

# Auxiliary domain method for solving multi-objective dynamic reliability problems for nonlinear structures

Lambros Katafygiotis<sup>†</sup>

*Department of Civil Engineering, Hong Kong University of Science and Technology,  
Clear Water Bay, Kowloon, Hong Kong, China*

Torgeir Moan<sup>‡</sup>

*Department of Marine Technology, NTNU, Trondheim, NO-7491, Norway*

Sai Hung Cheung<sup>‡‡</sup>

*Department of Civil Engineering, California Institute of Technology, 1200 East California  
Boulevard, Pasadena, California, CA 91125, USA*

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**Abstract.** A novel methodology, referred to as Auxiliary Domain Method (ADM), allowing for a very efficient solution of nonlinear reliability problems is presented. The target nonlinear failure domain is first populated by samples generated with the help of a Markov Chain. Based on these samples an auxiliary failure domain (AFD), corresponding to an auxiliary reliability problem, is introduced. The criteria for selecting the AFD are discussed. The emphasis in this paper is on the selection of the auxiliary linear failure domain in the case where the original nonlinear reliability problem involves multiple objectives rather than a single objective. Each reliability objective is assumed to correspond to a particular response quantity not exceeding a corresponding threshold. Once the AFD has been specified the method proceeds with a modified subset simulation procedure where the first step involves the direct simulation of samples in the AFD, rather than standard Monte Carlo simulation as required in standard subset simulation. While the method is applicable to general nonlinear reliability problems herein the focus is on the calculation of the probability of failure of nonlinear dynamical systems subjected to Gaussian random excitations. The method is demonstrated through such a numerical example involving two reliability objectives and a very large number of random variables. It is found that ADM is very efficient and offers drastic improvements over standard subset simulation, especially when one deals with low probability failure events.

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<sup>†</sup> Associate Professor, Corresponding author, E-mail: lambros@ust.hk

<sup>‡</sup> Professor, E-mail: tormo@ntnu.no

<sup>‡‡</sup> Graduate Student, E-mail: sai@caltech.edu

## 1. Introduction

The reliability analysis of structural systems subjected to uncertain dynamic loads is of great engineering interest and poses a challenging computational problem. The problem is the calculation of the probability of failure of the system, where failure is defined as the event that the response exceeds some specified thresholds. In this paper we consider nonlinear systems subjected to Gaussian random excitation. The input, and consequently the response, can be expressed in terms of a standard normal vector  $\mathbf{Z} \in R^{N_Z}$  comprised of uncorrelated standard normal variables  $Z_m$ ,  $m = 1, 2, \dots, N_Z$  (Katafygiotis and Cheung 2004a). Calculation of the probability of failure  $P_F = P(F)$  amounts to calculating the total probability volume corresponding to the failure domain  $F = \{\mathbf{Z} : g(\mathbf{Z}) < 0\}$ , where  $g(\mathbf{Z})$  denotes the limit state function (LSF).

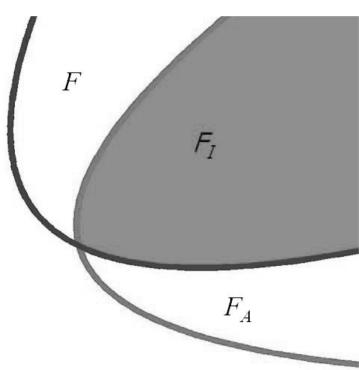
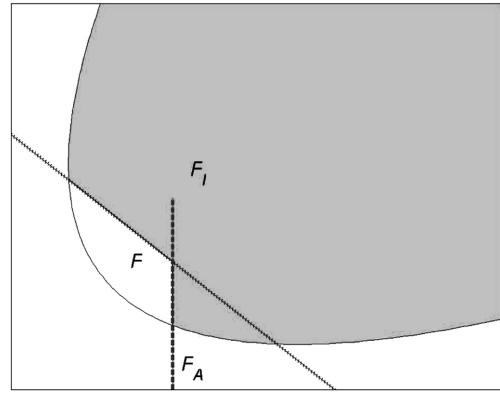
For most practical applications involving dynamical systems approximate analytical methods such as FORM (First Order Reliability Method) and SORM (Second Order Reliability Method) often lead to very inaccurate results. Monte Carlo simulations (MCS) offer the most robust method for calculating  $P_F$ . However, the drawback of MCS is that when one deals with small probabilities of failure an often prohibitively large computational effort is required. For example, if  $P_F = 10^{-6}$  one needs to use  $N = 10^8$  samples to achieve an accuracy corresponding to 10% coefficient of variation (C.O.V.). An excellent review on MCS can be found in (Proppe *et al.* 2003). Variance reduction methods, such as importance sampling (IS), have been proposed to improve computational efficiency. However, it is found that when dealing with high dimensional problems IS may yield extremely inaccurate estimates of  $P_F$  (Au and Beck 2002). The Controlled Monte Carlo simulation method (Pradlwarter and Schuëller 1999), Subset Simulation (SS) (Au and Beck 2001b), Spherical Subset Simulation ( $S^3$ ) (Katafygiotis and Cheung 2002, 2003) and Line-sampling (Koutsourelakis *et al.* 2004) were proposed for calculating high-dimensional reliability integrals for general nonlinear systems. A critical review of reliability estimation procedures for high dimensions can be found in (Schuëller *et al.* 2004).

It has been demonstrated that SS can offer significant improvements over MCS, especially when considering low probability failure events. A two-stage Approach (Katafygiotis and Cheung 2004b) was recently proposed for calculating the reliability of inelastic systems subjected to Gaussian inputs. This latter approach proved to further increase computational efficiency in the case where the nonlinear system exhibits significant elastic behavior. Finally, a generalization of the above two-stage approach, referred to as Auxiliary Domain Method (ADM), has been recently proposed (Katafygiotis and Cheung 2004c), that allows for even more drastic improvements to be realized for general nonlinear systems subjected to Gaussian excitations.

In the next section we review ADM. We continue by discussing the application of ADM in cases where the failure domain is defined by multiple objectives. A methodology leading to an almost optimal selection of auxiliary domain thresholds is outlined. Finally, a numerical example is presented involving the reliability of a 2-DOF bilinear system subjected to Gaussian random excitation to demonstrate the concepts and evaluate the performance of the proposed methodology.

## 2 Auxiliary domain method

Assume that the problem has been cast in a form involving an  $N_Z$ -dimensional standard normal random vector  $\mathbf{Z} = \{Z_1, Z_2, \dots, Z_{N_Z}\}^T$ . This is possible either by using a linear transformation of the

Fig. 1  $F$ ,  $F_I$  and  $F_A$ Fig. 2 Linear  $F_A$ 

original random variables if these are Gaussian, or through other approximate transformations in the case of most non-Gaussian random variables of practical interest (Liu and Der-Kiureghian 1986, Grigoriu 1995). In this section we assume that the failure domain  $F$  is defined as  $F = \{\mathbf{Z} : \max_t |Y(t|\mathbf{Z})| \geq b\}$ , where  $\max_t$  denotes the maximum over  $t$ , where the time variable  $t$  is assumed to belong in a fixed interval  $[0, T]$ . Here we assume that the response quantity of interest  $Y$  is a scalar, while in the next section we discuss the case where the failure is defined in terms of a vector response quantity  $\mathbf{Y}$  not exceeding corresponding threshold values  $\mathbf{b}$ .

Consider now an auxiliary reliability problem defined in terms of the same set of random variables  $\mathbf{Z} \in R^{N_Z}$ . Let  $F_A$  and  $g_A(\mathbf{Z})$  denote the corresponding auxiliary failure domain (AFD) and the corresponding auxiliary LSF, respectively. Let  $F_I$  denote the intersection between  $F$  and  $F_A$ , i.e.,  $F_I = F \cap F_A$ . Fig. 1 shows schematically the definitions of  $F$ ,  $F_A$ , and  $F_I$ . Fig. 2 depicts the case where  $F_A$  corresponds to a union of linear failure domains, which is the case assumed here. The target probability of failure  $P(F)$  can be expressed as a product of three terms as follows:

$$P(F) = P(F_A) \times \frac{P(F)}{P(F_I)} \times \frac{P(F_I)}{P(F_A)} \quad (1)$$

Note that the second term corresponds to the inverse of the conditional probability  $P(F_A|F)$ , while the third term corresponds to the conditional probability  $P(F|F_A)$ . One of the criteria in choosing the AFD is to be of a form that renders a very efficient calculation of the corresponding failure probability  $P(F_A)$ . The second term in (1) can be estimated by simulating  $N$  points in  $F$  distributed according to  $f_{\mathbf{Z}|F}$ :  $P(F)/P(F_I) \approx N/N_I$ , where  $N_I$  denotes the number of these  $N$  points belonging in  $F_I$ . Similarly, the third term in (1) can be estimated by simulating  $N'$  points in  $F_A$  distributed according to  $f_{\mathbf{Z}|F_A}$ :  $P(F_I)/P(F_A) \approx N'_I/N'$ , where  $N'_I$  denotes the number of these  $N'$  points belonging in  $F_I$ .

