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A dynamical stochastic finite element method based on the moment equation approach for the analysis of linear and nonlinear uncertain structures

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Abstract. A method for the dynamical analysis of FE discretized uncertain linear and nonlinear structures is presented. This method is based on the moment equation approach, for which the differential equations governing the response first and second-order statistical moments must be solved. It is shown that they require the cross-moments between the response and the random variables characterizing the structural uncertainties, whose governing equations determine an infinite hierarchy. As a consequence, a closure scheme must be applied even if the structure is linear. In this sense the proposed approach is approximated even for the linear system. For nonlinear systems the closure schemes are also necessary in order to treat the nonlinearities. The complete set of equations obtained by this procedure is shown to be linear if the structure is linear. The application of this procedure to some simple examples has shown its high level of accuracy, if compared with other classical approaches, such as the perturbation method, even for low levels of closures.

Keywords: uncertain structures; linear and nonlinear structures; moment equation approach; dynamic analysis.

1. Introduction

The study of uncertain structures has become more and more important in these last years. The uncertain structures are characterized by the fact that one or more of their mechanical and/or geometrical properties cannot be defined deterministically. The importance of this kind of study is above all related to some structural problems, as the structural reliability, for which neglecting the effective uncertain nature of the structural parameters is not possible. It is obvious that for these systems the traditional deterministic analyses cannot be applied, but alternative approaches have to be taken into account. If the uncertainty of the system parameters is due to imprecise information and the statistical data cannot be obtained, only the theory of fuzzy sets can be considered and the fuzzy finite element method must be used (Zadeh 1978, Rao and Savier 1995, De Lima and Ebecken 2000, Akpan *et al.* 2001). On the contrary, if it is possible to characterize the uncertain parameters stochastically, then the probabilistic approaches can be used. In this paper only this last case will be taken into consideration.

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Among the probabilistic approaches, the statistical ones, based on the Monte Carlo simulations, are the simplest from a theoretical point of view. In fact, they need the realizations of a sufficiently high number of samples of the uncertain parameters and the solution of the corresponding deterministic problems (Papadrakakis and Kotsopoulos 1999). However, increasing the structural degrees of freedom and the number of uncertain parameters, the computational effort attained by the statistical methods becomes very high, above all for nonlinear structures. For this reason, some alternative non-statistical methods have been proposed in the literature (Liu *et al.* 1987, Ghanem and Spanos 1991, Matthies *et al.* 1997, Sudret and DerKiureghian 2000, Schueller 2001, Noh 2004, Stefanou and Papadrakakis 2004). In particular, the perturbation approaches have had the greatest diffusion (Nakagiri and Hisada 1982, Elishakoff *et al.* 1995, Impollonia and Muscolino 2002, Van den Nieuwenhof and Coyette 2003). As a consequence, the stochastic finite element (SFE) approach is usually identified with the classical FE approaches coupled with the perturbation techniques. This happens in both the static and dynamic field. The fundamental drawback related to the use of the perturbation approaches lies on the consistent loss of accuracy when the level of uncertainty and of nonlinearity of the structural parameters increases. Consequently the results obtained by these approaches are acceptable only for very low level of uncertainty and nonlinearity.

Other non-statistical approaches, related to the case of linear structures, are based on the expansion methods of the structural stiffness matrix in order to perform explicitly its inversion. Some authors have used the Neumann expansion, both in the static and in the dynamic linear case (Yamazaki *et al.* 1988, Spanos and Ghanem 1989, Chakraborty and Dey 1998). A drawback common to all these approaches is that a sufficient accuracy is reached only for low levels of uncertainty and that they can be difficultly extended to the nonlinear structures. At last, in some works, the chaos expansion is used (Ghanem and Spanos 1990).

In this paper an approach is developed based on the consideration that, even if the structural system is linear, the relationship between the structural response and the random variables characterizing the uncertain parameters is nonlinear. As a consequence, a typical approach for nonlinear system, such as the moment equation approach (Lin 1967) coupled with a closure scheme (for example on the cumulants (Wu and Lin 1984)), can be advantageously applied. This approach shows the great advantage of being characterized by an accuracy level that can be improved by increasing the closure order, starting from the second one (Gaussian closure). Moreover the extension to the nonlinear structural systems is quite straightforward.

The applications of the proposed approach to some structural uncertain linear and nonlinear systems have relieved a better accuracy performance than the perturbation approaches, even for low closure orders (fourth at maximum).

