Structural Engineering and Mechanics, Vol. 25, No. 1 (2007) 107-126
DOI: http://dx.doi.org/10.12989/sem.2007.25.1.107

Bounds on plastic strains for elastic plastic structures in plastic shakedown conditions

Francesco Giambanco†, Luigi Palizzolo‡ and Alessandra Caffarelli‡†

Dipartimento di Ingegneria Strutturale & Geotecnica (DISEG), Università degli Studi di Palermo, Viale delle Scienze, 90128 – Palermo, Italy

(Received April 13, 2005, Accepted August 7, 2006)

Abstract. The problem related to the computation of bounds on plastic deformations for structures in plastic shakedown condition (alternating plasticity) is studied. In particular, reference is made to structures discretized by finite elements constituted by elastic perfectly plastic material and subjected to a special combination of fixed and cyclic loads. The load history is known during the steady-state phase, but it is unknown during the previous transient phase; so, as a consequence, it is not possible to know the complete elastic plastic structural response. The interest is therefore focused on the computation of bounds on suitable measures of the plastic strain which characterizes just the first transient phase of the structural response, whatever the real load history is applied. A suitable structural model is introduced, useful to describe the elastic plastic behaviour of the structure in the relevant shakedown conditions. A special bounding theorem based on a perturbation method is proposed and proved. Such theorem allows us to compute bounds on any chosen measure of the relevant plastic deformation occurring at the end of the transient phase for the structure in plastic shakedown; it represents a generalization of analogous bounding theorems related to the elastic shakedown. Some numerical applications devoted to a plane steel structure are effected and discussed.

Keywords: elastic plastic structures; cyclic loads; plastic shakedown; bounds; perturbation methods.

1. Introduction

As it is well known, structures, for the greater part (sometime the whole) of their lifetime, are subjected to the so-called load serviceability conditions which are usually slightly variable, while for short periods (often during just a few dozen seconds) they can suffer very high intensity loads characterized by great variability.

In the framework of the structural analysis or design problems, the analytical representation of a load condition as above described is often very difficult. On the other side, the scientific and practice engineering interest is substantially related to the possibility of ensure suitable safety factors for the structure with respect to prefixed mechanical and/or kinematical limit conditions.

As a consequence, very often structures are considered as subjected to (variable) actions which are usefully described as a simple combination of fixed loads and cyclic loads arbitrarily varying

† Full Professor, E-mail: fgiamban@diseg.unipa.it
‡ Associate Professor, Corresponding author, E-mail: palizzol@diseg.unipa.it
‡† PhD, E-mail: alecaffa@msn.com

within given limits. Such a representation is near enough to the real loading condition and, furthermore, it is related to a very special structure behaviour. Actually, in this condition, structures are characterized at first by a transient (short-term) response, and eventually by a steady-state (long-term) response, the latter being independent of the initial conditions and possessing the same periodicity features as the acting cyclic loads (Zarka and Casier 1979, Zarka *et al.* 1990, Polizzotto *et al.* 1990, Polizzotto 1994a,b).

In many cases of practical interest, structures are constituted by elastic plastic material possessing suitably wide ductility properties, so they can be usefully required to operate beyond their elastic limit when they are subjected to the action of a load combination as above described. Under such conditions, and for load intensities not exceeding suitable limits, the elastic shakedown theory provides useful tools in studying the behaviour of the relevant structure; in particular, the so-called shakedown limit load multiplier provides an effective safety factor for the structure (see, e.g., Melan 1938a,b, Koiter 1960, Polizzotto 1978, König 1987), as well as the so-called bounding techniques provide limits on suitably chosen measures of the plastic deformations related to the transient phase of the elastic shakedown response of the structure (see, e.g., König 1979, Capurso *et al.* 1979, Polizzotto 1982, Giambanco and Palizzolo 1997).

Furthermore, if the load multipliers exceed the elastic shakedown limit, then the structure is addressed towards a collapse condition, either due to a plastic shakedown behaviour (oligocyclic fatigue or alternating plasticity) or to a ratchetting behaviour (incremental collapse). Finally, for increasing values of the load multipliers the structure is eventually addressed towards an instantaneous collapse.

Therefore, depending on the load multiplier values a structure can exhibit: an elastic behaviour, an elastic shakedown behaviour, a plastic shakedown behaviour, a ratchetting behaviour, or it can be exposed to instantaneous collapse. In the space of the fixed and cyclic load multipliers it is very useful to make reference to the graphical representation of these behaviour zones on the so called Bree-like diagram (Fig. 1), whose knowledge is of crucial importance to establish if the assigned structure/load system safely operates with potentially different load conditions.

Obviously, above the elastic shakedown limit (and below the instantaneous collapse limit) it is

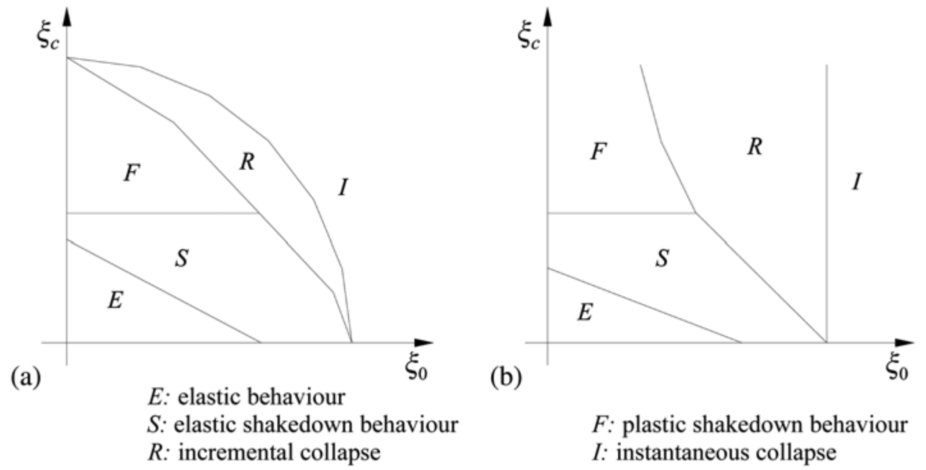

Fig. 1 Typical Bree-like diagram for: (a) mechanical cyclic loads, (b) thermal cyclic loads

preferable that the structure behaves in condition of plastic shakedown, rather than in condition of ratchetting; actually, in such condition, the structure is able to suffer a greater number of load cycles without loosing its functionality and usually the second order geometrical effects due to the occurring plastic strains can be considered as negligible.

As above stated, the plastic shakedown steady-state structural response possesses the same periodicity features as the loads and it can be determined by solving a sequence of linear complementarity problems related to the given load condition. The kinematical part of such a response provides the steady-state plastic strain history during the cycle and, as a consequence, it is possible to compute any chosen measure of such deformations, but, unfortunately, the same response does not provide any information about the plastic deformations occurring during the initial transient phase, and, as a consequence, the knowledge of such a response does not allow to check the respect of some ductility limits and/or functionality limits for the structure.

However, there are infinite admissible load histories through which the steady-state load condition is reached and, obviously, it is not possible to effect the elastic plastic analysis for each of them. If an admissible load path is arbitrarily chosen, then it is possible to effect a stepwise analysis and, furthermore, to compute the transient and steady-state elastic plastic response of the structure to the relevant load, but such a results can not be considered exhaustive about the structural behaviour in the defined load condition.

Since the knowledge of the elastic plastic structural behaviour during the transient phase plays an important role on the check related to some prefixed ductility limits and/or functionality limits for the structure, then, in order to obtain at least rough information about the plastic deformations occurring at the end of the transient phase, it is necessary to make reference to other analytical and/or numerical procedures and, in the relevant case, to suitable bounding techniques. These techniques allow us to evaluate bounds on some prefixed measures of the plastic deformations which the structure suffers at the end of the transient phase response, whatever the loading history is during the unknown transient phase load path.

To the author's knowledge, studies devoted to the formulation of bounding principles analogous to the above referred ones related to the elastic shakedown but holding for the present case of structures behaving in condition of plastic shakedown have never been effected; previously, (see, e.g., Giambanco and Palizzolo 1996) just a particular case of bounding technique has been treated, but related to the special case of load intensities slightly above the elastic shakedown limit.

The present paper, therefore, is mainly devoted to the formulation of a bounding principle useful in order to compute bounds on suitably chosen measures of the plastic deformations which characterize the transient phase response of an elastic perfectly plastic structure in condition of plastic shakedown.

Actually, if for structures behaving in condition of elastic shakedown it can be useful to have information, although rough, about the extent of the plastic deformations occurring during the transient phase, then such information is even more decisive when structures behave in a steady-state condition of alternating plasticity, due to the simultaneous presence of transient and steady-state phase plastic deformations.

In particular, the structure will be considered as subjected to a combination of fixed and perfect cyclic loads adopting the restrictive hypothesis of mechanical fixed loads and mechanical perfect cyclic loads, being the cyclic load a perfect one if for each value of the load intensity the opposite one occurs after half a cycle.

At first the structural model and the related elastic plastic behaviour will be described, making reference to the finite element discretized structure; then the steady-state response related to the cyclic loads will be described. In particular, the problem related to the determination of the bordeline between (elastic/plastic) shakedown domains and incremental/instantaneous collapse regions on the Bree diagram (see, e.g., Polizzotto *et al.* 2001, Ponter and Haofeng 2001, Giambanco *et al.* 2002, 2004) will be appropriately formulated.

Furthermore, a suitable bounding theorem based on a perturbation method will be formulated and proved, adopting for the load multipliers appropriate and quite general time functions, so as any admissible load history can be represented, and making use of the particular features related to the elastic plastic steady-state response of the structure subjected just to the perfect cyclic load. This theorem, devoted to structures behaving in condition of plastic shakedown, represents a generalization of other bounding principles, previously referenced and related to structures in condition of elastic shakedown, which can be deduced by the proposed one as suitable specialization through appropriate positions related to the load multiplier values.