In the next Sections we discuss the selection of the auxiliary domain and the calculation of the second and third term in the right hand side of Eq. (1).

## 2.1 Choice of auxiliary domain

The choice of AFD is based on the following principles: i) the failure probability  $P(F_A)$  can be

calculated very efficiently; ii) direct simulation of failure points in  $F_A$  distributed according to the conditional distribution  $f_{Z|F_A}$  can be performed very efficiently; and iii) there is significant overlap between the target failure domain  $F$  and the auxiliary failure domain  $F_A$ .

Conditions i) and ii) can be fulfilled if for example one chooses  $F_A$  to correspond to a linear failure domain or more generally to a union of linear failure domains. While other choices may be possible, herein we focus in the latter case and will assume that  $F_A$  corresponds to a union of linear failure domains. In this case  $P(F_A)$  can be calculated using minimal computational effort through the method described in (Katafygiotis and Cheung 2004a), while efficient direct simulation of samples in  $F_A$  distributed according to  $f_{Z|F_A}$  can be achieved through a procedure described in (Katafygiotis and Cheung 2004b). To ensure iii), i.e., a significant overlap between  $F$  and  $F_A$ , one needs to appropriately select the corresponding auxiliary problem. Specifically, one needs to: a) select an appropriate auxiliary linear system and b) select appropriate threshold values  $\mathbf{b}_A$  in the corresponding auxiliary problem. Points a) and b) are addressed below.

### 2.1.1 Choice of auxiliary system

A good selection may depend on experience gained from dealing with similar nonlinear problems. The optimal selection of the auxiliary system is currently an open question. Here we present some preliminary ideas that can be used to select a reasonable, although not optimal, auxiliary system in the case of nonlinear dynamic reliability problems involving a deterministic nonlinear dynamic system subjected to stochastic gaussian excitation.

A reasonable first choice can be to consider a “linearized” version of the nonlinear problem at hand. For example, if one studies the reliability of a nonlinear dynamic system, a reasonable first choice may be to consider as auxiliary system a corresponding linear system with stiffness equal to the initial elastic stiffness of the nonlinear system. Alternatively, slightly smaller value of stiffness may be chosen to account for stiffness degradations in the target nonlinear system. Similarly, the damping of the linear system can be chosen to be equal to the initial damping of the nonlinear system; larger values may also be justifiable to account for increased apparent damping due to hysteretic behavior. To obtain such modified values of stiffness and damping one may apply the method of equivalent linearization (Roberts and Spanos 1990). It is worth noting however that such equivalent linear system does not correspond to the optimal auxiliary linear system since the latter clearly must depend on the threshold level  $b$  (i.e., the larger the value of  $b$ , the “softer” the optimal auxiliary system should be), while the former does not.

It seems that the optimal, or almost optimal, selection of the auxiliary system is closely related to the problem of finding the design points of the nonlinear reliability problem at hand. Specifically, consider the case of white noise excitation where the random variable  $Z_i$  denotes the discrete input at time  $t_i = i\Delta t$ . Let  $\mathbf{Z}^*$  denote the design point corresponding to failure at time  $T = N_Z\Delta t$ . Then, it seems intuitive that one of the best choices for auxiliary linear SISO system can be obtained by requiring the design point  $\mathbf{Z}_A^*$  of the corresponding auxiliary reliability problem at time  $T$  to satisfy  $\mathbf{Z}_A^* = \mathbf{Z}^*$ . This is possible by selecting the impulse response function  $h_A(t)$  of the auxiliary linear SISO system to satisfy:  $h_A(i\Delta t) = cZ_{N_Z-i}^*$ , where  $c$  is an arbitrary scaling constant (i.e., the discrete impulse response function of the auxiliary linear system is chosen to be proportional to the design point vector after inverting the ordering of its elements) (Der Kiureghian 2000). It is noted that a SISO linear system is fully characterized by its impulse response function; seeking the form of the differential operator corresponding to a given impulse response function  $h_A(i\Delta t)$  is of no interest. Furthermore, if all design points of a nonlinear problem are known then the union of hyperplanes

corresponding to these design points is clearly a good AFD choice; the ADM can proceed without having to define the auxiliary dynamic system corresponding to this AFD. While it is shown that given  $\mathbf{Z}^*$  it is easy to find an almost optimal auxiliary linear system, it is stressed that finding the design point  $\mathbf{Z}^*$  is a very difficult problem in itself because it involves solving a very high-dimensional non-convex nonlinear optimization problem. Efficient methods for obtaining  $\mathbf{Z}^*$  (or approximations of it) have been proposed for certain categories of nonlinear systems (Koo and Der Kiureghian 2001, Koo *et al.* 2005).

While these methods are applicable to certain types of systems the authors have proposed a methodology for finding an approximation of the design point in the case of white noise input. The method has wide applicability, in the sense that it does not require an explicit formulation of the nonlinear input-output relationship, and can be summarized as follows. First the nonlinear system is subjected to an appropriately scaled impulse such that the absolute maximum of the system's response  $Y(t)$  over the fixed time interval  $[0, T]$  is equal to  $b$ , i.e.,  $\max_t |Y(t)| = b$ . This scaling is achieved iteratively. Next, the system is subjected to an input equal to a scaled version of the above determined response  $Y(t)$  but time-inverted, i.e., the input at time  $t$  is equal to  $\alpha Y(T - t)$ . As before, the scaling factor  $\alpha$  is determined iteratively so that the new absolute maximum response is equal to  $b$ . Consider now this input shifted to the right, such that the maximum absolute response occurs at time  $T$ . This shifted input corresponds to an approximation of the design point  $\mathbf{Z}^*$ . Inverting the order of the elements of this vector yields the discrete values of the impulse response function  $h_A(t)$  of the auxiliary SISO linear system. Note that generally this SISO linear system does not correspond to a SDOF oscillator because the function  $h_A(t)$  does not assume the form of the impulse response function of a SDOF oscillator. One is cautioned that the above procedure does only yield an approximation of  $\mathbf{Z}^*$  whose accuracy may not be acceptable in many cases. Thus, the resulting auxiliary SISO system is not optimal. To obtain a better estimate of  $\mathbf{Z}^*$  one may use this first approximation of  $\mathbf{Z}^*$  as a starting point in a follow-up optimization. Further research in developing methodologies that yield good approximations for the design point of general nonlinear systems will be invaluable. Such methodologies will enable an almost optimal selection of auxiliary system and, therefore, may optimize the performance of ADM.

### 2.1.2 Choice of threshold $b_A$

Let  $Y_A(t|\mathbf{Z})$  denote the response of the auxiliary linear system when subjected to an input specified by  $\mathbf{Z}$ . Note that if the auxiliary linear system corresponds to a physical model of the same complexity as the target nonlinear model, then  $Y_A$  will refer to the response of the auxiliary system at the same DOF as that corresponding to  $Y$ . However, if one chooses the auxiliary system to be the SISO linear system described in the previous paragraph, then  $Y_A$  refers to the single output of this SISO system. Given a threshold value  $b_A$  one then has a well defined auxiliary reliability problem with linear AFD  $F_A = \{\mathbf{Z} : \max_t |Y_A(t|\mathbf{Z})| \geq b_A\}$  and corresponding LSF  $g_A(\mathbf{Z}) = b_A - \max_t |Y_A(t|\mathbf{Z})|$ .