2. Preliminary concepts

The analysis of an uncertain structure usually requires its FE discretization and, if the uncertain structural parameters are modeled as continuous random fields, these have to be approximated by discrete random variables. In the literature many methods allow the discretization of the random fields in random variables (Li and DerKiureghian 1993). The approach used here needs the use of any of the point discretization methods, as, for example, the mid point method (DerKiureghian and Ke 1988) or the spatial averaging method (Vanmarcke and Grigoriu 1983) or the shape function method (Liu *et al.* 1986). By using one of these approaches, each uncertain parameter is assumed to be constant in each FE. Hence, one or more random variables define the structural uncertainty in each FE.

To simplify we make reference to those structures in which the uncertainties are only in the stiffness matrix. Moreover the dynamical external excitations are assumed to be deterministic. However, some more complicated cases can also be considered, such as the uncertain damping or mass matrices or the stochastic loads.

As a consequence of the above cited assumptions, the n ($n =$ structural DOFs) differential equations governing the structural motion of a linear structure can be written as:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}(\boldsymbol{\alpha})\mathbf{u}(t) = \mathbf{f}(t) \tag{1}$$

where $\boldsymbol{\alpha} = [\alpha_1\alpha_2\ldots\alpha_N]^T$ is the vector collecting the random variables characterizing the uncertainties in the various FEs; $\mathbf{u}(t)$ is the nodal displacement vector; $\mathbf{M}$ is the deterministic mass matrix; $\mathbf{C}$ is the deterministic damping matrix; $\mathbf{K}(\boldsymbol{\alpha})$ is the stochastic stiffness matrix; and $\mathbf{f}(t)$ is the deterministic nodal force vector.

By introducing the state variable vector $\mathbf{x}^T(t) = (\mathbf{u}^T(t) \;\; \dot{\mathbf{u}}^T(t))$, Eq. (1) can be rewritten in the following first order form:

$$\dot{\mathbf{x}}(t) = \mathbf{D}(\boldsymbol{\alpha})\mathbf{x}(t) + \mathbf{V}\mathbf{f}(t) \tag{2}$$

where:

$$\mathbf{D}(\boldsymbol{\alpha}) = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ -\mathbf{M}^{-1}\mathbf{K}(\boldsymbol{\alpha}) & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix}; \quad \mathbf{V} = \begin{pmatrix} \mathbf{0}_{n\times n} \\ \mathbf{M}^{-1} \end{pmatrix} \tag{3}$$

$\mathbf{I}_{n\times n}$ and $\mathbf{0}_{n\times n}$ being the identity and zero matrices, respectively, of order n.

If the structural system is characterized by nonlinearities of geometrical and/or mechanical types, the equation of motion (1) can be rewritten in the following form:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}_{NL}(\mathbf{u}, \dot{\mathbf{u}}, \boldsymbol{\alpha})\mathbf{u}(t) = \mathbf{f}(t) \tag{4}$$

where now the stiffness matrix $\mathbf{K}_{NL}$ depends on the response displacements and velocities. By introducing the state variable vector $\mathbf{x}(t)$, Eq. (4) takes on the form:

$$\dot{\mathbf{x}}(t) = \mathbf{D}_{NL}(\mathbf{x}, \boldsymbol{\alpha})\mathbf{x}(t) + \mathbf{V}\mathbf{f}(t) \tag{5}$$

where:

$$\mathbf{D}_{NL}(\mathbf{x}, \boldsymbol{\alpha}) = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ -\mathbf{M}^{-1}\mathbf{K}_{NL}(\mathbf{u}, \dot{\mathbf{u}}, \boldsymbol{\alpha}) & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix} \tag{6}$$

and $\mathbf{V}$ has the same expression as in Eq. (3).

3. Proposed approach: Linear structural system

As a first step, the following assumption about the dependence of the stiffness matrix on the uncertainties is made:

$$\mathbf{K}(\boldsymbol{\alpha}) = \mathbf{K}_0 + \sum_{i=1}^{N} \mathbf{K}_i \alpha_i; \quad \mathbf{K}_0 = \mathbf{K}(\mathbf{0}); \quad \mathbf{K}_i = \left. \frac{\partial \mathbf{K}(\boldsymbol{\alpha})}{\partial \alpha_i} \right|_{\boldsymbol{\alpha}=\mathbf{0}} \tag{7}$$

It is just the same assumption made on the stiffness matrix when the perturbation approaches are used. It is important to note that in many cases the dependence of the stiffness matrix on the uncertain parameters is really linear, so that, in these cases, the assumption given in Eq. (7) introduces no approximation in the results. In any case, it is always possible to perform a change of variables in order to make this dependence linear with respect to the new variables.