Finally, some numerical applications related to a steel structure will be described; the obtained results will be reported in order to emphasize the features and the applicability of the proposed bounding technique.

2. Structural model and elastic plastic behaviour

Let us consider a structure discretized into n finite elements. Let us assume that the elements exhibit an elastic perfectly plastic behaviour and that, for each element, plastic deformations can occur just at the plastic nodes (i.e., Gauss points), which are conceived as sources from which plastic strains spread within the element volume, according to fixed shape functions (see, e.g., Corradi 1983). It is worth noticing that plastic nodes and element nodes are generally not coincident. The elastic plastic behaviour of the typical structural element can be usefully described in terms of generalized variables (generalized stresses and strains). Let us suppose that the loads acting on the structure, $\boldsymbol{F} = \boldsymbol{F}(t)$, are variable in time quasi-statically and that they are defined within the time interval $(0 \le t \le t_f)$. The time t is not the physical time, but just some monotonically increasing parameter aimed at correctly specifying the loading sequence.

In the hypothesis of small displacements and homogeneous kinematical initial conditions, assuming that the elastic domain of the typical element is a convex and temperature-independent hyperpoliedric function, the elastic plastic behaviour of the structure at time t is described by the following equations:

$$\boldsymbol{K}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{p} = \boldsymbol{F} \tag{1a}$$

$$\boldsymbol{P} = \tilde{\boldsymbol{B}}\boldsymbol{u} - \boldsymbol{D}\boldsymbol{p} + \boldsymbol{P}^{*} \tag{1b}$$

$$\boldsymbol{\varphi} \equiv \tilde{\boldsymbol{N}}\boldsymbol{P} - \boldsymbol{R} \le \boldsymbol{0}, \quad \dot{\boldsymbol{\lambda}} \ge \boldsymbol{0}, \quad \tilde{\boldsymbol{\varphi}}\dot{\boldsymbol{\lambda}} = 0, \quad \tilde{\dot{\boldsymbol{\varphi}}}\dot{\boldsymbol{\lambda}} = 0 \tag{1c}$$

$$\dot{\boldsymbol{p}} = \boldsymbol{N}\dot{\boldsymbol{\lambda}} \tag{1d}$$

$$\boldsymbol{p} = \int_0^t \boldsymbol{N}\dot{\boldsymbol{\lambda}}(\bar{t})\,d\bar{t} \tag{1e}$$

where $\boldsymbol{u}$ is the structure node displacement vector, $\boldsymbol{p}$ the generalized plastic strain vector evaluated at the strain points, $\boldsymbol{K} = \tilde{\boldsymbol{C}}\boldsymbol{D}_e\boldsymbol{C}$ the external stiffness matrix with $\boldsymbol{C}$ compatibility matrix and $\boldsymbol{D}_e$ block diagonal element internal stiffness matrix, $\boldsymbol{B} = \tilde{\boldsymbol{C}}\boldsymbol{D}_e\boldsymbol{G}_p$ the pseudo-force matrix, being $\boldsymbol{G}_p$ a matrix which applied to plastic strains provides element nodal displacements, $\boldsymbol{F} = \overline{\boldsymbol{F}} + \boldsymbol{F}^*$ the equivalent nodal load vector with $\overline{\boldsymbol{F}}$ representing the vector of the loads directly acting upon the structure nodes and $\boldsymbol{F}^*$ representing the nodal load vector equivalent to the actions applied upon the elements, $\boldsymbol{P}$ the generalized stress response vector evaluated at the strain points, $\boldsymbol{P}^*$ the generalized stress response vector evaluated at the strain points but due just to the loads directly acting upon the structural elements, $\boldsymbol{D} = \tilde{\boldsymbol{G}}_p\boldsymbol{D}_e\boldsymbol{G}_p$ the block diagonal stiffness matrix related to the strain points, $\boldsymbol{\varphi}$ the yield function vector which also plays the role of plastic potential, $\boldsymbol{N}$ the block diagonal matrix of unit external normals to the yield surface, $\boldsymbol{R}$ the plastic resistance vector and, finally, $\dot{\boldsymbol{\lambda}}$ the plastic activation vector.

Substituting Eq. (1a) in Eq. (1b) and the modified version of Eq. (1b) in Eq. (1c), taking into account Eq. (1e) the solving set, that must be satisfied for all t $(0 \le t \le t_f)$, is obtained in the following form:

$$-\boldsymbol{\varphi} = \boldsymbol{R} - \tilde{\boldsymbol{N}}(\tilde{\boldsymbol{B}}\boldsymbol{K}^{-1}\boldsymbol{F} + \boldsymbol{P}^*) - \tilde{\boldsymbol{N}}(\tilde{\boldsymbol{B}}\boldsymbol{K}^{-1}\boldsymbol{B} - \boldsymbol{D})\boldsymbol{N}\int_0^t \dot{\boldsymbol{\lambda}}(\bar{t})d\bar{t} \tag{2a}$$

$$-\boldsymbol{\varphi} \ge \boldsymbol{0}, \quad \dot{\boldsymbol{\lambda}} \ge \boldsymbol{0}, \quad \tilde{\boldsymbol{\varphi}}\dot{\boldsymbol{\lambda}} = 0, \quad \tilde{\dot{\boldsymbol{\varphi}}}\dot{\boldsymbol{\lambda}} = 0 \tag{2b}$$

where just the independent unknown vectors $\boldsymbol{\varphi}$ and $\dot{\boldsymbol{\lambda}}$ appear.

Problem (2) refers to a structure discretized into finite elements and with a discrete yield surface, but it is yet continuous with respect to the time t. At least in principle Eqs. (2), plus the appropriate initial conditions at $t = 0$, can be integrated with respect to time t in order to obtain the unknown vectors $\boldsymbol{\varphi}$ and $\dot{\boldsymbol{\lambda}}$, and therefore, through Eqs. (1), the elastic plastic structural response. Anyway, in practice, in order to obtain a numerical solution to problem (2), it is necessary to discretize the problem also with respect to the time, for example subdividing the time axis into m time subintervals with the same width $\delta t = t_f/m$. During the typical subinterval k (and, namely $(k-1)\delta t \le t \le k\delta t$) the unknown time function $\dot{\boldsymbol{\lambda}}(\tau)$, $\forall \tau = (t-(k-1)\delta t) \in (0, \delta t)$, is modelled in such a way it can be expressed in terms of a time independent unknown nonnegative vector $\boldsymbol{Y}^k$, i.e.:

$$\dot{\boldsymbol{\lambda}}(\tau) = \boldsymbol{g}(\tau)\boldsymbol{Y}^k, \quad \boldsymbol{Y}^k \ge \boldsymbol{0} \tag{3a}$$

where $\boldsymbol{g}(\tau)$ is a suitable square dimensional matrix with nonnegative time function entries and such that

$$\int_0^{\delta t} \boldsymbol{g}(\tau)d\tau = \boldsymbol{I} \tag{3b}$$

being $\boldsymbol{I}$ the identity matrix. $\boldsymbol{Y}^k$ is often called the plastic activation intensity vector.

Due to the plastic activation intensity modelling described by Eqs. (3), elastic unloading can occur just at the m prescribed discretization instants; namely, every element will remain either in the elastic regime or in the elastic plastic one during each step, and the last of Eq. (2b) becomes meaningless. Although Eqs. (2) can not be satisfied at some instant $\tau \in (0, \delta t)$, however, they must be satisfied in the following integrated, holonomic form:

$$\boldsymbol{Z}^k \equiv -\boldsymbol{\varphi}^k = -\int_0^{\delta t} \tilde{\boldsymbol{g}}(\tau)\boldsymbol{\varphi}(\tau)d\tau = \boldsymbol{S}\boldsymbol{Y}^k + \boldsymbol{b}^k \quad (4a)$$

$$\boldsymbol{Z}^k \geq \boldsymbol{0}, \quad \boldsymbol{Y}^k \geq \boldsymbol{0}, \quad \tilde{\boldsymbol{Y}}^k \boldsymbol{Z}^k = 0 \quad (4b)$$

Very often, in order to simplify the plastic activation modelling, a stepwise constant shape for $\dot{\boldsymbol{\lambda}}$ is chosen, i.e., $\boldsymbol{g}(\tau) = \boldsymbol{g} = \boldsymbol{I}(1/\delta t)$ is suitably assumed. In such a case, it results:

$$\boldsymbol{S} = -\tilde{\boldsymbol{N}}(\tilde{\boldsymbol{B}}\boldsymbol{K}^{-1}\boldsymbol{B} - \boldsymbol{D})\boldsymbol{N} = -\tilde{\boldsymbol{N}}\tilde{\boldsymbol{G}}_p(\boldsymbol{D}_e\boldsymbol{C}\boldsymbol{K}^{-1}\tilde{\boldsymbol{C}}\boldsymbol{D}_e - \boldsymbol{D}_e)\boldsymbol{G}_p\boldsymbol{N} \quad (5a)$$

$$\boldsymbol{b}^k = \boldsymbol{R} + \boldsymbol{S}\sum_{j=1}^{k-1}\boldsymbol{Y}^j - \tilde{\boldsymbol{N}}\tilde{\boldsymbol{B}}\boldsymbol{K}^{-1}\boldsymbol{F}^k - \tilde{\boldsymbol{N}}\boldsymbol{P}^{*k} \quad (5b)$$

where $-\boldsymbol{S}$ is a time independent symmetric structural matrix which transforms the plastic activation intensities $\boldsymbol{Y}^k$ into the plastic potentials $\boldsymbol{\varphi}^k$, $\boldsymbol{b}^k$ is a known vector depending on the pertinent loading at step k, on the increments of the plastic activation intensity vectors accumulated at step $k-1$, and on the known constant plastic resistance vector $\boldsymbol{R}$. In virtue of the effected time discretization, problem (2) transforms into the following sequence of linear complementarity problems:

$$\boldsymbol{Z}^k \geq \boldsymbol{0}, \quad \boldsymbol{Y}^k \geq \boldsymbol{0}, \quad \tilde{\boldsymbol{Y}}^k \boldsymbol{Z}^k = 0 \quad (k = 1, 2, \dots, m) \quad (6)$$

In the present case matrix $\boldsymbol{S}$ is positive semi-defined (see, e.g., Maier 1968), and as a consequence, neither the existence of a bounded solution $\boldsymbol{Y}^k$, nor its uniqueness is ensured (see, e.g., Cottle 1992). If Eqs. (6) admit an unbounded solution $\boldsymbol{Y}^k$ (at least somewhere in the structure), instantaneous collapse occurs; if they admit a vanishing solution $\boldsymbol{Y}^k$, the full structure is elastic; finally, if they admit a finite no vanishing solution $\boldsymbol{Y}^k$, the structure exhibits an elastic plastic behaviour. In this last case, any two solutions to the same problem can differ at least by a stressless (i.e., compatible, corresponding to a mechanism) set of plastic deformations (see, e.g., Maier 1968).