As discussed earlier, the second term in (1) can be estimated by simulating  $N$  points in  $F$  distributed according to  $f_{\mathbf{Z}|F}$ :  $P(F)/P(F_I) \approx N/N_I$ , where  $N_I$  denotes the number of these  $N$  points belonging in  $F_I$ . The simulation of points distributed according to  $f_{\mathbf{Z}|F}$  is discussed in the next Section. Note that in order to calculate  $N_I$  the values  $g_A(\mathbf{Z}^{(i)})$ ,  $i = 1, \dots, N$  need to be calculated, that is,  $N$  linear analyses need to be performed. Clearly, the accuracy in estimating the ratio  $P(F)/P(F_I)$  for a given number of samples  $N$  decreases as  $P(F_A|F) = P(F_I)/P(F)$  decreases. This observation is used as the basis for the selection of the auxiliary threshold value  $b_A$ . Specifically,  $b_A$  is chosen such that  $N_I/N$  is equal to a prespecified ratio  $p_1$  close to one. Note that a smaller value of  $p_1$  implies

larger threshold values  $b_A$  and, therefore, smaller  $P(F_A)$ . Although the smallest uncertainty in the estimation of  $P(F)/P(F_I)$  is obtained when choosing  $p_1 = 1$ , the effect of  $p_1$  on the last term in (1) is not obvious. In order to investigate the effect of  $p_1$  on the accuracy of  $P(F)$  in the numerical example we consider three different values for  $p_1$ : 1, 0.8, and 0.5.

**Simulation of points according to  $f_{Z|F}$**  Here we discuss the generation of a set of points  $\mathcal{Z} = \{\mathbf{Z}^{(i)}, i = 1, \dots, N\}$ , distributed according to the conditional distribution  $f_{Z|F}$ . This can be achieved through a Markov Chain having  $f_{Z|F}$  as its stationary distribution. The starting point of this chain is selected such as to belong in the failure domain  $F$ ; however, it is usually not required to be distributed according to  $f_{Z|F}$ . Therefore, a certain burn-in time is needed before the generated points follow the target stationary distribution. Assuming that the burn-in length of the chain consists of  $K$  points, the total chain length to be generated is equal to  $K + N$ . Thus, an extended chain  $\mathcal{Z}^e = \{\mathbf{Z}^{(i)}, i = -(K - 1), \dots, N\}$  is generated; the first  $K$  points of this chain are discarded and the remaining chain corresponds to the target set  $\mathcal{Z}$ . Once a starting point  $\mathbf{Z}^{(-K+1)}$  has been selected the chain  $\mathcal{Z}^e$  can be generated using an appropriate algorithm, such as the modified Metropolis Hastings algorithm presented in (Au and Beck 2001b).

Next we discuss the selection of a starting point for the chain  $\mathcal{Z}^e$ . The only criterion used for this selection is that the starting point  $\mathbf{Z}^{(-K+1)}$  must correspond to a failure point. For this one needs to first simulate a standard random vector  $\mathbf{Z}$ ; in the unlikely case where  $\mathbf{Z}$  happens to be a failure point one can readily set  $\mathbf{Z}^{(-K+1)} = \mathbf{Z}$ ; in the more probable case where  $\mathbf{Z}$  is not a failure point a scaling factor  $\lambda > 1$  is sought, based on iterations, such that  $\mathbf{Z}^{(-K+1)} = \lambda\mathbf{Z}$  is a failure point. The rationale for such a scaling approach is that since  $\mathbf{Z}$  corresponds to a loading input, scaling it results in more energy being pumped into the system causing it eventually to fail.

An alternative approach to scaling  $\mathbf{Z}$  is to generate a Markov chain  $\mathcal{Z}' = \{\mathbf{Z}'^{(i)}, i = 1, \dots, L\}$  such that the corresponding sequence of LSF values  $\{g(\mathbf{Z}'^{(i)}), i = 1, \dots, L\}$  is monotonically decreasing; the chain is started at  $\mathbf{Z}'^{(1)} = \mathbf{Z}$ , with  $g(\mathbf{Z}) > 0$ , and is propagated until a point having negative LSF value, i.e., a failure point, is reached. The rule for propagation of such chain can be chosen as follows: given a current point  $\mathbf{Z}'^{(i)}$ , a candidate point  $\mathbf{Z}'$  is generated distributed according to a chosen proposal distribution centered at  $\mathbf{Z}'^{(i)}$ . If  $g(\mathbf{Z}') < g(\mathbf{Z}'^{(i)})$  then  $\mathbf{Z}'^{(i+1)} = \mathbf{Z}'$ ; otherwise,  $\mathbf{Z}'^{(i+1)} = \mathbf{Z}'^{(i)}$ . If  $g(\mathbf{Z}') < 0$ , then  $L = i + 1$  and  $\mathbf{Z}'^{(-K+1)} = \mathbf{Z}'$ . In the numerical example presented in this paper this latter procedure has been followed to generate a starting point  $\mathbf{Z}^{(-K+1)} \in F$ ; for the generation of the chain  $\mathcal{Z}'$  a standard uniform distribution centered at a current point has been used in order to generate the next candidate point  $\mathbf{Z}'$ . The computational effort associated with finding the starting point  $\mathbf{Z}^{(-K+1)}$  is found to be relatively low (see numerical example).

The burn-in length  $K$  is estimated by monitoring the lengths  $|\mathbf{Z}^{(i)}|$  of the leading points in the chain  $\mathcal{Z}^e$ . In this regard, it is noted that the length  $R = |\mathbf{Z}|$  of an  $N_z$ -dimensional standard normal vector  $\mathbf{Z}$  has the following property:  $R^2$  follows a chi-square distribution with  $N_z$  degrees of freedom. It follows that for large  $N_z$  most of the points  $\mathbf{Z}$  in the standard normal space belong in a relatively narrow spherical ring: almost 99.7% of all points satisfy  $\sqrt{N_z - 3\sqrt{2N_z}} \leq R \leq \sqrt{N_z + 3\sqrt{2N_z}}$  (Katafygiotis and Cheung 2002, 2003); e.g., for  $N_z = 1000$ ,  $29.42 \leq R \leq 33.68$ . However, the starting point  $\mathbf{Z}^{(-K+1)}$ , due to its construction, is located further from the origin, outside this high-probability spherical ring. Therefore, it is reasonable to assume that the initial part of the chain, comprised of points located outside this ring, cannot be possibly distributed according to the target stationary distribution and, therefore, must be discarded. Therefore, the value of  $K$  is suggested to be taken at least equal to the number of points in the leading part of the chain satisfying  $R > \sqrt{N_z + 3\sqrt{2N_z}}$ .

It is noted that while one is able to “map” the failure domain through such a set of points, this

information by itself does not allow for the calculation of the probability of failure  $P(F)$ . Having said that however it is also important to note that because the samples are distributed according to  $f_{Z|F}$ , one is able to estimate the relative probabilities of failure corresponding to any two subsets of  $F$  by calculating the ratio of the number of samples belonging in these two subsets.

## 2.2 Calculation of $P(F_I)/P(F_A)$

The last term to be discussed in Eq. (1) is the term  $P(F_I)/P(F_A) = P(F|F_A)$ . In order to calculate this term, a set of points  $\mathcal{Z}' = \{\mathbf{Z}'^{(i)}, i=1, \dots, N'\}$ , distributed according to  $f_{Z|F_A}$  is generated. Since  $F_I \subseteq F_A$  and since the points are generated according to  $f_{Z|F_A}$ , this term can be estimated as:

$$P(F|F_A) = \frac{P(F_I)}{P(F_A)} \approx \frac{N'_I}{N'} \quad (2)$$

where  $N'_I$  denotes the total number of points in the set  $\mathcal{Z}'$  that belong in  $F_I$ , i.e., that lead the nonlinear system to fail. Assuming a small sample size, e.g.,  $N' = 100$ ,  $P(F|F_A)$  can be estimated through (2) with sufficient accuracy (e.g., C.O.V.  $\leq 30\%$ ) only if it is of order 0.1 or larger. If the estimate of  $P(F|F_A)$  is found to be less than 0.1, one alternative for increasing the accuracy in the estimation of this ratio to an acceptable level (say 30%), is to increase the number of simulated points to a level prescribed by Monte Carlo simulations (say  $10 \times P^{-1}(F|F_A)$ ). However, if  $P(F|F_A)$  is too small, the number of required simulations by such an approach may be undesirably high. Therefore, in cases where  $P(F|F_A) < 0.1$ , an alternative procedure based on the general framework of SS (Au and Beck 2001b) is proposed as described next.

The idea is establishing a sequence of failure domains  $F_A \supset F_1 \supset F_2 \supset \dots \supset F_K$  with  $F_K \equiv F_I$  and expressing  $P(F|F_A)$  as a product of a sequence of conditional probabilities:

$$P(F|F_A) = P(F_1|F_A) \prod_{i=2}^K P(F_i|F_{i-1}) \quad (3)$$

The LSF corresponding to  $F_1$ , denoted here as  $LSF_1$ , is determined so that  $P(F_1|F_A) = p_0$  (e.g.,  $p_0 = 0.1$ ). To establish  $LSF_1$ , a number of points are simulated in  $F_A$  according to the conditional distribution  $f_{Z|F_A}$  and  $LSF_1$  is determined as a scaled version of the target LSF such that the percentage of points belonging in  $F_1$  is equal to  $p_0$ . Recall that  $F_A$  is comprised of a union of linear domains. A procedure which allows for the simulation of samples in such types of domains can be found in (Katafygiotis and Cheung 2004b).