Taking into account this hypothesis, Eq. (2) can be rewritten as follows:

$$\dot{\mathbf{x}}(t) = \left(\mathbf{D}_0 + \sum_{i=1}^{N} \mathbf{D}_i \alpha_i \right) \mathbf{x}(t) + \mathbf{V}\mathbf{f}(t) \tag{8}$$

where:

$$\mathbf{D}_0 = \begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{I}_{n \times n} \\ -\mathbf{M}^{-1}\mathbf{K}_0 & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix}; \quad \mathbf{D}_i = \begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{0}_{n \times n} \\ -\mathbf{M}^{-1}\mathbf{K}_i & \mathbf{0}_{n \times n} \end{pmatrix} \tag{9}$$

The presence of the random variables α_i in Eq. (8) makes the response vector $\mathbf{x}(t)$ as a vector stochastic process. It can be characterized, from a probabilistic point of view, by the knowledge of the corresponding moments of various orders. Each of these moments is governed by a first order differential equation that can be easily found starting from the motion Eq. (8).

3.1 Mean response

The equation governing the mean of $\mathbf{x}(t)$ is obtained by applying the mean operator $E[\cdot]$ to both the members of Eq. (8), that is:

$$\dot{E}[\mathbf{x}(t)] = \mathbf{D}_0 E[\mathbf{x}(t)] + \sum_{i=1}^{N} \mathbf{D}_i E[\alpha_i \mathbf{x}(t)] + \mathbf{V}\mathbf{f}(t) \tag{10}$$

The solution of this equation requires the knowledge of the cross-moments $E[\alpha_i \mathbf{x}(t)]$ that are governed by first order differential equations whose expression can be easily found to be:

$$\dot{E}[\alpha_i \mathbf{x}(t)] = \mathbf{D}_0 E[\alpha_i \mathbf{x}(t)] + \sum_{j=1}^{N} \mathbf{D}_j E[\alpha_i \alpha_j \mathbf{x}(t)] + \mathbf{V}\mathbf{f}(t) E[\alpha_i] \tag{11}$$

The expressions of Eqs. (10) and (11) show that the exact evaluation of the response mean requires the solution of an infinite hierarchy of differential equations. However, an approximate solution can be obtained by applying a closure scheme (for example, the cumulant neglect closure (Wu and Lin 1984)) on the cross-moments between the response $\mathbf{x}(t)$ and the random variables α_i. If the second order cumulant neglect closure scheme is applied, the third order cross-moments appearing into

Eq. (11) can be obtained as function of the first and second order ones by imposing that the corresponding third order cumulants are zero. This implies that (Lin 1967, Ibrahim 1985):

$$E[\alpha_i\alpha_j x_m(t)] = \sum_3 E[\alpha_i]E[\alpha_j x_m(t)] - 2E[\alpha_i]E[\alpha_j]E[x_m(t)] \tag{12}$$

where $x_m(t)$ is the generic element of $\mathbf{x}(t)$ and the number under the summation sign refers to the number of possible terms generated in the form of the indicated expression without allowing permutation of terms, that is:

$$\sum_3 E[\alpha_i]E[\alpha_j x_m(t)] = E[\alpha_i]E[\alpha_j x_m(t)] + E[\alpha_j]E[\alpha_i x_m(t)] + E[x_m(t)]E[\alpha_i\alpha_j] \tag{13}$$

It is important to note that the moments $E[\alpha_i]$ and $E[\alpha_i\alpha_j]$ appearing in Eq. (13) are known quantities, the probability density function of α_i being given. This implies that the equations giving the mean response, that are Eqs. (10), (11) and (12), are linear. Thus, they can be easily solved by any numerical procedure for the solution of linear first order differential equations.