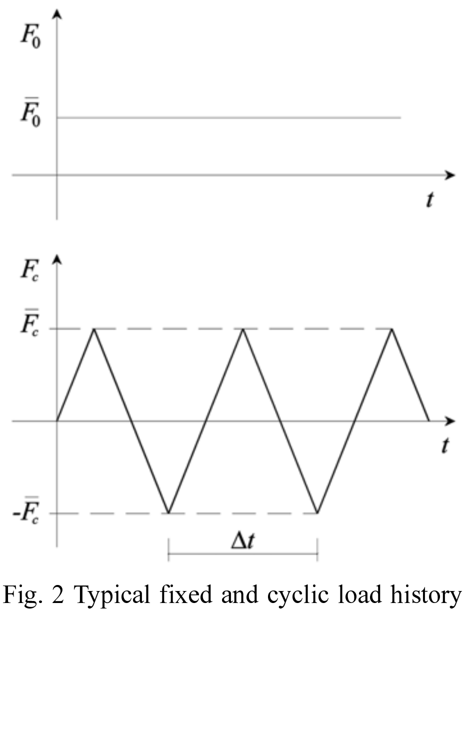

Fig. 2 Typical fixed and cyclic load history

3. Cyclic loads and steady-state structural response

Very often engineering practice structures are subjected to the contemporaneous action of fixed and cyclic loads. Therefore, let us suppose that the acting load $\boldsymbol{F} = \boldsymbol{F}(t)$, $(0 \le t \le t_f)$, is represented by the combination of a reference mechanical fixed load $\boldsymbol{F}_0(t) = \overline{\boldsymbol{F}}_0$ and a reference mechanical cyclic load $\boldsymbol{F}_c(t)$ of period Δt (Fig. 2).

In addition, let us assume that the cyclic load identifies with a convex polygonal shaped loading path with vertices corresponding to an even number b of mutually independent load vectors, denoted with $\boldsymbol{F}_{ci}$, $\forall i \in I(b) \equiv \{1, 2, \ldots, b\}$. Furthermore, let us assume the hypothesis that the cyclic load is a perfect one, namely for each basic load condition an opposite one exists in the load space. Finally, let us introduce the two scalars $\xi_0 \ge 0$ and $\xi_c \ge 0$, which represent the fixed and the cyclic load multipliers, respectively, so that $\xi_0 \boldsymbol{F}_0$ and $\xi_c \boldsymbol{F}_c$ are the amplified fixed and cyclic loads.

For an assigned loading history (as for example, applying at first the fixed load and successively the cyclic load starting from an arbitrary vertex) the elastic plastic response of the structure can be obtained by a step-by-step analysis effected for a given number of cycles, following the same procedure as in the previous section.

As it is well known (see, e.g., Zarka and Casier 1979, Zarka *et al.* 1990, Polizzotto *et al.* 1990, Polizzotto 1994a,b), in the described load condition the response of the relevant structure follows two subsequent phases: first a transient (short-term) response and eventually a steady-state (long-term) response. The latter exhibits the same periodicity features as the cyclic loads and it is independent of the initial conditions and of the special chosen load path. On the contrary, for each cycle of the loading history, the steady-state response just depends on the sequence of the b amplified basic load conditions, $\boldsymbol{F}_i = \xi_0 \boldsymbol{F}_0 + \xi_c \boldsymbol{F}_{ci}$, $\forall i \in I(b)$, obtained as combination of the amplified fixed and cyclic loads.

As a consequence, the elastic plastic steady-state response of the structure in the cycle can be obtained by an analysis effected just for the b basic load conditions, i.e.:

$$\boldsymbol{Z}_i = \boldsymbol{S}\boldsymbol{Y}_i + \boldsymbol{b}_i \qquad \forall i \in I(b) \tag{7a}$$

$$\boldsymbol{Z}_i \ge \boldsymbol{0}, \quad \boldsymbol{Y}_i \ge \boldsymbol{0}, \quad \tilde{\boldsymbol{Y}}_i \boldsymbol{Z}_i = 0 \qquad \forall i \in I(b) \tag{7b}$$

where

$$\boldsymbol{b}_i = \boldsymbol{R} - \tilde{\boldsymbol{N}}\tilde{\boldsymbol{B}}\boldsymbol{K}^{-1}\boldsymbol{F}_i - \tilde{\boldsymbol{N}}\boldsymbol{P}_i^* \qquad \forall i \in I(b) \tag{7c}$$

and $\boldsymbol{Y}_i$ is the vector of plastic activation intensities related to the i-th basic load condition.

The increment of plastic strain in the cycle (ratchet strain) can be expressed as:

$$\Delta \boldsymbol{p} = \sum_{i=1}^{b} \boldsymbol{N}\boldsymbol{Y}_i \tag{7d}$$

According to a terminology previously introduced in (Polizzotto *et al.* 1990), the plastic strain process related to the long-term response is referred to as the Plastic Accumulation Mechanism (PAM), that is a plastic strain rate cycle resulting in a compatible plastic strain field (the ratchet strain $\Delta \boldsymbol{p}$). Depending on the type of PAM and for loads above the purely elastic limit, three different types of steady-state responses are usually distinguished:

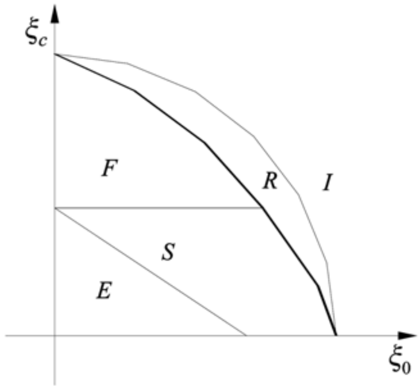

Fig. 3 Borderline (thick line) between the elastic/plastic shakedown domains (zones $S+F$) and the incremental/instantaneous collapse regions (zones $R+I$)

(i) elastic shakedown (or simply shakedown), when the PAM is a trivial one, i.e., the plastic strain rates vanish identically and the structural response is eventually elastic $(\boldsymbol{Y}_i = 0, \forall i \in I(b))$;

(ii) plastic shakedown (or alternating plasticity collapse, or oligocyclic fatigue), when the PAM is a non-trivial one, but the plastic strain field resulting in the cycle is nought $\left(\Delta \boldsymbol{p} = \sum_{i=1}^{b} \boldsymbol{N}\boldsymbol{Y}_i = \boldsymbol{0}\right)$;

(iii) ratchetting (or incremental plastic collapse), when the PAM is a non-trivial one, and the ratchet strain is non-vanishing, at least somewhere in the structure volume, causing the plastic strain to progressively increase.

For the purposes of the present paper it is very useful to consider the steady-state elastic plastic response of the structure subjected just to the amplified perfect cyclic loads $\xi_c \boldsymbol{F}_{ci}$, $\forall i \in I(b)$, and separately the purely elastic response of the same structure to the amplified fixed loads $\xi_0 \boldsymbol{F}_0$; moreover, in order to describe the shakedown behaviour of the structure it can be useful to determine, on the Bree diagram (Fig. 3), the borderline between the (elastic/plastic) shakedown domains (zones $S+F$ of the Bree-diagram) and the incremental/instantaneous collapse regions (zones $R+I$ of the Bree-diagram) of the relevant structure, solving the following problem (see, e.g., Giambanco *et al.* 2004):

$$\boldsymbol{K}\boldsymbol{u}_{ci} - \boldsymbol{F}_{ci} = \boldsymbol{0} \qquad \forall i \in I(b) \tag{8a}$$

$$\boldsymbol{P}_{ci} = \tilde{\boldsymbol{B}}\boldsymbol{u}_{ci} + \boldsymbol{P}^*_{ci} \qquad \forall i \in I(b) \tag{8b}$$

$$-\boldsymbol{\varphi}_{ci} = \boldsymbol{R} - \xi_c^a \tilde{\boldsymbol{N}}\boldsymbol{P}_{ci} + \boldsymbol{S}\boldsymbol{Y}_{ci} \qquad \forall i \in I(b) \tag{8c}$$

$$-\boldsymbol{\varphi}_{ci} \geq \boldsymbol{0}, \quad \boldsymbol{Y}_{ci} \geq \boldsymbol{0}, \quad \tilde{\boldsymbol{Y}}_{ci}\boldsymbol{\varphi}_{ci} = 0 \qquad \forall i \in I(b) \tag{8d}$$

$$\boldsymbol{K}\boldsymbol{u}_0 - \boldsymbol{F}_0 = \boldsymbol{0} \tag{9a}$$

$$\boldsymbol{P}_0 = \tilde{\boldsymbol{B}}\boldsymbol{u}_0 + \boldsymbol{P}^*_0 \tag{9b}$$