In the subsequent steps  $LSF_i$  is determined from  $LSF_{i-1}$  so as to ensure  $P(F_i|F_{i-1}) = p_0$ . For this, a number of points are simulated through MCMC method in  $F_{i-1}$  distributed according to the conditional distribution  $f_{Z|F_{i-1}}$ . Then,  $LSF_i$  is determined as a scaled version of the target LSF such that the percentage of the simulated points belonging in  $F_i$  is equal to  $p_0$ . If  $F_i \subset F_I$ , then  $K = i$ ,  $F_K = F_I$ , and  $P(F_K|F_{K-1}) = P(F_I|F_{K-1})$ ; the latter is  $> p_0$  and is determined as the ratio of points simulated in  $F_{K-1}$  that lie in  $F_I$ .

Summarizing,  $P(F)$  is calculated from (1) with the first term being calculated using efficient available algorithms, the second term being set equal to a predetermined value  $p_1$  by appropriately selecting the threshold value  $b_A$ , and the last term being calculated through (2) or, in case this term is found to be smaller than 0.1, through the modified SS expression (3).

### 3. Selection of AFD in multi-objective reliability problems

Consider now the case where the failure domain  $F$  is defined as  $F = \bigcup_{j=1}^M F_j$ , where  $F_j = \{\mathbf{Z} : \max_t |Y_j(t|\mathbf{Z})| \geq b_j\}$ . That is, we consider the case where failure is defined as the event where the absolute value of at least one of the response quantities  $Y_j$ ,  $j = 1, \dots, M$ , exceeds its corresponding threshold value  $b_j$  at some time  $t$  within the time interval of interest. Based on the discussion of the previous section, one is able to select a set of  $M$  linear auxiliary systems (one for each response quantity  $Y_j$ ) with corresponding impulse response functions  $h_{jA}(t)$ ,  $j = 1, \dots, M$ . Let  $b_{jA}$ ,  $j = 1, \dots, M$ , denote a set of thresholds defining the AFD  $F_A = \bigcup_{j=1}^M F_{jA}$ , where  $F_{jA} = \{\mathbf{Z} : \max_t |Y_{jA}(t|\mathbf{Z})| \geq b_{jA}\}$ . In this section we discuss a strategy for an optimal selection of these threshold values  $\mathbf{b}_A = [b_{1A}, \dots, b_{MA}]^T$ .

It is assumed that the ratio  $p_1 = N_I/N$ , which serves as an estimate of the ratio  $P(F_I)/P(F)$ , has been preselected. Then, for the given set of simulated points  $\mathcal{Z} = \{\mathbf{Z}^{(i)}, i = 1, \dots, N\}$  we are faced with the task of selecting  $\mathbf{b}_A$  so that  $N_I = p_1 N$  points out of this set belong in  $F_A$ . Note that in the case where  $M=1$  the selection of  $b_A$  is uniquely determined. Specifically, defining  $\hat{Y}_A^{(i)} = \max_t |Y_A(t|\mathbf{Z}^{(i)})|$ ,  $i = 1, \dots, N$ , the value of  $b_A$  can be determined as the largest possible value such that  $N_I$  values out of the set  $\{\hat{Y}_A^{(i)}, i = 1, \dots, N\}$  are larger or equal to  $b_A$ . However, in the case where  $M > 1$  the selection of  $\mathbf{b}_A$  is non-unique. For example, one can set arbitrary  $M - 1$  components of  $\mathbf{b}_A$  to be equal to infinity and adjust the remaining component following the aforementioned procedure for  $M = 1$ . The objective here is to develop a methodology that will lead to a good choice of  $\mathbf{b}_A$ , in the sense that it minimizes  $P(F_A)$  while ensuring that  $F_A$  contains  $N_I = p_1 N$  points out of the set  $\mathcal{Z}$ . The requirement of minimizing  $P(F_A)$  is imposed in order to ensure maximum overlap between  $F_A$  and  $F$  and, consequently, a small number of steps in the modified subset simulation procedure when calculating the conditional probability  $P(F|F_A)$ .

Consider first the case where  $p_1 = 1$ , i.e., all points in  $\mathcal{Z}$  are to be included in  $F_A$ . A basic algorithm for selecting  $\mathbf{b}_A$ , although clearly not optimal, is to choose the value of each threshold component  $b_{jA}$ ,  $j = 1, \dots, M$ , as follows:  $b_{jA} = \min\{\hat{Y}_{jA}^{(i)}, i = 1, \dots, N\}$ , where  $\hat{Y}_{jA}^{(i)} = \max_t |Y_{jA}(t|\mathbf{Z}^{(i)})|$ . Based on this algorithm, each threshold value  $b_{jA}$  is chosen as the maximum value such that all points  $\mathbf{Z}^{(i)}$ ,  $i = 1, \dots, N$ , belong in the corresponding auxiliary failure domain  $F_{jA}$ . For the purpose of comparing various alternative algorithms later in the numerical example let us denote this algorithm as  $A_1$ .

An obvious improvement on algorithm  $A_1$ , referred to later as algorithm  $A_2$ , is to base the selection of  $b_{jA}$  on only the subset of points that belong in  $F_j$ . Specifically,  $A_2$  is based on choosing the values of  $b_{jA}$ ,  $j = 1, \dots, M$ , as follows:  $b_{jA} = \min\{\hat{Y}_{jA}^{(j_k)}, k = 1, \dots, N_j\}$ , where  $\tilde{\mathcal{Z}}^{(j)} = \{\mathbf{Z}^{(j_k)}, k = 1, \dots, N_j \leq N\} = \mathcal{Z} \cap F_j$ . Since  $\{\hat{Y}_{jA}^{(j_k)}, k = 1, \dots, N_j\} \subseteq \{\hat{Y}_{jA}^{(i)}, i = 1, \dots, N\}$ , it follows that  $\min\{\hat{Y}_{jA}^{(j_k)}, k = 1, \dots, N_j\} \geq \min\{\hat{Y}_{jA}^{(i)}, i = 1, \dots, N\}$ , i.e., the threshold values  $b_{jA}$  obtained by  $A_2$  are larger than or equal to the corresponding values obtained by  $A_1$ . Therefore,  $A_2$  yields a better auxiliary domain, having smaller value of  $P(F_A)$  and, therefore, leading to a greater overlap between  $F_A$  and  $F$ .

It is important to note that both of the above algorithms  $A_1$  and  $A_2$  cannot be easily generalized to cases where  $p_1 \neq 1$ . Specifically, consider an approach where one attempts to determine the individual thresholds  $b_{jA}$  by requiring  $p'N$  points out of the set  $\mathcal{Z}$  (in the case of  $A_1$ ) or out of  $\tilde{\mathcal{Z}}^{(j)}$  (in the case of  $A_2$ ) to belong in  $F_{jA}$ . It can be easily seen that it is difficult to determine the value  $p'$  corresponding to a prespecified value  $p_1$  (except when  $p_1 = 1$ , in which case  $p' = p_1 = 1$ ).

Next we describe an algorithm that further improves on  $A_2$  and leads to an optimal, or almost

optimal, choice of  $\mathbf{b}_A$ . The proposed algorithm, referred to hereafter as  $A_P$ , is applicable for any value  $p_1 \leq 1$  such that  $p_1 N$  is an integer.

