chopra 1995). For comparison with the suggested method, Φ is assumed as the first mode shape of a uniform case, and calculated separately for each case. $T_{Rayleigh}$ shown in Table 2 are results of evaluating Eq. (32) once without iteration. The overhead of Φ assumption also highlights an advantage of using the suggested method in this study; Calculation of mode shape is not required.

Examination of relative errors corresponding to the methods reveals that for the case of T_{Kim}, the error is bounded within 3.13%, while for the case of $T_{Rayleigh}$, the error is bounded within 6.43%. The result exemplifies the practical accuracy of the suggested method. Note that the 0% error for the case of the 3 story building is due to the significant digit trade-off; if the full machine representable number is used, the error evaluates as −0.1798% for T_{Kim}. For quicker and simpler calculation, an engineer may opt for using an average value of m_i, and skip the calculation of $\hat{k}_i$, since the maximum variation of the story masses are 8.14, 6.16, and 5.80% respectively for the cases of 3, 9, and 20 story buildings. For the case of the simpler calculation of T_{Kim} using the average mass, results show −0.840, 1.01, and 2.91% of relative error for the three cases. As is expected, neglecting the small difference in mass hardly affects the quality of the output. In the case of the 9, and 20 story buildings, the error actually reduces when the average mass is used. More importantly, the method still gives more accurate result than using Rayleigh's quotient.

5. Example 2 : Application to buildings with a damaged or rehabilitated strory

Buildings can be modeled as mass-spring systems where the $\hat{k}_i$ values represent the equivalent story stiffnesses. If local damage or rehabilitation is applied at the ith story, the equivalent stiffness of the story is altered to $\delta\hat{k}_i$ where δ is a scale factor ($\delta < 1$ for a damaged story, and $\delta > 1$ for a rehabilitated story). Let T_1 be the original fundamental period before damage or rehabilitation, and T_2 be the period after the alteration. Also, let $T_2 = \sqrt{T_1^2 - \Delta T^2}$, and equivalently $\Delta T^2 = T_1^2 - T_2^2$.

For the altered story, the original term $(n-i+1)/\hat{k}_i$ in Eq. (26) changes to $(n-i+1)/(\delta\hat{k}_i)$, while all the other terms in the summed series vanish after subtraction. Hence, the square difference ΔT^2 can be calculated as

$$\begin{aligned}\Delta T^2 &= T_1^2 - T_2^2\\ &= 4\pi^2\frac{m}{n(n+1)}\left(\frac{n-i+1}{\hat{k}_i} - \frac{n-i+1}{\delta\hat{k}_i}\right)\left[1-\cos\left(\frac{\pi}{2n+1}\right)\right]^{-1}\\ &= 4\pi^2\left(\frac{m}{\hat{k}_i}\right)\left[\frac{n-i+1}{n(n+1)}\right]\left(\frac{\delta-1}{\delta}\right)\left[1-\cos\left(\frac{\pi}{2n+1}\right)\right]^{-1}\end{aligned} \tag{33}$$

As a numerical example, a 10 story building with linearly varying stiffness is considered. The linearly varying stiffness is set to be $\hat{k}_i = [1-0.5(i-1)/(n-1)]\cdot\hat{k}_i$ so that $\hat{k}_n = 0.5\hat{k}_1$. Stiffness at each story is increased and decreased 30%, and the normalized values of $\mathcal{T}_{exact} = \sqrt{\hat{k}_1/m}\sqrt{T_{exact}^2 - \Delta T_{exact}^2}$ and $\mathcal{T}_{approx} = \sqrt{\hat{k}_1/m}\sqrt{T_{exact}^2 - \Delta T_{approx}^2}$ are calculated by assuming the exact fundamental period before the alteration is known. Then, the relative error is calculated for each case. Table 3 shows the result from the calculation.

The results show that, for the presented case, the approximate formula in Eq. (33) can be effectively applied to the calculation of the fundamental periods of building structures after damage

Table 3 Results for the numerical example of a 10 story building with linearly varying stiffness $\hat{k}_i = [1-0.5(i-1)/(n-1)] \cdot \hat{k}_1$

	Comparison of T_{exact} and T_{approx}					
	30% increased stiffness			30% decreased stiffness		
Story level	T_{exact}	T_{approx}	Error (%)	T_{exact}	T_{approx}	Error (%)
1	45.28	45.27	−0.02	47.57	47.55	−0.04
2	45.26	45.31	0.10	47.59	47.48	−0.23
3	45.27	45.35	0.17	47.56	47.41	−0.32
4	45.32	45.40	0.17	47.47	47.32	−0.32
5	45.40	45.45	0.11	47.32	47.22	−0.22
6	45.52	45.52	0.01	47.12	47.10	−0.04
7	45.65	45.59	−0.13	46.88	46.97	0.19
8	45.80	45.68	−0.25	46.61	46.81	0.42
9	45.93	45.79	−0.32	46.36	46.61	0.56
10	46.04	45.92	−0.26	46.16	46.38	0.47

or rehabilitation. With the maximum error at 0.56% shown in Table 3, the approximation is also shown to be highly accurate.