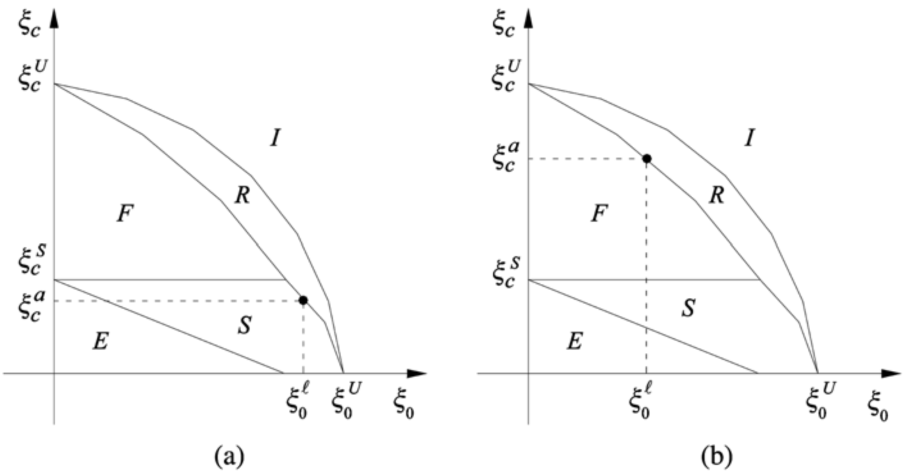

Fig. 4 Determination of the shakedown boundary: (a) elastic shakedown, (b) plastic shakedown

$$\xi_0^\ell(\xi_c^a) = \max_{(\xi_0,\, Y_0)} \xi_0 \qquad \text{subject to} \qquad (9c)$$

$$-\boldsymbol{\varphi}_i^S \equiv -\boldsymbol{\varphi}_{ci} - \xi_0 \tilde{\boldsymbol{N}}\boldsymbol{P}_0 + \boldsymbol{S}\boldsymbol{Y}_0 \geq \boldsymbol{0}, \quad \boldsymbol{Y}_0 \geq \boldsymbol{0} \qquad \forall i \in I(b) \qquad (9d)$$

for a suitably chosen number of assigned cyclic multiplier values ξ_c^a, such that $0 \leq \xi_c^a < \xi_x^U$, being ξ_c^U the limit cyclic load multiplier value above which the structure subjected just to the amplified perfect cyclic loads instantaneously collapses (Fig. 4).

In Eqs. (8)-(9), besides the already defined symbols, $\boldsymbol{P}_{ci}$ and $\boldsymbol{u}_{ci}$ are the purely elastic response just to the i-th reference cyclic load in terms of generalized stresses evaluated at the strain points and in terms of structure node displacements, being $\boldsymbol{P}_{ci}^*$ the generalized stress response vectors evaluated at the strain points due to the i-th cyclic load directly acting upon the structural elements, while $\boldsymbol{\varphi}_{ci}$ and $\boldsymbol{Y}_{ci}$ are the analogous of $\boldsymbol{\varphi}_i$ and $\boldsymbol{Y}_i$ but related to the purely cyclic load. Furthermore, $\boldsymbol{P}_0$, $\boldsymbol{u}_0$ and $\boldsymbol{P}_0^*$ are the analogous of $\boldsymbol{P}_{ci}$, $\boldsymbol{u}_{ci}$ and $\boldsymbol{P}_{ci}^*$, but related to the reference fixed load, $\boldsymbol{\varphi}_i^S$ is the vector of plastic potentials for the structure at the limit state of (elastic/plastic) shakedown (depending on value of ξ_c^a) related to the i-th basic load condition, while $\boldsymbol{Y}_0$ is a time independent vector of plastic activation intensities related with the selfstress field at the (elastic/plastic) shakedown limit.

If $0 \leq \xi_c^a \leq \xi_c^S$ is assumed, being ξ_c^S the elastic shakedown limit load multiplier for the structure subjected just to the cyclic load (Fig. 4a), then Eqs. (8) admit the vanishing solution $\boldsymbol{Y}_{ci} = \boldsymbol{0}$, $\forall i \in I(b)$, and in the steady-state phase the whole structural behaviour is eventually elastic. In this case the couple of values $[\xi_0^\ell(\xi_c^a),\, \xi_c^a]$, deduced solving problem (9), represents a point of the borderline between the elastic shakedown domain and the incremental/instantaneous collapse regions. Otherwise, if $\xi_c^S < \xi_c^a < \xi_c^U$ is assumed (Fig. 4b), then Eqs. (8) admit a non-vanishing solution, $\boldsymbol{Y}_{ci}$, at least for some $i \in I(b)$, and the structure eventually exhibits a steady-state elastic plastic behaviour, so that the couple of values $[\xi_0^\ell(\xi_c^a),\, \xi_c^a]$ represents a point of the borderline between the plastic shakedown domain and the incremental/instantaneous collapse regions. Anyway, in this last case, the increment of plastic strain in the cycle is nought, i.e.:

$$\Delta \boldsymbol{p} = \sum_{i=1}^{b} \boldsymbol{N}\boldsymbol{Y}_{ci} = \boldsymbol{0} \tag{10}$$

The results obtained by means of the previously described analyses are very consistent but, unfortunately, they are not complete enough in order to give a definitive judgement on the structural safety with respect to some appropriate prefixed ductility and/or functionality limits. Actually, as already stated, the solution of the above described problems can not provide any information about the elastic plastic response which the structure exhibits during the transient phase; however, such an information can not be determined even making recourse to a step by step analysis because, usually, the load path is unknown. As a consequence, in order to obtain at least some rough indication about such a response, it is possible to make reference to the bounding techniques.

In the following section a special bounding theorem, based on a perturbation method and devoted to structure in plastic shakedown, is proposed and proved; it represents a generalization of other known bounding principles formulated in the context of the elastic shakedown.

4. Bounding principle

As previously stated the unknowledge of the loading path through which the steady-state load condition is reached makes impossible the determination of the actual elastic plastic transient response of the structure which depends just on the loading history. However, it must be noticed that the unknowledge of the loading path is usually due to the unknowledge of the load multiplier intensity variation, being the reference basic load intensities generally given.

Therefore, let us suppose that the cyclic load multiplier ξ_c can vary as unknown function of the time t, and let us denote with $\overline{\xi}_c$ its maximum assigned value. Furthermore, let us assume that $\xi_0(t) = \overline{\xi}_0 \; \forall t \geq 0$. This last simplifying hypothesis can be accepted accounting for the very smooth expected variability of the relevant fixed loads. Obviously, $\overline{\xi}_0$ and $\overline{\xi}_c$ have to be considered as known values; actually, they represent the safety factors with respect to some prescribed limit conditions for the relevant structure. As a consequence, any load history is characterized by the load multipliers defined as follows:

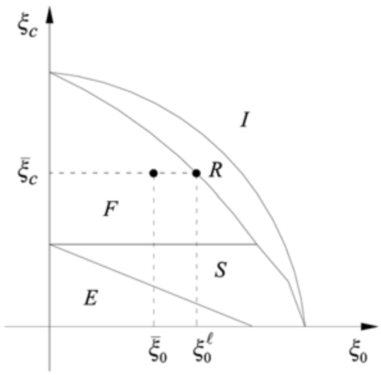

Fig. 5 $\overline{\xi}_0$, $\overline{\xi}_c$: plastic shakedown load multipliers; $\overline{\xi}_0^{\ell}$, $\overline{\xi}_c$: plastic shakedown *limit* load multipliers

$$\xi_0(t) = \overline{\xi}_0, \quad \xi_c(t) \leq \overline{\xi}_c, \qquad (0 \leq t < t_1) \tag{11a}$$

$$\xi_0(t) = \overline{\xi}_0, \quad \xi_c(t) = \overline{\xi}_c, \qquad (t \geq t_1) \tag{11b}$$

where t_1 is a suitably chosen instant such that for each $t \geq t_1$ *the loads become steady-state*. Furthermore, in order to prescribe the desired structural behavior, let us assume that the couple of multiplier values $(\overline{\xi}_0, \overline{\xi}_c)$ identifies a point belonging to the plastic shakedown region of the Bree diagram of the structure (Fig. 5). As it will be shown hereafter, it can be useful to identify the instant t_1 as a multiple of the load period Δt, i.e., $t_1 = n_1 \Delta t$, being n_1 a suitably chosen integer.