For each response quantity of interest  $Y_j$ ,  $j = 1, \dots, M$ , and for each input  $\mathbf{Z}^{(i)}$ ,  $i = 1, \dots, N$ , we calculate the maximum absolute response  $\hat{Y}_{jA}^{(i)} = \max_t |Y_{jA}(t|\mathbf{Z}^{(i)})|$ ,  $i = 1, \dots, N$ . Let  $\{\hat{Y}_{jA}^{(j_k)}, k = 1, \dots, N\}$  denote the ordered set comprised of the values  $\hat{Y}_{jA}^{(i)}$ ,  $i = 1, \dots, N$  ordered in a descending order. Note that the set of indices  $\{j_1, \dots, j_N\}$  corresponds to a permutation of the set  $\{1, \dots, N\}$ , where this permutation is dependent on the index  $j$  (i.e., the permutation depends on the response quantity under consideration). Let now  $\{P(F_{jA}(\hat{Y}_{jA}^{(j_k)})), k = 1, \dots, N\}$  denote the ordered set of failure probabilities corresponding to the failure events  $F_{jA}(\hat{Y}_{jA}^{(j_k)}) = \{\mathbf{Z} : \max_t |Y_j(t|\mathbf{Z})| \geq \hat{Y}_{jA}^{(j_k)}\}$ . Note that since the set of thresholds  $\{\hat{Y}_{jA}^{(j_k)}, k = 1, \dots, N\}$  are ordered in descending order, the corresponding set  $P\{(F_{jA}(\hat{Y}_{jA}^{(j_k)})), k = 1, \dots, N\}$  is ordered in ascending order. Recall that each of the values  $P(F_{jA}(\hat{Y}_{jA}^{(j_k)})), k = 1, \dots, N$  can be calculated using efficient algorithms (e.g., Katafygiotis and Cheung 2004a). However, since  $N$  may be of the order of hundred or more, in order to reduce the computational effort it is sufficient to calculate these  $N$  values using an approximation of  $P(F_{jA}(b_{jA}))$  as a function of the threshold level  $b_{jA}$ . Such approximation can be obtained by selecting an appropriate parametric expression for this function and interpolating through a set of points  $\{(b_{jA}^{(r)}, P(F_{jA}(b_{jA}^{(r)}))), r = 1, \dots, R\}$  where  $R \ll N$ , and where the interpolation values  $b_{jA}^{(r)}$  are selected to belong in the range of interest  $[\hat{Y}_{jA}^{(j_N)}, \hat{Y}_{jA}^{(j_1)}]$ . A recommended expression, involving three parameters  $a_0$ ,  $a_1$  and  $a_2$ , is:  $\ln(P(F_{jA}(b_{jA}))) = a_2 b_{jA}^2 + a_1 b_{jA} + a_0$ , i.e., the logarithm of the probability of failure is assumed to vary quadratically with respect to the threshold level.

Note that  $P(F_A) \leq \sum_{j=1}^M P(F_{jA})$ . Rather than minimizing  $P(F_A)$  here we propose to minimize its upper bound corresponding to the right hand side of the latter expression. The proposed algorithm  $A_P$  involves a sequence of  $N_1 = p_1 N$  steps. In the description of the algorithm that follows below we adopt the following notation:  $\mathbf{b}_A^{(l)}$  denotes the value of  $\mathbf{b}_A$  at the end of the  $l$ th step;  $F_A^{(l)} = F_A(\mathbf{b}_A^{(l)})$  denotes the corresponding auxiliary failure domain, and  $F_{jA}^{(l)} = F_{jA}(b_{jA}^{(l)})$ ;  $\mathcal{Z}^{(l)}$  denotes the set of those points in  $\mathcal{Z}$  that belong in  $F_A^{(l)}$ , i.e.,  $\mathcal{Z}^{(l)} = \mathcal{Z} \cap F_A^{(l)}$ ; and  $\mathcal{Z}^{(j,l)}$  denotes the set of those points in  $\mathcal{Z}$  that belong in  $F_{jA}^{(l)}$ , i.e.,  $\mathcal{Z}^{(j,l)} = \mathcal{Z} \cap F_{jA}^{(l)}$ . Clearly,  $\cup_{j=1}^M \mathcal{Z}^{(j,l)} = \mathcal{Z}^{(l)}$  while the sets  $\mathcal{Z}^{(j,l)}$ ,  $j = 1, \dots, M$ , are generally not mutually exclusive.

The algorithm is initialized by assigning a threshold value  $\mathbf{b}_A^{(0)}$ , where  $\mathbf{b}_{jA}^{(0)} = \infty$ ,  $j = 1, \dots, M$  (with corresponding failure probabilities  $P(F_{jA}^{(0)}) = 0$ ,  $j = 1, \dots, M$ ). The corresponding AFD  $F_A^{(0)}$  is, therefore, guaranteed to not contain any of the points in  $\mathcal{Z}$ , i.e.,  $\mathcal{Z}^{(0)}$  is an empty set. The basic idea of the algorithm is to successively build up the set of failure points from an empty set at the very beginning to a set containing  $N_1$  points at the end. During each step, the threshold values are modified in a way such that the corresponding set of failure points includes all previous points plus one, i.e., the set  $\mathcal{Z}^{(l)}$  is chosen to include all elements in  $\mathcal{Z}^{(l-1)}$  plus one additional point. Thus, the set  $\mathcal{Z}^{(l)}$  contains exactly  $l$  points, and also satisfies:  $\mathcal{Z}^{(l-1)} \subset \mathcal{Z}^{(l)}$ ,  $l = 1, \dots, N_1$ . The question asked during the  $l$ th step of the proposed algorithm  $A_P$  is how to adjust the threshold from its current value  $\mathbf{b}_A^{(l-1)}$  to some new value  $\mathbf{b}_A^{(l)}$  such that: 1) the resulting  $\mathcal{Z}^{(l)}$  satisfies the above conditions (i.e.,  $\mathcal{Z}^{(l)}$  contains exactly  $l$  points and satisfies:  $\mathcal{Z}^{(l-1)} \subset \mathcal{Z}^{(l)}$ ), and 2) this adjustment is optimal in the sense of minimizing the target function  $\sum_{j=1}^M P(F_{jA}^{(l)})$ .

The answer to the above question is obtained by considering  $M$  different alternative values for  $\mathbf{b}_A^{(l)}$ . Once these alternatives are determined, the one with the smallest corresponding value  $\sum_{j=1}^M P(F_{jA}^{(l)})$  is selected as the final choice for  $\mathbf{b}_A^{(l)}$ . Thus, the only issue that remains to be discussed is the selection of the  $M$  alternative values for  $\mathbf{b}_A^{(l)}$ . Next, the procedure for selecting the  $j$ -th such alternative value for  $\mathbf{b}_A^{(l)}$ , given  $\mathbf{b}_A^{(l-1)}$ , is discussed.

First, the value of the  $j$ -th threshold is “forward” adjusted from its current value  $b_{jA}^{(l-1)}$  to a new smaller value  $b_{jA}^{(l)}$ , while keeping the remaining values  $b_{mA}^{(l-1)}$ ,  $m \neq j$ , fixed. Here “forward” adjustment refers to a reduction of threshold value and is chosen such as to ensure that one additional point fails. For this we consider the ordered (descending order) set  $\{\hat{Y}_{jA}^{(j_s)}, k=1, \dots, N\}$ . The value  $b_{jA}^{(l)}$  is then selected to be equal to  $\hat{Y}_{jA}^{(j_s)}$ , where  $s = \min_k \{k = 1, \dots, N | \mathbf{Z}^{(j_k)} \notin \mathcal{Z}^{(l-1)}\}$ . Another way of expressing the same thing is that  $b_{jA}^{(l)}$  is selected as the largest value  $\hat{Y}_{jA}^{(i)}$ ,  $i = 1, \dots, N$ , when considering only the points that do not belong in  $\mathcal{Z}^{(l-1)}$ , i.e.,  $b_{jA}^{(l)} = \max \{\hat{Y}_{jA}^{(i)}, i = 1, \dots, N | \mathbf{Z}^{(i)} \notin \mathcal{Z}^{(l-1)}\}$ . Clearly, simply adjusting the value of  $b_{jA}^{(l)}$  in the above manner, while keeping all other values fixed, ensures that the set  $\mathcal{Z}^{(l)}$  will contain one more failure point than  $\mathcal{Z}^{(l-1)}$ , as desired. Note that  $b_{jA}^{(l)} < b_{jA}^{(l-1)}$ . Also, note that  $\mathcal{Z}^{(j, l-1)} \subset \mathcal{Z}^{(j, l)}$ . However, note that the difference in the number of elements in the latter two sets may be larger than one. This is because the threshold  $b_{jA}^{(l-1)}$  corresponds to a value  $\hat{Y}_{jA}^{(j_{k_1})}$  and  $b_{jA}^{(l)}$  corresponds to a value  $\hat{Y}_{jA}^{(j_{k_2})}$ , where  $k_2 > k_1$ , but where not necessarily  $k_2 = k_1 + 1$ . For example, if  $\mathbf{Z}^{(j_{(k_1+1)})} \in \mathcal{Z}^{(l-1)}$ , one clearly needs to jump more than one space in the sequence  $\{\hat{Y}_{jA}^{(j_k)}, k = 1, \dots, N\}$  in order to produce an overall additional failure point. As a consequence, in such case the number of elements in  $\mathcal{Z}^{(j, l)}$  will increase by more than one. Whenever this happens, the possibility for “backward” adjustment of the remaining thresholds arises. Here, “backward” refers to increasing, rather than decreasing, a threshold value. Specifically, each of the remaining thresholds  $b_{mA}^{(l-1)}$ ,  $m \neq j$  is adjusted to the value:  $b_{mA}^{(l)} = \min \{\hat{Y}_{mA}^{(i)}, i = 1, \dots, N | \mathbf{Z}^{(i)} \in \mathcal{Z}^{(m, l-1)}, \mathbf{Z}^{(i)} \notin \mathcal{Z}^{(j, l)}\}$ . This completes the description of the procedure for generating the  $j$ -th alternative value for  $\mathbf{b}_A^{(l)}$ . As mentioned earlier the alternative with the smallest corresponding value  $\sum_{j=1}^M P(F_{jA}^{(l)})$  is selected as the final choice for  $\mathbf{b}_A^{(l)}$ .