6. Example 3 : Extension to continuous cases

When $u_i(t)$ is taken as a sampled response of a continuous function $u(x, t)$ in a uniform grid spaced $h = H/(n+1/2)$ in $x \in (0, H)$, the response vector becomes $\mathbf{u} = \{u(0, t), u(h, t), ..., u(nh, t)\}^T$. Then, $\mathbf{E}^T\mathbf{u}$ in Eq. (7) at each row in continuous sense is

$$u_i(t) - u_{i-1}(t) = u(ih, t) - u(ih-h, t) \tag{34}$$

Eq. (34) can be expanded in a Taylor series about h, to give

$$\begin{aligned} u(ih, t) - u(ih-h, t) &= u(x, t) - u(x-h, t) \\ &= u(x, t) - \left[u(x, t) - h \cdot u_x(x, t) + \frac{h^2}{2!} \cdot u_{xx}(x, t) - \dots\right] \\ &= h \cdot u_x(x, t) + h \cdot O(h) \end{aligned} \tag{35}$$

where $u_x(x, t)$ denotes $\partial u(x, t)/\partial x$, u_{xx} denotes $\partial u^2(x, t)/\partial x^2$, and $O(h)$ is the leading error term in order of h.

The spring constant k_i in continuous sense is

$$k_i = \frac{E_i A_i}{h} = \frac{E(x)A(x)}{h} = \frac{K(x)}{h} \tag{36}$$

where $E(x)$ is the Young's modulus, and $A(x)$ is the area of the cross section at x. Accordingly, mass m_i can be represented by $h\rho(x)A(x)$, where $\rho(x)$ is the mass density at x. By noting that each row of **Eu** can be expanded to give $u(x,t)-u(x+h,t) = -h\cdot\partial u(x,t)/\partial x + h\cdot O(h)$, the rows of Eq. (7) can be expanded to give a differential expression, such that

$$h\rho(x)A(x)u_{tt}(x,t) - h\left\{\frac{K(x)}{h}[h\cdot u_x(x,t) + h\cdot O(h)]\right\}_x + h\cdot O(h) = 0$$

$$\Leftrightarrow u_{tt}(x,t) - \frac{1}{\rho(x)A(x)}[K(x)u_x(x,t)]_x + O(h) = 0 \tag{37}$$

where subscripts denote partial derivatives.

As the structure described by Eq. (37) reaches a continuum $(h \to 0)$, the leading error term $O(h)$ vanishes, and Eq. (7) transforms to a governing equation of a cantilever rod, that is

$$u_{tt}(x,t) - \frac{1}{\rho(x)A(x)}[K(x)u_x(x,t)]_x = 0 \tag{38}$$

with boundary conditions imposed at $x = 0$ and $x = H$ as

$$\begin{aligned} u(0,t) &= 0 \\ u_x(1,t) &= 0 \end{aligned} \tag{39}$$

By applying the same technique to derive Eq. (37), Eq. (17) becomes

$$\begin{aligned} \lambda &= \frac{m_1 + m_2 + \ldots + m_n}{\hat{k}_1} + \frac{m_1 + m_2 + \ldots + m_{n-1}}{\hat{k}_2} + \frac{m_n}{\hat{k}_n} \\ &= \int_0^H \frac{1}{K(x)}\int_0^{H-x}\rho(\xi)A(\xi)d\xi dx \end{aligned} \tag{40}$$

The correction factor Ψ when $n \to \infty$ is

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{\Psi} &= \lim_{n\to\infty} n(n+1)\left[1-\cos\left(\frac{\pi}{2n+1}\right)\right] \\ &= \lim_{n\to\infty} n(n+1)\left[\frac{\pi^2}{2(2n+1)^2} - \frac{\pi^4}{4!(2n+1)^4} + \ldots\right] \\ &= \frac{\pi^2}{8} \end{aligned} \tag{41}$$