Keeping in mind Eqs. (8)-(9), making reference to the cyclic load as a known function of time t as previously described, taking into account definitions (11a,b), remembering that the maximum values of the load multipliers $\overline{\xi}_0$ and $\overline{\xi}_c$ characterize the plastic shakedown condition for the structure and denoting with $t_2 = n_2 \Delta t > t_1$ a suitably chosen subsequent instant such that for $t \geq t_2$ *the structural response is steady-state*, the following relations hold true:

$$\boldsymbol{K u}_c(t) - \boldsymbol{F}_c(t) = \boldsymbol{0} \qquad (t \geq t_2) \tag{12a}$$

$$\boldsymbol{P}_c(t) = \tilde{\boldsymbol{B}} \boldsymbol{u}_c(t) + \boldsymbol{P}_c^*(t) \qquad (t \geq t_2) \tag{12b}$$

$$-\boldsymbol{\varphi}_c \equiv \boldsymbol{R} - \overline{\xi}_c \tilde{\boldsymbol{N}} \boldsymbol{P}_c(t) + \boldsymbol{S Y}_c(t) \geq \boldsymbol{0}, \quad \dot{\boldsymbol{Y}}_c(t) \geq \boldsymbol{0}, \quad \tilde{\boldsymbol{\varphi}}_c \dot{\boldsymbol{Y}}_c = \dot{\tilde{\boldsymbol{\varphi}}}_c \dot{\boldsymbol{Y}}_c = 0 \qquad (t \geq t_2) \tag{12c}$$

$$\dot{\boldsymbol{p}}_c(t) = \boldsymbol{N} \dot{\boldsymbol{Y}}_c(t), \quad \boldsymbol{p}_c(t) = \int_0^t \dot{\boldsymbol{p}}_c(\overline{\tau}) d\overline{\tau} \qquad (t \geq t_2) \tag{12d}$$

$$\int_{n\Delta t}^{(n+1)\Delta t} \boldsymbol{N} \dot{\boldsymbol{Y}}_c(\tau) d\tau = \boldsymbol{0} \qquad (n \geq n_2) \tag{12e}$$

$$\boldsymbol{K u}_0 - \boldsymbol{F}_0 = \boldsymbol{0} \tag{12f}$$

$$\boldsymbol{P}_0 = \tilde{\boldsymbol{B}} \boldsymbol{u}_0 + \boldsymbol{P}_0^* \tag{12g}$$

$$-\overline{\boldsymbol{\varphi}}^S \equiv \boldsymbol{R} - \overline{\xi}_c \tilde{\boldsymbol{N}} \boldsymbol{P}_c(t) - \overline{\xi}_0 \tilde{\boldsymbol{N}} \boldsymbol{P}_0 + \boldsymbol{S Y}_c(t) + \boldsymbol{S Y}_0 \geq \boldsymbol{0}, \quad \boldsymbol{Y}_0 \geq \boldsymbol{0} \qquad (t \geq t_2) \tag{12h}$$

where Eq. (12e) ensure the vanishing of the plastic strain increment in the cycle, according to the cyclic load path previously described and to the assigned value of the cyclic load multiplier. Furthermore, in Eqs. (12), besides the already defined symbols, $\boldsymbol{u}_c(t), \boldsymbol{P}_c(t)$ and $\boldsymbol{P}_c^*(t)$ are the purely elastic response of the structure, subjected just to the cyclic loads $\boldsymbol{F}_c(t)$, in terms of node displacements and generalized stresses, respectively; $\dot{\boldsymbol{Y}}_c(t), \boldsymbol{p}_c(t)$ are the plastic activation vector and the generalized plastic strains related to the plastic shakedown behaviour of the structure subjected just to the amplified cyclic loads $\overline{\xi}_c \boldsymbol{F}_c(t)$, respectively, while $\boldsymbol{\varphi}_c$ and $\overline{\boldsymbol{\varphi}}^S$ represent the plastic potential related to the structure subjected just to the amplified cyclic loads in a steady-state condition of alternating plasticity and the analogous plastic potential of the same structure but related to the full presence of the acting (amplified fixed and cyclic) loads, respectively, according with the described load history.

Vector $\boldsymbol{Y}_0$, appearing in Eq. (12h), can be deduced by the solution to the following maximum problem:

$$\boldsymbol{K}\boldsymbol{u}_0 - \boldsymbol{F}_0 = \boldsymbol{0} \tag{13a}$$

$$\boldsymbol{P}_0 = \tilde{\boldsymbol{B}}\boldsymbol{u}_0 + \boldsymbol{P}_0^* \tag{13b}$$

$$\xi_0^{\ell}(\overline{\xi}_c) = \max_{(\xi_0,\, \boldsymbol{Y}_0)} \xi_0 \quad \text{subject to} \tag{13c}$$

$$-\boldsymbol{\varphi}^{S\ell} \equiv \boldsymbol{R} - \overline{\xi}_c \tilde{\boldsymbol{N}}\boldsymbol{P}_c(t) - \xi_0 \tilde{\boldsymbol{N}}\boldsymbol{P}_0 + \boldsymbol{S}\boldsymbol{Y}_c(t) + \boldsymbol{S}\boldsymbol{Y}_0 \geq \boldsymbol{0}, \quad \boldsymbol{Y}_0 \geq \boldsymbol{0}, \quad n_2\Delta t \leq t \leq (n_2+1)\Delta t \tag{13d}$$

Actually, taking in mind Eqs. (9), vector $\boldsymbol{Y}_0$ deduced by the solution to problem (13) ensures the satisfaction of Eq. (13d) for $\xi_c(t) = \overline{\xi}_c$ and $\xi_0^{\ell} \geq \xi_0$ (Fig. 5). As a consequence, remembering positions (11), Eq. (13d) implies that even the following relation holds true:

$$-\boldsymbol{\varphi}^{S} \equiv \boldsymbol{R} - \xi_c(t)\tilde{\boldsymbol{N}}\boldsymbol{P}_c(t) - \overline{\xi}_0 \tilde{\boldsymbol{N}}\boldsymbol{P}_0 + \boldsymbol{S}\boldsymbol{Y}_c(t) + \boldsymbol{S}\boldsymbol{Y}_0 \geq \boldsymbol{0}, \quad \forall t \geq 0 \tag{14}$$

It is worth noticing that relation (12h) can be easily deduced from this last inequality as specialization through suitable position related with $\xi_c(t)$ and for $\forall t \geq 0$.

As previously stated, Eqs. (12)-(14) do not provide any information about the elastic plastic structural response during the transient phase. Therefore, taking in mind the unknowledge of the history which describes the load multiplier value variation, and with the aim of obtaining at least some indication about the elastic plastic response which the structure exhibits during the transient phase, it is possible to make reference to a suitable bounding theorem based on a perturbation method. The bounding technique here proposed and proved, which represents a special generalization of analogous principles formulated in the context of the elastic shakedown (see, e.g., Polizzotto 1982), is devoted to the computation of a bound on a chosen measure of the plastic deformations which eventually occurr at the end of the transient phase.

With this aim, let us consider the relevant finite element elastic perfectly plastic structure subjected to the described combination of amplified fixed and cyclic loads. For each $t \geq 0$ the elastic plastic response is governed by the following equations:

$$\boldsymbol{K}\boldsymbol{u}_c(t) - \boldsymbol{F}_c(t) = \boldsymbol{0} \tag{15a}$$

$$\boldsymbol{P}_c(t) = \tilde{\boldsymbol{B}}\boldsymbol{u}_c(t) + \boldsymbol{P}_c^*(t) \tag{15b}$$

$$-\boldsymbol{\varphi} \equiv \boldsymbol{R} - \xi_c(t)\tilde{\boldsymbol{N}}\boldsymbol{P}_c(t) - \overline{\xi}_0 \tilde{\boldsymbol{N}}\boldsymbol{P}_0 + \boldsymbol{S}\boldsymbol{\lambda}(t) \geq \boldsymbol{0}, \quad \dot{\boldsymbol{\lambda}}(t) \geq \boldsymbol{0}, \quad \tilde{\boldsymbol{\varphi}}\dot{\boldsymbol{\lambda}} = \tilde{\dot{\boldsymbol{\varphi}}}\dot{\boldsymbol{\lambda}} = 0 \tag{15c}$$

$$\dot{\boldsymbol{p}}^a(t) = \boldsymbol{N}\dot{\boldsymbol{\lambda}}(t), \quad \boldsymbol{p}^a(t) = \int_0^t \boldsymbol{N}\dot{\boldsymbol{\lambda}}(t)\,d\overline{t} \tag{15d}$$

where $\boldsymbol{P}_0$ is computed by means of Eqs. (12f,g) and $\boldsymbol{p}^a = \boldsymbol{p}^a(t)$ is the actual plastic deformation vector.

Let us assume now that $\dot{\boldsymbol{p}}^a(t)$ can be computed as sum of the plastic strain rates due to the amplified cyclic load $\dot{\boldsymbol{p}}_c(t)$, plus a new unknown plastic strain vector, $\dot{\boldsymbol{p}}(t)$, substantially related to the transient phase, i.e.:

$$\dot{\boldsymbol{p}}^{a}(t) = \dot{\boldsymbol{p}}_c(t) + \dot{\boldsymbol{p}}(t) \tag{16a}$$

that, in terms of plastic activations, is analoguos to the following relation:

$$\boldsymbol{N}\dot{\boldsymbol{\lambda}}(t) = \boldsymbol{N}\dot{\boldsymbol{Y}}_c(t) + \boldsymbol{N}\dot{\boldsymbol{Y}}(t) \tag{16b}$$

where $\dot{\boldsymbol{Y}}_c(t) \geq \boldsymbol{0}$ is related to the purely cyclic loads while $\dot{\boldsymbol{Y}}(t) \geq \boldsymbol{0}$ is a new unknown function of the time t.

Since $\dot{\boldsymbol{Y}}_c(t)$ is a known history of plastic activations, as a consequence of positions (16), Eq. (15c) transform into:

$$-\boldsymbol{\varphi} \equiv \boldsymbol{R} - \xi_c(t)\tilde{\boldsymbol{N}}\boldsymbol{P}_c(t) - \overline{\xi}_0\tilde{\boldsymbol{N}}\boldsymbol{P}_0 + \boldsymbol{S}\boldsymbol{Y}_c(t) + \boldsymbol{S}\boldsymbol{Y}(t) \geq \boldsymbol{0} \tag{17a}$$

$$\dot{\boldsymbol{Y}}(t) \geq \boldsymbol{0}, \quad \tilde{\boldsymbol{\varphi}}\dot{\boldsymbol{Y}} = \tilde{\dot{\boldsymbol{\varphi}}}\dot{\boldsymbol{Y}} = 0 \quad \forall t \geq 0 \tag{17b}$$

Basing, as previously stated, on the perturbation method of the bounding theory, let us introduce the linear perturbation mode vector $\hat{\boldsymbol{R}}$ and the related perturbation multiplier $\omega > 0$ (see, e.g., Polizzotto 1982).