If a better accuracy is required, the fourth order cumulant neglect closure scheme can be applied. This requires that the equations governing the third and fourth order cross-moments have to be considered, besides of Eqs. (10) and (11). They have the following form:

$$\dot{E}[\alpha_i\alpha_j\mathbf{x}(t)] = \mathbf{D}_0 E[\alpha_i\alpha_j\mathbf{x}(t)] + \sum_{k=1}^{N}\mathbf{D}_k E[\alpha_i\alpha_j\alpha_k\mathbf{x}(t)] + \mathbf{Vf}(t)E[\alpha_i\alpha_j] \tag{14}$$

$$\dot{E}[\alpha_i\alpha_j\alpha_k\mathbf{x}(t)] = \mathbf{D}_0 E[\alpha_i\alpha_j\alpha_k\mathbf{x}(t)] + \sum_{l=1}^{N}\mathbf{D}_l E[\alpha_i\alpha_j\alpha_k\alpha_l\mathbf{x}(t)] + \mathbf{Vf}(t)E[\alpha_i\alpha_j\alpha_k] \tag{15}$$

The fifth cross-moments appearing in Eq. (15) can be obtained as functions of the first four ones by imposing that the corresponding fifth order cumulants are zero, that is (Lin 1967, Ibrahim 1985):

$$\begin{aligned}E[\alpha_i\alpha_j\alpha_k\alpha_l x_m(t)] = {} & \sum_5 E[\alpha_i]E[\alpha_j\alpha_k\alpha_l x_m(t)] - 2\sum_{10} E[\alpha_i]E[\alpha_j]E[\alpha_k\alpha_l x_m(t)] \\ & + 6\sum_{10} E[\alpha_i]E[\alpha_j]E[\alpha_k]E[\alpha_l x_m(t)] - 2\sum_{15} E[\alpha_i]E[\alpha_j\alpha_k]E[\alpha_l x_m(t)] \\ & + \sum_{10} E[\alpha_i\alpha_j]E[\alpha_k\alpha_l x_m(t)] - 24E[\alpha_i]E[\alpha_j]E[\alpha_k]E[\alpha_l]E[x_m(t)]\end{aligned} \tag{16}$$

It is important to note that in this expression the moments of the random variables α_i are known. As a consequence, in Eq. (16) the unknown quantities appear linearly. Hence, even in this case, the mean response can be obtained as the solution of a system of linear equations. It is clear that more accurate results can be obtained by adopting greater order closure schemes.

3.2 Second order response moments

The second order response moments are governed by a first order differential equation having the following form (Di Paola *et al.* 1992):

$$\dot{E}[\mathbf{x}(t) \otimes \mathbf{x}(t)] = (\mathbf{D}_0 \oplus \mathbf{D}_0)E[\mathbf{x}(t) \otimes \mathbf{x}(t)] + \sum_{i=1}^{N} (\mathbf{D}_i \oplus \mathbf{D}_i)E[\alpha_i \mathbf{x}(t) \otimes \mathbf{x}(t)] \\ + (\mathbf{V}\mathbf{f}(t) \oplus \mathbf{V}\mathbf{f}(t))E[\mathbf{x}(t)] \tag{17}$$

where the symbol $\otimes$ indicates the Kronecker product (Brewer 1978) that is defined as follows: if $\mathbf{A}_{m \times n}$ and $\mathbf{B}_{p \times q}$ are a $(m \times n)$ and a $(p \times q)$ matrix, respectively, then the Kronecker product $\mathbf{A}_{m \times n} \otimes \mathbf{B}_{p \times q}$ is a matrix $\mathbf{C}_{mp \times nq}$ of order $(mp \times nq)$ built by multiplying each element of $\mathbf{A}_{m \times n}$ by the entire matrix $\mathbf{B}_{p \times q}$, that is:

$$\mathbf{C}_{mp \times nq} = \begin{pmatrix} a_{11}\mathbf{B} & \dots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \dots & a_{mn}\mathbf{B} \end{pmatrix} \tag{18}$$

At last, in Eq. (17) the symbol $\oplus$ indicates the Kronecker sum defined as follows:

$$\begin{aligned} \mathbf{D}_r \oplus \mathbf{D}_r &= \mathbf{D}_r \otimes \mathbf{I}_{2n \times 2n} + \mathbf{I}_{2n \times 2n} \otimes \mathbf{D}_r; \quad r = 0, i; \\ \mathbf{V}\mathbf{f}(t) \oplus \mathbf{V}\mathbf{f}(t) &= \mathbf{V}\mathbf{f}(t) \otimes \mathbf{I}_{2n \times 2n} + \mathbf{I}_{2n \times 2n} \otimes \mathbf{V}\mathbf{f}(t) \end{aligned} \tag{19}$$