Hence, the fundamental period of the continuous cantilever rod in Eq. (38) can be approximated by

$$\begin{aligned} T &= 2\pi\sqrt{\lambda\Psi} \\ &= 2\pi\sqrt{\frac{8}{\pi^2}\int_0^H\frac{1}{K(x)}\int_0^{H-x}\rho(\xi)A(\xi)d\xi dx} \\ &= 4\sqrt{2}\sqrt{\int_0^H\frac{1}{K(x)}\int_0^{H-x}\rho(\xi)A(\xi)d\xi dx} \end{aligned} \tag{42}$$

For the special case of $K(x) = K = EA$, $A(x) = A$, and $\rho(x) = \rho$, Eq. (42) becomes

$$T = 4\sqrt{2}\sqrt{\frac{\rho A}{K}\int_0^H (H-x)dx}$$
$$= 4\sqrt{2}\sqrt{\frac{H^2}{2c^2}} = \frac{4H}{c} \tag{43}$$

where $c^2 = K/(\rho A) = E/\rho$ is the wave speed intrinsic to the material property of the structure. The result in Eq. (43) also agrees with the result suggested by Iwan (1995).

An interesting comparison can be made between the approximate formula in Eq. (42) and the exact solution of a special case. Kumar and Sujith (1997) derived an exact solution of the governing equation shown in Eq. (38), where the area varies as $A(x) = A_0\sin^2(ax+b)$. The solution is given by a transcendental equation, that is

$$[a/\tan(aH+b)]\tan kH = k \tag{44}$$

where k is defined as

$$k = \frac{\rho}{E}\left(\frac{2\pi}{T}\right)^2 + a^2 \tag{45}$$

For the case where $E(x) = E$ and $\rho(x) = \rho$, Eq. (42) can be written as

$$T = 4\sqrt{2}\sqrt{\frac{\rho H^2}{E}}\sqrt{\int_0^1 \frac{1}{A(x)}\int_0^{1-x} A(\xi)d\xi dx} \tag{46}$$

By setting $\beta = \sqrt{\rho H^2/E}$, Eq. (46) reduces to $T = \beta B$ where B is the product of $4\sqrt{2}$ and the square rooted integration part in Eq. (46). Note that β is in the unit of 1/velocity, and B is in the unit of length. Hence, T can be conveniently compared in terms of β. For the case where $a = 1$, $b = 1$, and min[$A(x)$]/max[$A(x)$] = 0.708, exact solution by Kumar and Sujith gives $T_{exact} = 4.140\beta$ sec, while the approximate formula in Eq. (46) gives $T_{approx} = 4.026\beta$ sec. Relative error of the fundamental period is $\varepsilon = (T_{approx} - T_{exact})/T_{exact} = -2.747\%$. Hence, the approximate formula exhibits high accuracy.

Another example of the comparison where $a = 2$ and $b = 1$ gives $T_{approx} = 4.473\beta$ sec, and $T_{exact} = 2.924\beta$ sec. The relative error of the fundamental period calculates as $\varepsilon = 52.95\%$. The approximation outputs large error. The reason is suspected to be the fact that the ratio min[$A(x)$]/max[$A(x)$] = 0.0199, which implies extreme stiffness drop. Hence, a note should be made that if the min/max ratio of the given system is extremely low, usage of the approximation in Eq. (42) should be avoided.

7. Example 4 : Extension to higher modes

The formula for estimating the fundamental period can be extended to the evaluation of all the natural periods. The correction factor Ψ in Eq. (24), derived from the exact solution of the uniform case, is in fact (Kim 2003)

$$\frac{1}{\Psi_j} = n(n+1)\left\{1 - \cos\left[\frac{(2j-1)\pi}{2n+1}\right]\right\} \tag{47}$$