It is worth noticing that suitably choosing the perturbation vector $\hat{\boldsymbol{R}}$ it is possible to obtain bounds on different quantities related to the actual process, while the value of ω influences the stringentness of the bounds to be computed.

Introducing the perturbation quantities in Eqs. (14), the following relations hold:

$$-\hat{\boldsymbol{\varphi}}^{S} \equiv \boldsymbol{R} - \omega\hat{\boldsymbol{R}} - \xi_c(t)\tilde{\boldsymbol{N}}\boldsymbol{P}_c(t) - \overline{\xi}_0\tilde{\boldsymbol{N}}\boldsymbol{P}_0 + \boldsymbol{S}\boldsymbol{Y}_c(t) + \boldsymbol{S}\hat{\boldsymbol{Y}}(t) \geq \boldsymbol{0}, \quad \hat{\boldsymbol{Y}}_0 \geq \boldsymbol{0}, \quad \forall t \geq 0 \tag{18}$$

where $\hat{\boldsymbol{\varphi}}^{S}$ is the perturbed yield function, while vector $\hat{\boldsymbol{Y}}_0$ can be deduced by the solution to the following maximum problem:

$$\boldsymbol{K}\boldsymbol{u}_0 - \boldsymbol{F}_0 = \boldsymbol{0} \tag{19a}$$

$$\boldsymbol{P}_0 = \tilde{\boldsymbol{B}}\boldsymbol{u}_0 + \boldsymbol{P}_0^{*} \tag{19b}$$

$$\hat{\xi}_0(\overline{\xi}_c) = \max_{(\xi_0,\, \hat{\boldsymbol{Y}}_0)} \xi_0 \quad \text{subject to} \tag{19c}$$

$$-\hat{\boldsymbol{\varphi}}^{S} \equiv \boldsymbol{R} - \omega\hat{\boldsymbol{R}} - \overline{\xi}_c\tilde{\boldsymbol{N}}\boldsymbol{P}_c(t) - \xi_0\tilde{\boldsymbol{N}}\boldsymbol{P}_0 + \boldsymbol{S}\boldsymbol{Y}_c(t) + \boldsymbol{S}\hat{\boldsymbol{Y}}_0(t) \geq \boldsymbol{0}, \quad \hat{\boldsymbol{Y}}_0 \geq \boldsymbol{0}, \quad \forall t \geq 0 \tag{19d}$$

It is worth noticing that Eqs. (15) and (17) concern the *actual* elastic plastic response of the structure, while Eqs. (18) represent the *fictitious* plastic shakedown conditions related to the perturbed plastic potential.

Utilizing Eq. (17b) and taking in mind that the opposite of the perturbed plastic potential (Eq. (19d)) must be nonnegative, as well as the introduced unknown $\dot{\boldsymbol{Y}}$, the following inequality holds:

$$\tilde{\boldsymbol{\varphi}}\dot{\boldsymbol{Y}} - \tilde{\hat{\boldsymbol{\varphi}}}^{S}\dot{\boldsymbol{Y}} \geq 0, \qquad \forall t \geq 0 \tag{20}$$

If into inequality (20) the plastic potential vectors $\boldsymbol{\varphi}$ and $\hat{\boldsymbol{\varphi}}^S$ are expressed through their relevant form deduced by Eqs. (17) and (18), the following inequality can be written $(\forall t \geq 0)$:

$$-\tilde{\boldsymbol{R}}\dot{\boldsymbol{Y}}(t) + \xi_c(t)\tilde{\boldsymbol{P}}_c(t)\boldsymbol{N}\dot{\boldsymbol{Y}}(t) + \overline{\xi}_0\tilde{\boldsymbol{P}}_0(t)\boldsymbol{N}\dot{\boldsymbol{Y}}(t) - \tilde{\boldsymbol{Y}}_c(t)\boldsymbol{S}\dot{\boldsymbol{Y}}(t) - \tilde{\boldsymbol{Y}}(t)\boldsymbol{S}\dot{\boldsymbol{Y}}(t) + \tilde{\boldsymbol{R}}\dot{\boldsymbol{Y}}(t) +$$

$$-\xi_c(t)\tilde{\boldsymbol{P}}_c(t)\boldsymbol{N}\dot{\boldsymbol{Y}}(t) - \overline{\xi}_0\tilde{\boldsymbol{P}}_0\boldsymbol{N}\dot{\boldsymbol{Y}}(t) + \tilde{\boldsymbol{Y}}_c(t)\boldsymbol{S}\dot{\boldsymbol{Y}}(t) + \tilde{\hat{\boldsymbol{Y}}}_0\boldsymbol{S}\dot{\boldsymbol{Y}}(t) - \omega\tilde{\hat{\boldsymbol{R}}}\dot{\boldsymbol{Y}}(t) \geq 0 \tag{21}$$

which, with appropriate simplifications, reduces to:

$$[\tilde{\hat{\boldsymbol{Y}}}_0 - \tilde{\boldsymbol{Y}}(t)]\boldsymbol{S}\dot{\boldsymbol{Y}}(t) - \omega\tilde{\hat{\boldsymbol{R}}}\dot{\boldsymbol{Y}}(t) \geq 0 \qquad \forall t \geq 0 \tag{22}$$

Vector $\hat{\boldsymbol{Y}}_0$ is time independent, so that its time derivative $\dot{\hat{\boldsymbol{Y}}}_0$ is certainly nought. As a consequence, the inequality (22) can be rewritten in the following analogous form:

$$\omega\tilde{\hat{\boldsymbol{R}}}\dot{\boldsymbol{Y}}(t) \leq -[\tilde{\hat{\boldsymbol{Y}}}_0 - \tilde{\boldsymbol{Y}}(t)]\boldsymbol{S}[\dot{\hat{\boldsymbol{Y}}}_0 - \dot{\boldsymbol{Y}}(t)] \tag{23}$$

Integrating inequality (23) from the initial instant $t = 0$ to the selected time $t_2 = n_2\Delta t$, in correspondence of which the structural response is eventually steady-state

$$\int_0^{n_2\Delta t} \omega\tilde{\hat{\boldsymbol{R}}}\dot{\boldsymbol{Y}}(t)dt \leq -\int_0^{n_2\Delta t} [\tilde{\hat{\boldsymbol{Y}}}_0 - \tilde{\boldsymbol{Y}}(t)]\boldsymbol{S}[\dot{\hat{\boldsymbol{Y}}}_0 - \dot{\boldsymbol{Y}}(t)]dt \tag{24}$$

one obtains:

$$\omega\tilde{\hat{\boldsymbol{R}}}\boldsymbol{Y}(n_2\Delta t) - \omega\tilde{\hat{\boldsymbol{R}}}\boldsymbol{Y}(0) \leq$$

$$-\frac{1}{2}[\tilde{\hat{\boldsymbol{Y}}}_0 - \tilde{\boldsymbol{Y}}(n_2\Delta t)]\boldsymbol{S}[\hat{\boldsymbol{Y}}_0 - \boldsymbol{Y}(n_2\Delta t)] + \frac{1}{2}[\tilde{\hat{\boldsymbol{Y}}}_0 - \tilde{\boldsymbol{Y}}(0)]\boldsymbol{S}[\hat{\boldsymbol{Y}}_0 - \boldsymbol{Y}(0)] \tag{25}$$

Remembering that on $\boldsymbol{Y}(t)$ homogeneous initial conditions have been imposed and that $\omega > 0$, inequality (25) can be simplified as follows:

$$\tilde{\hat{\boldsymbol{R}}}\boldsymbol{Y}(n_2\Delta t) \leq \frac{1}{2\omega}\tilde{\hat{\boldsymbol{Y}}}_0\boldsymbol{S}\hat{\boldsymbol{Y}}_0 - \frac{1}{2\omega}[\tilde{\hat{\boldsymbol{Y}}}_0 - \tilde{\boldsymbol{Y}}(n_2\Delta t)]\boldsymbol{S}[\hat{\boldsymbol{Y}}_0 - \boldsymbol{Y}(n_2\Delta t)] \tag{26}$$

Finally, on the grounds of the positivity of the quadratic form depending on $\boldsymbol{Y}(n_2\Delta t)$, the bounding inequality (26) can be reinforced as follows:

$$\tilde{\hat{\boldsymbol{R}}}\boldsymbol{Y}(n_2\Delta t) \leq \frac{1}{2\omega}\tilde{\hat{\boldsymbol{Y}}}_0\boldsymbol{S}\hat{\boldsymbol{Y}}_0 \tag{27}$$

Inequality (27) represents the searched bounding principle on the chosen measure of the plastic deformations produced during the transient phase: whatever the real load history during the transient phase, the measure of the *real* plastic deformation related to the actual elastic plastic response, on the left hand side in inequality (27), results not greater than the bounding quantity, on the right hand side of the same inequality, related to the *fictitious* elastic plastic process.

It is worth noticing that inequality (27) holds in both cases of elastic and plastic shakedown; actually, in the first case the previous proof can be easily effected assuming $\dot{\boldsymbol{Y}}_c(\tau) = \boldsymbol{0}$, $\forall\,\tau \in (0, \Delta t)$ in order to obtain the same relation (27).

Appropriate choices of the perturbation vector $\hat{\boldsymbol{R}}$ provide bounds on any measure of the plastic deformations related to the structure transient response as function of $\hat{\boldsymbol{Y}}_0$. It is worth noticing that the choice of the time $t_2 = n_2\Delta t$ ensures the measure of quantities related to the transient phase; actually, the structure behaves in condition of plastic shakedown and, as a consequence, the plastic strain increment in the cycle is nought at every $n\Delta t \ (n \geq n_2)$.

5. Application

As an application the steel frame plotted in Fig. 6 has been studied, where its geometry and load condition is represented.