#### 4. Illustrative example

Consider a two-story shear building (Fig. 3) with equation of motion:

$$M\ddot{\mathbf{y}}(t) + C\dot{\mathbf{y}}(t) + \mathbf{F}_s(\mathbf{y}(t)) = -\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} U(t) \quad (4)$$

where the response vector  $\mathbf{y}(t)$  is comprised of the first-floor displacement  $Y_1(t)$  and roof displacement  $Y_2(t)$  and  $U(t)$  denotes the horizontal ground acceleration; the mass matrix  $M$  is

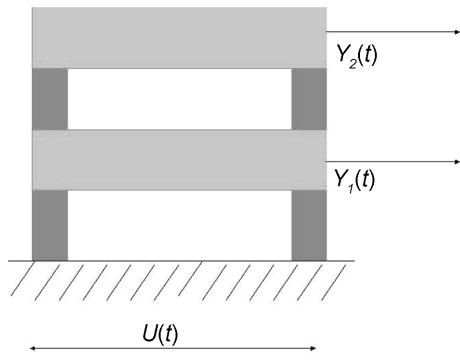


Fig. 3 Two-story shear building model

diagonal with elements  $m_1 = m_2 = 10000$  kg; the damping matrix  $C$  is:

$$C = \begin{pmatrix} 3371.91 & -1123.97 \\ -1123.97 & 2247.94 \end{pmatrix} \text{kg/s} \quad (5)$$

and the stiffness force vector  $\mathbf{F}_s(\mathbf{y}(t))$  is given as follows:

$$\mathbf{F}_s(\mathbf{y}(t)) = \begin{pmatrix} F_1 - F_2 \\ F_2 \end{pmatrix} \quad (6)$$

where the first and second inter-story restoring forces  $F_1$  and  $F_2$ , respectively, are bilinear functions of the corresponding inter-story drifts:

$$\begin{aligned} F_1 &= k_1^{(1)} Y_1, \text{ if } |Y_1(t)| < D_y^{(1)} \\ &= k_1^{(2)} Y_1 + \text{sgn}(Y_1)(k_1^{(1)} - k_1^{(2)})D_y^{(1)}, \quad \text{otherwise} \end{aligned} \quad (7)$$

$$\begin{aligned} F_2 &= k_2^{(1)}(Y_2 - Y_1), \text{ if } |Y_2(t) - Y_1(t)| < D_y^{(2)} \\ &= k_2^{(2)}(Y_2 - Y_1) + \text{sgn}(Y_2 - Y_1)(k_2^{(1)} - k_2^{(2)})D_y^{(2)}, \quad \text{otherwise} \end{aligned} \quad (8)$$

where  $\text{sgn}(x)$  is equal to 1 if  $x > 0$ , -1 if  $x < 0$ , and 0 if  $x = 0$ ;  $D_y^{(1)} = D_y^{(2)} = 2$  cm and the stiffness parameters are  $k_1^{(1)} = k_2^{(1)} = 40000 \pi^2 \text{N/m}$  and  $k_1^{(2)} = k_2^{(2)} = r_0 k_1^{(1)}$ , with stiffness reduction factor  $r_0 = 0.25$ . For small vibration amplitudes the system behaves linearly having two modes of vibration with modal frequencies  $f_1 = 0.618$  Hz and  $f_2 = 1.618$  Hz and corresponding modal damping ratios both equal to 2%.

The ground acceleration  $U(t)$  is assumed to be stationary Gaussian white noise with spectral density  $S = 10^{-4} \text{m}^2/\text{s}^3$ . A discrete input signal of duration  $T = 10$  sec with sampling interval  $\Delta t = 0.01$  sec is considered. The discrete data  $U(t_k) = U(k)$ ,  $k = 0, 1, \dots, N_T$ , ( $N_T = 1000$ ) are simulated with the help of a standard normal vector  $\mathbf{Z} \in R^{N_Z}$ , where  $N_Z = N_T + 1$ , as follows:  $U(k) = \sqrt{2\pi S/\Delta t} Z_{k+1}$ ,  $k = 0, 1, \dots, N_T$ . The target failure event  $F$  is defined as the event where either the first floor displacement  $Y_1(t)$  or the roof displacement  $Y_2(t)$  exceeds the threshold level  $b_1 = 4$  cm and  $b_2 = 6.5$  cm, respectively, at any discrete time instant during the 10 sec duration of the ground shaking, i.e.:

$$F = \bigcup_{i=1}^2 \bigcup_{k=1}^{N_T} \{ \mathbf{Z}: Y_i(k|\mathbf{Z}) < -b_i \vee Y_i(k|\mathbf{Z}) > b_i \} = \{ \mathbf{Z}: Y_{\max}(\mathbf{Z}) > 1 \} \quad (9)$$

where  $Y_{\max}(\mathbf{Z}) = [\max]_i [\max]_k \{ Y_i(k|\mathbf{Z}) \} / b_i \}$ . The system is assumed to start with zero initial conditions, i.e.,  $\mathbf{y}(0) = \dot{\mathbf{y}}(0) = 0$ . Eq. (9) states that the target failure domain is given by the union of 4000 nonlinear LSF, where each such LSF is defined in the 1001-dimensional  $\mathbf{Z}$  parameter space. MCS yields the value  $P(F) = 1.60 \times 10^{-5}$  with an accuracy of 17% C.O.V.

The ADM is employed assuming a linear auxiliary system corresponding to the small amplitude linear system. The mass and damping matrix of the auxiliary system are chosen to be the same as in the corresponding nonlinear system. The stiffness force vector of the auxiliary system is given by:

$$\mathbf{F}_s(\mathbf{y}(t)) = K\mathbf{y}(t) \quad (10)$$

where the stiffness matrix  $K$  is:

$$K = \begin{pmatrix} k_1^{(1)} + k_2^{(1)} & -k_2^{(1)} \\ -k_2^{(1)} & k_2^{(1)} \end{pmatrix} \quad (11)$$

Let  $Y_{1A}(t|\mathbf{Z})$  and  $Y_{2A}(t|\mathbf{Z})$  denote the displacements of the first and second floor, respectively, of the auxiliary system subjected to the same ground acceleration  $U(t|\mathbf{Z})$ , and let  $b_{1A}$  and  $b_{2A}$  denote the threshold values of the corresponding auxiliary reliability problem. The auxiliary failure domain  $F_A$  is defined analogously to (9) as follows:

$$F_A = \bigcup_{i=1}^2 \bigcup_{k=1}^{N_T} \{\mathbf{Z}: Y_{iA}(k|\mathbf{Z}) < -b_{iA} \vee Y_{iA}(k|\mathbf{Z}) > b_{iA}\} = \{\mathbf{Z}: Y_{A,\max}(\mathbf{Z}) > 1\} \quad (12)$$

where  $Y_{A,\max}(\mathbf{Z}) = [\max]_i [\max]_k \{Y_{iA}(k|\mathbf{Z})\} / b_{iA}\}$ .