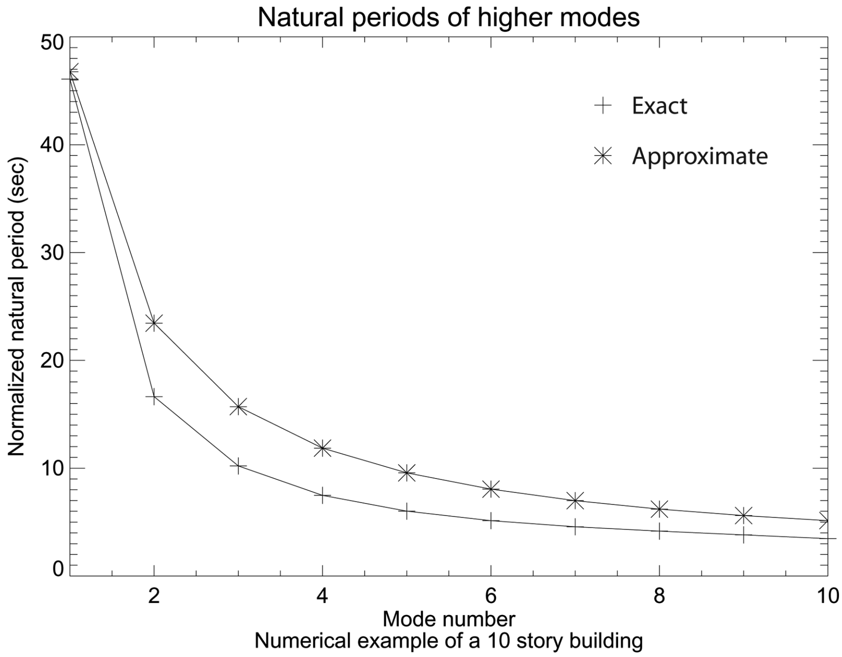

Fig. 4 Comparison of $T_{j,\,exact}$ and $T_{j,\,approx}$ for the numerical example of a 10 story building

where j denotes the mode number. Accordingly, periods T_j of all modes can be approximated by plugging in Eq. (47) to Eq. (26), such that

$$T_j = 2\pi\sqrt{\frac{m}{n(n+1)}\left[\frac{n}{\hat{k}_1}+\frac{(n-1)}{\hat{k}_2}+\ldots+\frac{1}{\hat{k}_n}\right]\left\{1-\cos\left[\frac{(2j-1)\pi}{2n+1}\right]\right\}^{-1}} \tag{48}$$

The continuous counterpart of Eq. (48) for the cantilever rod described by Eq. (38) is

$$\begin{aligned} T_j &= 2\pi\sqrt{\frac{8}{(2j-1)^2\pi^2}\int_0^H\frac{1}{K(x)}\int_0^{H-x}\rho(\xi)d\xi dx} \\ &= \frac{4\sqrt{2}}{2j-1}\sqrt{\int_0^H\frac{1}{K(x)}\int_0^{H-x}\rho(\xi)d\xi dx} \end{aligned} \tag{49}$$

As a numerical example, natural periods of the 10 story building used in Example 1 are presented. The normalized natural periods are shown in Fig. 4 for $T_{j,exact}$, and $T_{j,approx}$ by Eq. (48) at the jth mode. In the figure, the approximate and the exact periods are shown to match closely at j = 1, and diverge as j becomes large. Hence, the approximation is the most accurate at the lowest mode (j = 1) as is expected. Since the formulation is based on the approximation of the fundamental period, accuracy for the higher modes is not guaranteed. However, it is worthwhile to note that the curve diverges and then appears to start converging again for large n. The reason behind the trend of accuracy recovery at large n is uncertain, however.

8. Conclusions

A one line formula to estimate the fundamental periods of mass-spring systems with varying mass and stiffness is presented. The formula holds advantage over conventional methods in that

- it does not require calculation of mode shapes
- it is composed in a relatively simple form
- it does not require iteration

The proposed formula in numerical test (for the case of $\hat{k}_{\min} \geq 0.1\hat{k}_{\max}$) showed rational behavior in error trend. The accuracy was shown to increase at larger n, and probability of exceedance from the mean error was shown to be bounded. With the error range within ±4% at most, with confidence level of 90%, the approximate formula may prove adequate for practical application.

The formula, in the given examples of application, is expected to carry the same level of accuracy that was shown in the numerical test. Also, the examples suggest that the range of application may be wide enough for general usage whenever a quick estimate of the fundamental period is necessary. The applications may feature varying masses, continuous cases, or combination of each. Especially, the accuracy was tested on random structures, which means that the formula may even prove suitable for the application to random vibration analysis. Indeed, in the pool of generated structures, there were cases where the stiffness distribution was highly irregular. The highly irregular distribution necessarily involved jumps of stiffness, which means that the formula may also be applied to the analysis of attached masses, or substructures that bring about discontinuity in stiffness distribution.

In Example 3, the formula for estimating the fundamental period is extended to the evaluation of all the periods. Although convenient, extra caution should be paid in using the formula, since the accuracy of the formula for higher modes is not explicitly analyzed through numerical try-outs. The formula for all modes may be used as an initial estimation, when initial approximation is required in numerical iteration to obtain exact eigenvalues of the non-uniform systems.

In all aspects, however, the error analysis and the examples shown in this study are far from exhaustive. Thus, more case studies involving the application of the proposed formula may be necessary to increase the credibility.

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