The frame is subjected to a fixed uniformly distributed vertical load $q_0 = 35$ kN/m acting on the beams and to perfect cyclic horizontal loads applied at each floor, represented in Fig. 6, and considered in the computational stage, as concentrated nodal forces $F_j (j = 1, 2, 3, 4)$, whose intensities, deduced according to the Italian technical rules, are given as follows: $F_1 = \pm 4.74$ kN, $F_2 = \pm 9.74$ kN, $F_3 = \pm 14.6$ kN, $F_4 = \pm 19.46$ kN.

The structure is discretized into beam-type finite elements, constituted by elastic perfectly plastic

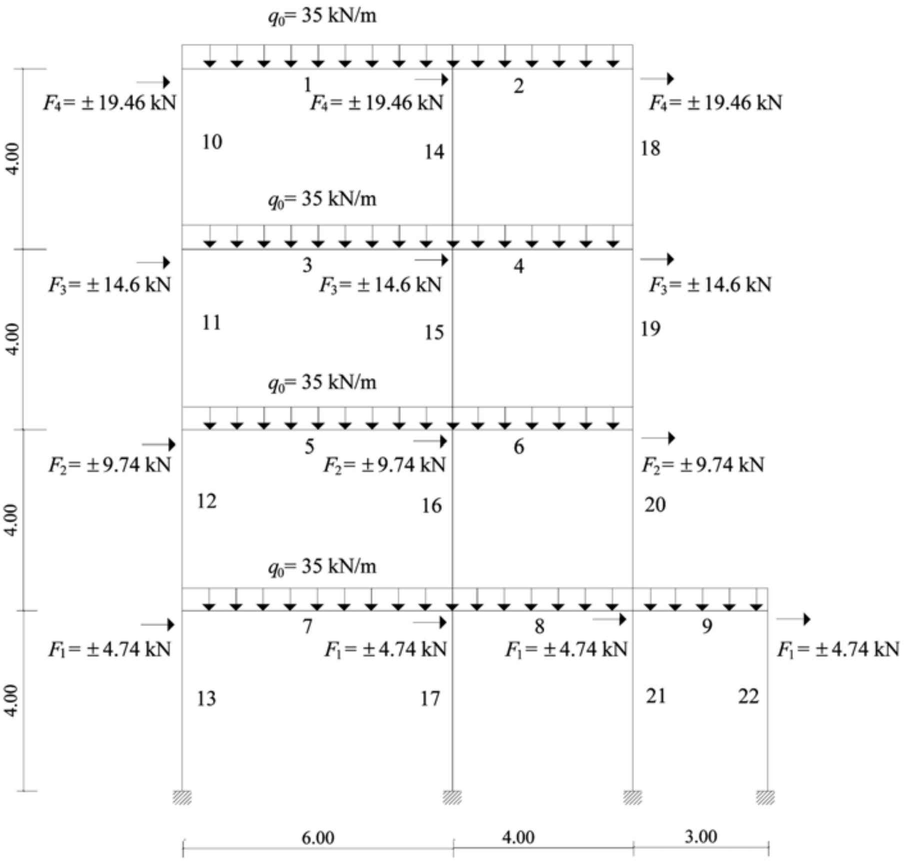

Fig. 6 Steel frame: geometry and load conditions

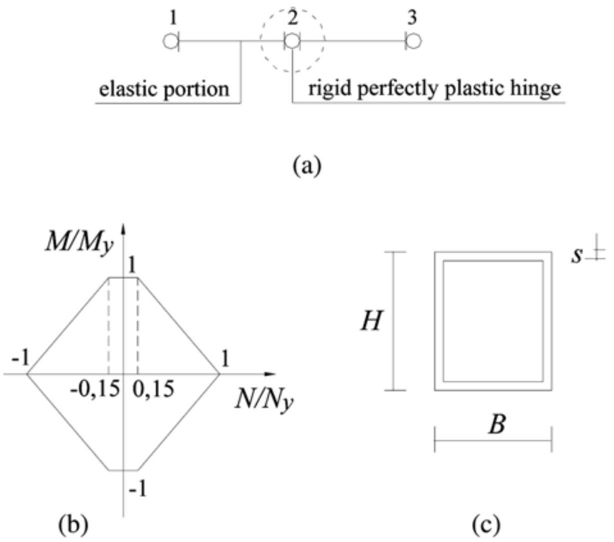

Fig. 7 Typical structural element: (a) elastic portions and rigid perfectly plastic hinge location, (b) dimensionless rigid perfectly plastic hinge domain, (c) rectangular box cross-section

material. In particular, two rigid perfectly plastic hinges are located at the extremes of all elements, which are considered as purely elastic, while an additional rigid perfectly plastic hinge is located in the middle point of each beam (Fig. 7a). The material is characterized by the following mechanical properties: Young modulus $E = 20600$ kN/cm^2, yield stress $\sigma_y = 23.5$ kN/cm^2.

The interaction between the bending moment M and the axial force N has been taken into account and in Fig. 7(b) the relevant dimensionless rigid plastic domain of the typical rigid perfectly plastic hinge is plotted in the plane (N/N_y, M/M_y), being N_y and M_y the yield axial force and bending moment, respectively.

The elements of the frame are characterized by a rectangular box cross-section as represented in Fig. 7(c). All the frame elements (beams and columns) have the same width, $B = 200$ mm, and the same height, $H = 400$ mm. On the contrary, the constant thickness of each element is provided by the solution of an appropriate (minimum volume) plastic shakedown design problem previously effected and reported in (Benfratello *et al*. 2006).

Table 1 Limit load multiplier values and optimal element thicknesses (mm)

$\bar{\xi}_0$	$\bar{\xi}_c$	s_1	s_2	s_3	s_4
1	4	5.66	4.00	7.39	10.78
s_5	s_6	s_7	s_8	s_9	s_{10}
9.37	19.21	8.22	14.50	14.02	4.00
s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}
5.59	6.15	6.68	6.65	12.66	14.29
s_{17}	s_{18}	s_{19}	s_{20}	s_{21}	s_{22}
14.96	4.00	7.16	13.79	14.72	14.02

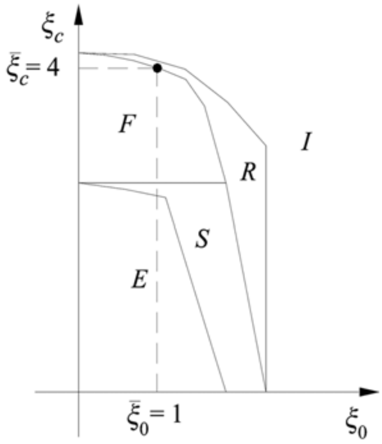

Fig. 8 Bree diagram of the minimum volume plastic shakedown design

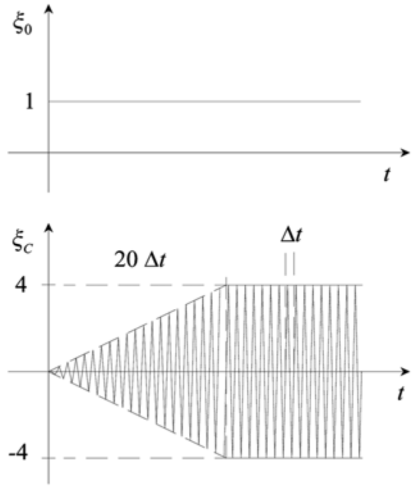

Fig. 9 Load multiplier history

In particular, the referenced optimal design of the structure plotted in Fig. 6 has been determined assuming $\bar{\xi}_0 = 1$ and $\bar{\xi}_c = 4$ as fixed load multiplier and cyclic load multiplier values, respectively. The obtained optimal thicknesses, s_n, are reported in Table 1, together with the limit load multiplier values.

First of all, the Bree diagram of the optimal structure has been determined and plotted in Fig. 8. It is worth noticing that the couple of values $(\bar{\xi}_0 = 1, \bar{\xi}_c = 4)$ individuates a limit plastic shakedown condition as required by the imposed design constraints. Furthemore, in order to investigate about the complete elastic plastic response, a stepwise elastic plastic analysis of the structure has been worked out, assuming that the steady-state referred limit values of the load multipliers are reached following the history represented in Fig. 9. It is worth noticing that the path plotted in Fig. 9 represents just one among the infinite admissible paths through which the steady-state condition is reached. Finally, in order to determine a check parameter related to the ductility behaviour of the structure in such a limit load condition, the residual horizontal displacement of the fourth floor, related to the transient phase, has been computed: $u_4^{res} = 53.2$ mm. Such a parameter (plastic displacement) has been chosen as suitable measure of the plastic deformations occurring during the transient phase to be bounded utilizing the proposed principle.