In order to choose  $b_{1A}$  and  $b_{2A}$  a set  $\mathcal{Z} = \{\mathbf{Z}^{(i)}, i = 1, \dots, N = 500\}$  comprised of samples distributed according to  $f_{\mathbf{Z}|F}$  is generated. This is achieved by first finding a failure point and then using this as the starting point generating a Markov Chain that has  $f_{\mathbf{Z}|F}$  as its stationary distribution. The modified Metropolis Hastings algorithm presented in (Au and Beck 2001b) using a standard normal proposal distribution for each component has been chosen to generate such chain. To account for the burn-in time of the chain, the chain is run for six hundred points:  $\mathcal{Z}^e = \{\mathbf{Z}^{(i)}, i = -99, \dots, 500\}$  and the first  $K = 100$  points are discarded. The starting failure point is obtained using the procedure described in section 2.1.2, namely by running a chain with monotonically decreasing corresponding LSF values. Fig. 4 shows the computational effort associated with finding the starting point in twenty different runs. It can be seen that the required computational effort ranges from 5 to 23 dynamic analyses. The burn-in length  $K$  is estimated by monitoring the lengths  $|\mathbf{Z}^{(i)}|$  of the leading points in the chain  $\mathcal{Z}^e$ . Fig. 5 depicts these lengths as a function of the sample number in the chain for twenty different runs. It can be seen that in most of the runs the first approximately  $K = 100$  points in the chain are located outside the spherical ring with inner radius 29.42 and outer radius 33.68 where 99.7% of the samples are expected to be distributed. This implies that the samples in this leading part of the chain are clearly not distributed according to  $f_{\mathbf{Z}|F}$  and, therefore, should be discarded. Based on the above discussion the computational effort associated with the generation of the set  $\mathcal{Z}$  amounts to a total of 605-623 nonlinear dynamic analyses.

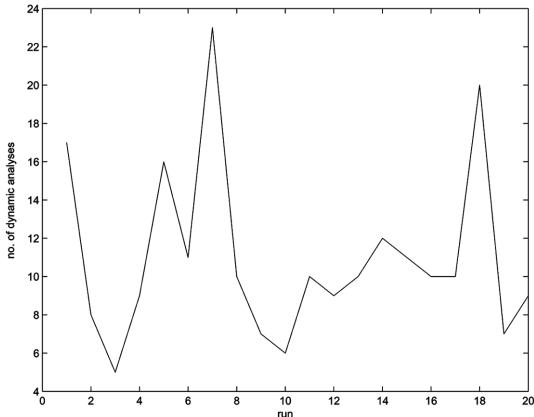


Fig. 4 Number of analyses for locating initial failure point

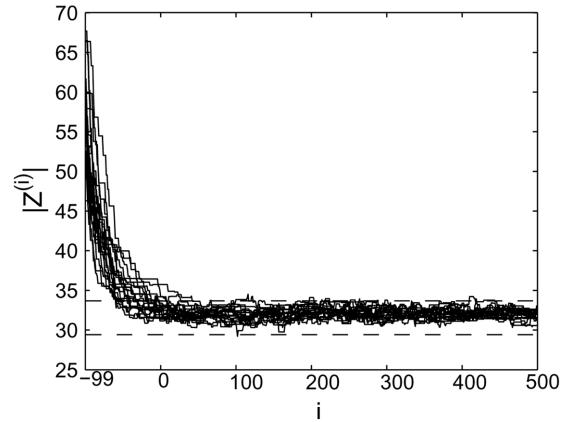
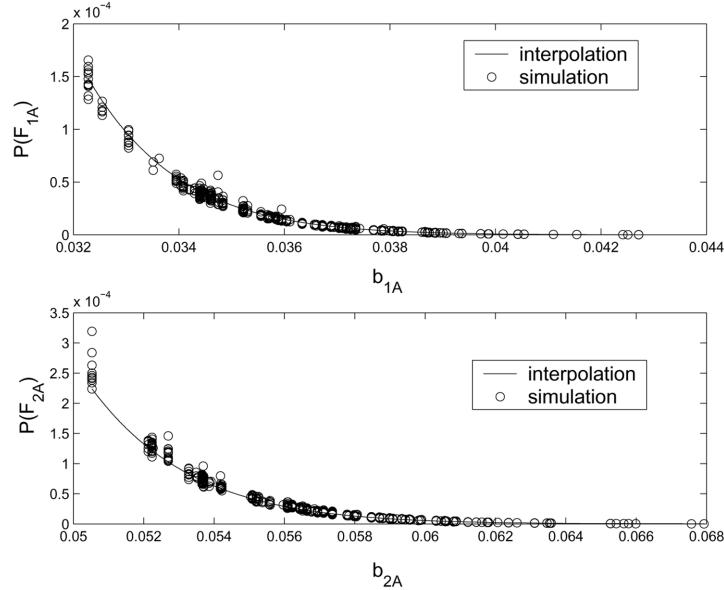


Fig. 5 Evolution of sample distance from origin

Fig. 6 Approximation of  $P(F_{jA}(b_{jA}))$ Table 1  $b_{1A}$ ,  $b_{2A}$  and  $P(F_A)$  using algorithms  $A_1$ ,  $A_2$  and  $A_P$ 

	$A_1$	$A_2$	$A_P$
$b_{1A}$ (m)	0.03228	0.03228	0.03228
$b_{2A}$ (m)	0.05053	0.05349	$\infty$
$P(F_A)$	$2.466 \times 10^{-4}$	$1.531 \times 10^{-4}$	$1.428 \times 10^{-4}$

Next, for each sample point  $\mathbf{Z}^{(i)} \in \mathcal{Z}$ ,  $i = 1, \dots, N = 500$ , the corresponding maximum linear responses  $\hat{Y}_{1A}^{(i)} = \max_k |Y_{1A}(k|\mathbf{Z}^{(i)})|$ ,  $\hat{Y}_{2A}^{(i)} = \max_k |Y_{2A}(k|\mathbf{Z}^{(i)})|$ , are calculated. This step involves a total of 500 linear dynamic analyses. Based on these sample points and for the chosen auxiliary linear system, the next task is to define the threshold values  $b_{1A}$  and  $b_{2A}$  needed to fully define the auxiliary reliability problem.

In Fig. 6, the circular dots give the values of  $P(F_{jA}(\hat{Y}_{jA}^{(j)})$ ,  $k = 1, 2, \dots, N$ ,  $j = 1, 2$  obtained using the simulation method presented in (Katafygiotis and Cheung 2004a); the solid curve gives the corresponding values when using an approximation of  $P(F_{jA}(b_{jA}))$  as a function of the threshold level  $b_{jA}$  based on the parametric expression suggested in Section 3, i.e., assuming  $\ln(P(F_{jA}(b_{jA})))$  to be a quadratic function of  $b_{jA}$ , and interpolating through 10 data points between  $\hat{Y}_{jA}^{(j_N)}$  and  $\hat{Y}_{jA}^{(j_1)}$ . From this figure it can be seen that the approximation works well requiring only a few data points. Note that the determination of each data point  $(b_{jA}, P(F_{jA}(b_{jA})))$  involves only one linear dynamic analysis.

Table 1 gives  $b_{1A}$ ,  $b_{2A}$  and  $P(F_A)$  obtained using algorithms  $A_1$ ,  $A_2$  and  $A_P$  (described in Section 3) for the case  $p_1 = 1$  making use of MCMC samples from one simulation run. It can be seen that for this MCMC sample set,  $b_{1A}$  obtained using the three algorithms are the same; the proposed algorithm  $A_P$  gives the largest value for  $b_{2A}$  while the algorithm  $A_1$  gives the smallest such value. As a result, the proposed algorithm  $A_P$  gives the smallest  $P(F_A)$  while the algorithm  $A_1$  gives the

largest  $P(F_A)$ . This shows that the proposed algorithm  $A_P$  is the best algorithm for selecting the thresholds defining the AFD because it leads to the largest  $P(F|F_A)$ , and therefore its estimation, assuming fixed accuracy, requires the smallest number of simulations. Although due to space limitations this fact is illustrated here using only one simulation run, it is obvious that  $A_P$  is by construction always the best algorithm for selecting the thresholds defining the AFD.

In the following analysis, the threshold values  $b_{1A}$  and  $b_{2A}$  are selected using the proposed algorithm  $A_P$ . For a chosen combination of  $b_{1A}$  and  $b_{2A}$ ,  $P(F_A)$  can then be determined using minimal computational effort following the method presented in (Katafygiotis and Cheung 2004a). Here we consider three choices for  $p_1$ : 1, 0.8, 0.5.