Assuming $\xi_c = \bar{\xi}_c = 4$ and, therefore, moving within the plastic shakedown domain, four different values have been alternatively assigned to the chosen bound, and in particular: b = 20 mm; 50 mm; 100 mm; 150 mm, obtaining four different values of the fixed load multiplier related to the relevant perturbed plastic shakedown limit condition, i.e., $\hat{\xi}_0$ = 0.349; 0.599; 0.647; 0.723, respectively, solving the following problem:

$$\boldsymbol{K}\boldsymbol{u}_{ci} - \boldsymbol{F}_{ci} = \boldsymbol{0} \qquad \forall i \in I(b) \qquad (28a)$$

$$\boldsymbol{P}_{ci} = \tilde{\boldsymbol{B}}\boldsymbol{u}_{ci} + \boldsymbol{P}^*_{ci} \qquad \forall i \in I(b) \qquad (28b)$$

$$-\boldsymbol{\varphi}_{ci} = \boldsymbol{R} - \bar{\xi}_c\tilde{\boldsymbol{N}}\boldsymbol{P}_{ci} + \boldsymbol{S}\boldsymbol{Y}_{ci} \qquad \forall i \in I(b) \qquad (28c)$$

$$-\boldsymbol{\varphi}_{ci} \geq \mathbf{0}, \quad \boldsymbol{Y}_{ci} \geq \mathbf{0}, \quad \tilde{\boldsymbol{Y}}_{ci}\boldsymbol{\varphi}_{ci} = 0 \qquad \forall i \in I(b) \tag{28d}$$

$$\boldsymbol{K}\boldsymbol{u}_0 - \boldsymbol{F}_0 = \mathbf{0} \tag{29a}$$

$$\boldsymbol{P}_0 = \tilde{\boldsymbol{B}}\boldsymbol{u}_0 + \boldsymbol{P}_0^* \tag{29b}$$

$$\hat{\xi}_0(\overline{\xi}_c) = \max_{(\xi_0,\, \hat{\boldsymbol{Y}}_0)} \xi_0 \qquad \text{subject to} \tag{29c}$$

$$-\hat{\boldsymbol{\varphi}}_i^S \equiv \boldsymbol{R} - \omega\hat{\boldsymbol{R}} - \overline{\xi}_c\tilde{\boldsymbol{N}}\boldsymbol{P}_{ci} - \xi_0\tilde{\boldsymbol{N}}\boldsymbol{P}_0 + \boldsymbol{S}\boldsymbol{Y}_{ci} + \boldsymbol{S}\hat{\boldsymbol{Y}}_0 \geq \mathbf{0} \qquad \forall i \in I(b) \tag{29d}$$

$$\hat{\boldsymbol{Y}}_0 \geq \mathbf{0} \tag{29e}$$

$$\tilde{\hat{\boldsymbol{Y}}}_0\boldsymbol{S}\hat{\boldsymbol{Y}}_0 \leq 2\,\omega b \tag{29f}$$

It is worth noticing that $\omega = 5$ has been chosen in all the effected computations, resulting such a value the optimal one (in order to maximize $\hat{\xi}_0$) among a discrete set of prefixed suitably chosen values.

The results obtained by the analyses related to the four different new couples of load multiplier values are summarized in Table 2, always in terms of residual horizontal displacement of the fourth floor occurring during the transient phase.

Table 2 Imposed bound values (mm) and obtained load multipliers and residual displacements (mm)

b	---	150	100	50	20
ξ_0	1	0.723	0.647	0.599	0.349
ξ_c	4	4	4	4	4
u_4^{res}	53.20	27.05	18.95	8.06	7.74

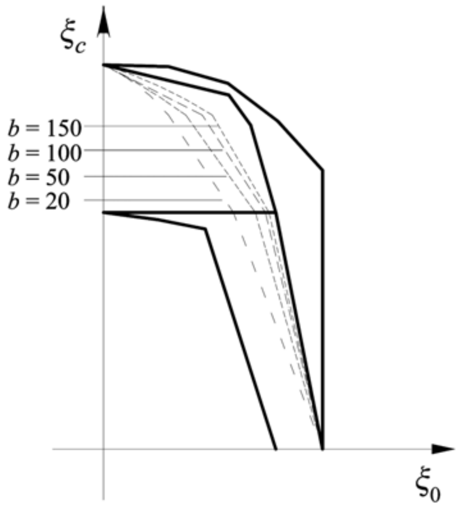

Fig. 10 Bree diagrams obtained taking into account the four assigned bound values (mm)

In order to provide more complete information about the elastic/plastic behaviour of the structure in the above referred perturbed conditions, in Fig. 10 the borderlines related to the elastic/plastic shakedown limit with constraints on plastic deformation, obtained taking into account the four previously assigned bound values, are plotted.

6. Conclusions

In the present paper a special formulation of a bounding principle devoted to compute bounds on suitable measures of the plastic deformations which characterize the transient phase response of elastic perfectly plastic structures subjected to a combination of fixed and perfect cyclic loads in condition of plastic shakedown, has been proposed.

At first, the utilized structural model and the related elastic plastic behaviour have been described in an appropriate form, as well as the problem to be solved in order to determine the borderline between elastic/plastic shakedown domains and incremental/instantaneous collapse regions on the Bree diagram has been suitably formulated. In particular, the latter maximum problem has been solved by determining, separately, the elastic plastic response of the relevant structure subjected just to the amplified perfect cyclic loads and, successively, the purely elastic response of the same structure to the amplified fixed loads.

Furthermore, the relations describing and governing the elastic plastic steady-state response of the structure in condition of plastic shakedown have been reported, assuming suitably chosen special time functions for the load multipliers. Due to the unknowledge of the load multiplier history through which the steady-state condition is reached, such a response can not provide any useful information regarding the transient phase response of the structure and, as a consequence, in order to have although rough evaluation of this response, it has been necessary to make reference to the bounding theory.

So, a suitable bounding theorem has been proposed and proved; it allows to compute bounds on any chosen measure of the plastic deformations related to the transient phase of an elastic perfectly plastic structure subjected to the described load conditions and exhibiting a steady-state plastic shakedown behaviour. It represents a generalization of analogous bounding theorems related to the elastic shakedown theory and based on a perturbation method.

Finally, in the framework of the numerical applications, reference has been made to steel structures. A four floor frame, with element constituted by rectangular box cross-section and subjected to a combination of fixed and perfect cyclic loads amplified by assigned values of the load multipliers, has been studied in order to compute bounds on a suitable measure of the plastic deformations which characterize the transient phase response of the structure. In particular, the chosen quantity to be bounded has been the maximum horizontal residual displacement of the fourth floor. The effected numerical application confirmed the theoretical expectations.

References

Benfratello, S., Cirone, L. and Giambanco, F. (2006), “A multicriterion design of steel frames with shakedown constraints”, *Comput. Struct*., **84**, 269-282.

Capurso, M., Corradi, L. and Maier, G. (1979), “Bounds on deformations and displacements in shakedown theory”, *Matériaux et structures sous chargement cyclique, Ass. amicale des ingénieurs anciens eléves de l’E.N.P.C.*, Paris, 231-244.

Corradi, L. (1983), “A displacement formulation for the finite element elastic-plastic problem”, *Meccanica*, **18**, 77-91.

Cottle, R.W., Spang, J.S. and Stone, R.E. (1992), *The Linear Complementarity Problem*, Academic Press.

Giambanco, F. and Palizzolo, L. (1996), “Computation of bounds on chosen measures of real plastic deformation for beams”, *Comput. Struct*., **61**(1), 171-182.

Giambanco, F. and Palizzolo, L. (1997), “Optimal bounds on plastic deformations for bodies constituted by temperature-dependent elastic hardening material”, *J. Appl. Mech.*, ASME, **64**(3), 510-518.

Giambanco, F., Palizzolo, L. and Caffarelli, A. (2002), “Optimal design of elastic plastic structures: Elastic and plastic shakedown formulations”, *Proc. of the “Intensive Workshop” – Optimal Design of Materials and Structures*, Paris, France.

Giambanco, F., Palizzolo, L. and Caffarelli, A. (2004), “Computational procedures for plastic shakedown design of structures”, *J. of Structural and Multidisciplinary Optimization*, **28**, 317-329.

Koiter, W.T. (1960), “General theorem for elastic-plastic solids”, In *Progress in Solid Mechanics*, Vol. I, Ed. I.N. Sneddon and R. Hill, North Holland, 165-219.

König, J.A. (1979), “On upper bounds to shakedown loads”, *Z. Angew. Math*., **59**, 349-354.

König, J.A. (1987), *Shakedown of Elastic Plastic Structures*, PWN-Polish Scientific Publishers, Warsaw and Elsevier, Amsterdam.

Maier, G. (1968), “A quadratic programming approach for certain classes of nonlinear structural problems”, *Meccanica*, **3**, 121-130.

Melan, E. (1938a), “Der Spannungszustand eines Hencky-Mises’shen continuum bei verändilicher Belastung”, *Stz. Ber. AK. Wiss Wien*, **147**, 73.

Melan, E. (1938b), “Zur Plastzität del räumlichen kontinuum”, *Ing. Arch*., **9**, 116.

Polizzotto, C. (1978), “A unified approach to quasi-static shakedown problem for elastic-plastic solids with a piecewise linear yield surface”, *Meccanica*, **13**, 109-120.

Polizzotto, C. (1982), “A unified treatment of shakedown theory and related bounding techniques”, *S. M. Archives*, **7**(1), 19-75.

Polizzotto, C., Borino, G. and Fuschi, P. (1990), “On the steady-state response of elastic perfectly plastic solids to cyclic loads”, In M. Kleiber, J.A. König (ed.), *Inelastic Solids and Structures*, Swansea (U.K.): Pineridge Press, 473-488.

Polizzotto, C. (1994a), “On elastic plastic structures under cyclic loads”, *Eur. J. Mech. A/Solids*, **13/4**, 149-173.

Polizzotto, C. (1994b), “Steady states and sensitivity analysis in elastic-plastic structures subjected to cyclic loads”, *Int. J. Solids Struct*., **31**, 953-970.

Polizzotto, C., Borino, G. and Fuschi, P. (2001), “Assessment of the elastic/plastic shakedown load boundary for structures subjected to cyclic loading”, *Proc. of XV Congresso Nazionale AIMETA*, Taormina, Italy.

Ponter, A.R.S. and Haofeng, C. (2001), “A minimum theorem for cyclic load in excess of shakedown, with application to the evaluation of a ratchet limit”, *Eur. J. Mech. A/Solids*, **20**, 539-553.

Zarka, J. and Casier, J. (1979), “Elastic-plastic response of structure to cyclic loading: Practical rules”, In S. Nemat Nasser (ed.), *Mechanics Today*, Oxford: Pergamon Press, 93-198.

Zarka, J., Frelat, J., Inglebert, G. and Kasnat-Navidi, P. (1990), “A new approach in inelastic analysis of structure”, *Ecole Polytechnique Palaisau*, France.