Twenty simulation runs are implemented to study the statistical performance of the proposed ADM method. Fig. 7 shows  $b_{1A}$  (solid curve) and  $b_{2A}$  (dotted curve) for each simulation run. It can be seen that the choice of  $b_{1A}$  and  $b_{2A}$  varies for different runs. Specifically, note that for some of the simulation runs the auxiliary domain is essentially defined using only one of the two constraints,

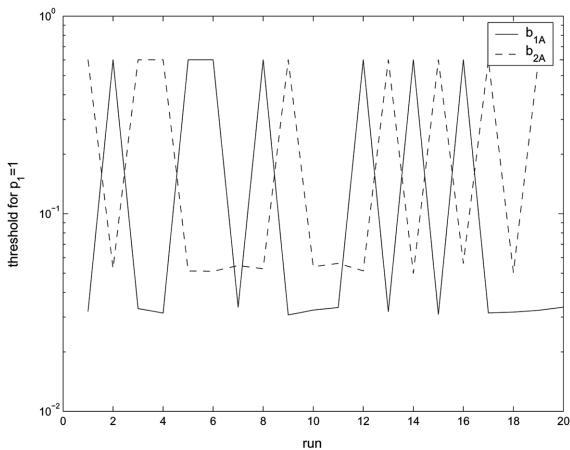


Fig. 7  $b_{1A}$  and  $b_{2A}$  for  $p_1 = 1$

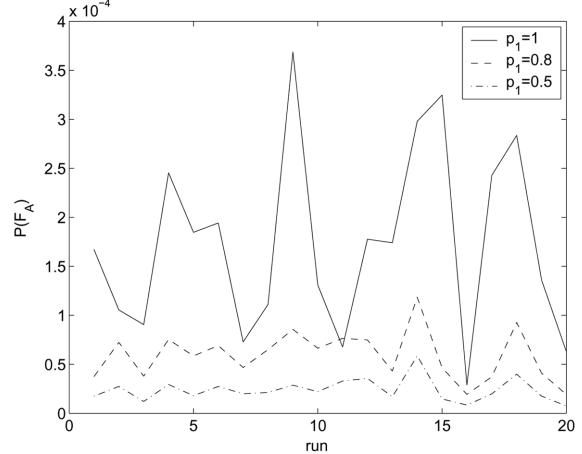


Fig. 8  $P(F_A)$  for 20 simulation runs

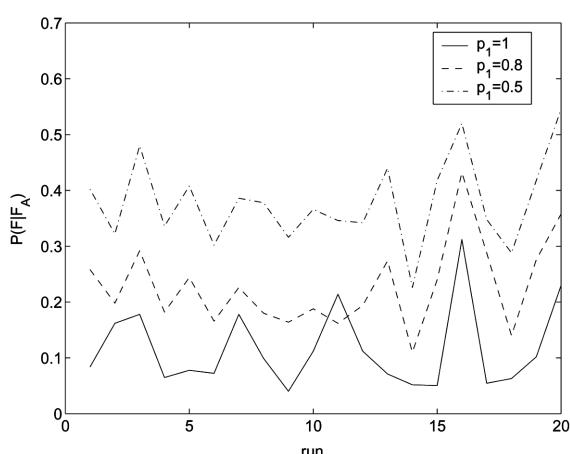


Fig. 9  $P(F|F_A)$  for 20 simulation runs

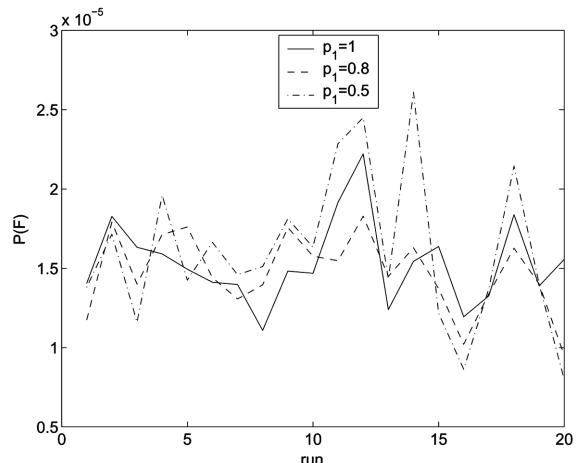


Fig. 10  $P(F)$  for 20 simulation runs

Table 2 Statistical results using: ADM; SS; MCS

	ADM- $p_1 = 1$	$p_1 = 0.8$	$p_1 = 0.5$	SS	MCS
$E[\tilde{P}(F)]$	$1.53 \times 10^{-5}$	$1.47 \times 10^{-5}$	$1.62 \times 10^{-5}$	$1.70 \times 10^{-5}$	$1.60 \times 10^{-5}$
C.O.V [ $\tilde{P}(F)$ ]	17.23%	16.74%	29.99%	17.85%	17.01%
Computations	1359	1112	1112	27600	$2.1626 \times 10^6$

i.e., in certain cases  $b_{1A}$  is very large, implying only the second constraint is active, while in other cases the opposite happens. On the other hand there are cases where both constraints seem to be active. In each simulation run the proposed algorithm selects the best combination of threshold values that lead to an auxiliary domain that contains all simulated sample points and at the same time has the smallest value of  $P(F_A)$  possible.

Figs. 8, 9 and 10 show the values of  $P(F_A)$ ,  $P(F|F_A)$  and  $P(F)$ , respectively, corresponding to the earlier three  $p_1$  values for different simulation runs. From Fig. 8 it can be seen that larger  $p_1$  values result in larger  $P(F_A)$  values, as expected. Fig. 9 shows that a larger  $p_1$  gives a smaller  $P(F|F_A)$  as also expected. In all three cases of  $p_1$ ,  $P(F|F_A)$  is found to be between 0.54 and 0.03. Therefore,  $P(F|F_A)$  is calculated using either one step (Eq. (2)) or two steps (Eq. (3) with  $K = 2$ ).

The statistical variability of the estimates  $\tilde{P}(F)$  for  $P(F)$  is assessed using 20 simulation runs. The statistical results corresponding to ADM using three different  $p_1$  values (columns 2-4), standard SS (column 5), and MCS (column 6), are summarized in Table 2. The mean  $E[\tilde{P}(F)]$ , the coefficient of variation  $C.O.V[\tilde{P}(F)]$  and the average total number of computations required to calculate  $\tilde{P}(F)$  by each method are given in the second, third and fourth rows of this table, respectively. Here each structural nonlinear dynamic analysis is counted as one computation. By comparing the values of  $E[\tilde{P}(F)]$  obtained by the different methods, we conclude that all methods yield practically unbiased results. It is observed that ADM gives best results (smallest C.O.V.) when  $p_1 = 0.8$ . Using only about 1359 nonlinear dynamic analyses, the C.O.V obtained by ADM is already within high levels of accuracy (less than 17.23%). Note that in addition to these nonlinear dynamic analyses another 500 linear dynamic analyses are required when calculating the linear responses of the points  $\mathbf{Z}^{(i)} \in \mathcal{Z}$ . The computational effort associated with these linear analyses is not reflected in Table 2.

Table 2 shows that for the same accuracy, ADM outperforms MCS and standard SS by over 1591 times and 20 times, respectively. If one includes the computational effort associated with the linear dynamic analyses required by the ADM these performance ratios are expected to be somewhat smaller. It is worth stressing here that while ADM requires calculation of the conditional probability  $P(F|F_A)$ , standard SS proceeds with the direct calculation of  $P(F)$ . In a case such as in this example where  $P(F)$  is of the order of  $10^{-5}$ , standard SS would, therefore, require about five steps to converge. Because the uncertainties introduced in each step accumulate, standard SS requires a much larger number of simulations in each step in order to achieve the same final accuracy as ADM which requires modified SS with a maximum of two steps.

## 5. Conclusions

This paper presents a novel methodology referred to as Auxiliary Domain Method (ADM) for solving dynamic reliability problems for nonlinear structures. Specifically, deterministic nonlinear

structures subjected to stochastic dynamic loads are considered herein. The ADM is based on the introduction of an auxiliary failure domain (AFD) that significantly intersects the target failure domain. The AFD is chosen to have a linear structure facilitating a very efficient calculation of its failure probability as well as a very efficient simulation of its failure points. Assuming that an auxiliary domain is appropriately chosen ADM can benefit through a modified SS procedure involving less steps and, therefore, being more efficient than standard SS. The methodology and issues related to the selection of an appropriate auxiliary domain for reliability problems involving single or multiple failure conditions have been presented.

It has been demonstrated with a numerical example that the method can offer significant computational savings compared to other existing methods when dealing with reliability problems involving weakly nonlinear systems. The optimal selection of an AFD and an appropriate selection in the case of strongly nonlinear systems needs further investigation. It is worth noting that there is a trade-off between efficiency and generality, namely, while the ADM provides improved computational efficiency in solving the particular type of reliability problems it lacks the generality of standard SS which allows for uncertainties in the system to also be considered.

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