If the second order cumulant neglect closure is applied on the cross-moments between the response $\mathbf{x}(t)$ and the random variables α_i, then the quantities $E[\alpha_i x_m(t) x_p(t)]$, appearing in Eq. (17) as elements of the vector $E[\alpha_i \mathbf{x}(t) \otimes \mathbf{x}(t)]$, can be obtained by setting zero the corresponding third order cumulant. This leads to the following result:

$$E[\alpha_i x_m(t) x_p(t)] = \sum_3 E[\alpha_i]E[x_m(t)x_p(t)] - 2E[\alpha_i]E[x_m]E[x_p(t)] \tag{20}$$

It is easy to recognize that this equation requires the knowledge of quantities already evaluated in the subsection 3.1. Moreover it does not introduce any non-linearity in the evaluation of the second order moments of the response.

If one wants to improve the accuracy of the results by considering the fourth order cumulant neglect closure on the cross-moments, first the differential equations governing the cross-moments $E[\alpha_i \mathbf{x}(t) \otimes \mathbf{x}(t)]$ have to be written. It is easy to show that they have the following form:

$$\dot{E}[\alpha_i \mathbf{x}(t) \otimes \mathbf{x}(t)] = (\mathbf{D}_0 \oplus \mathbf{D}_0)E[\alpha_i \mathbf{x}(t) \otimes \mathbf{x}(t)] + \sum_{j=1}^{N} (\mathbf{D}_j \oplus \mathbf{D}_j)E[\alpha_i \alpha_j \mathbf{x}(t) \otimes \mathbf{x}(t)] \\ + (\mathbf{V}\mathbf{f}(t) \oplus \mathbf{V}\mathbf{f}(t))E[\alpha_i \mathbf{x}(t)] \tag{21}$$

The quantities $E[\alpha_i \alpha_j \mathbf{x}(t) \otimes \mathbf{x}(t)]$ appearing in this equation are governed by the following differential equation:

$$\dot{E}[\alpha_i \alpha_j \mathbf{x}(t) \otimes \mathbf{x}(t)] = (\mathbf{D}_0 \oplus \mathbf{D}_0)E[\alpha_i \alpha_j \mathbf{x}(t) \otimes \mathbf{x}(t)] + \sum_{k=1}^{N} (\mathbf{D}_k \oplus \mathbf{D}_k)E[\alpha_i \alpha_j \alpha_k \mathbf{x}(t) \otimes \mathbf{x}(t)] \\ + (\mathbf{V}\mathbf{f}(t) \oplus \mathbf{V}\mathbf{f}(t))E[\alpha_i \alpha_j \mathbf{x}(t)] \tag{22}$$

The fifth order cross-moments of the type $E[\alpha_i \alpha_j \alpha_k x_m(t) x_p(t)]$ appearing in this equation are evaluated by applying the closure scheme as follows (Lin 1967, Ibrahim 1985):

$$E[\alpha_i\alpha_j\alpha_k x_m(t)x_p(t)] = \sum_{5} E[\alpha_i]E[\alpha_j\alpha_k x_m(t)x_p(t)] - 2\sum_{10} E[\alpha_i]E[\alpha_j]E[\alpha_k x_m(t)x_p(t)]$$
$$+ 6\sum_{10} E[\alpha_i]E[\alpha_j]E[\alpha_k]E[x_m(t)x_p(t)] - 2\sum_{15} E[\alpha_i]E[\alpha_j\alpha_k]E[x_m(t)x_p(t)] \tag{23}$$
$$+ \sum_{10} E[\alpha_i\alpha_j]E[\alpha_k x_m(t)x_p(t)] - 24E[\alpha_i]E[\alpha_j]E[\alpha_k]E[x_m(t)]E[x_p(t)]$$

Even in this case, it is easy to show that Eq. (23) requires the knowledge of quantities already evaluated in the previous subsection and that it does not introduce any non-linearity in the set of governing equations. Thus, taking into account the results obtained for the evaluation of the mean response, the second order moments can be evaluated by solving Eqs. (17), (21), (22) and (23) by any numerical rule valuable for linear differential equations.

4. Proposed approach: Nonlinear structural system

For a nonlinear structural system let us consider the following assumption:

$$\mathbf{K}_{NL}(\mathbf{u}, \alpha) = \mathbf{K}_L(\alpha) + \hat{\mathbf{K}}_{NL}(\mathbf{u}, \alpha) = \mathbf{K}_0 + \sum_{i=1}^{N} \mathbf{K}_i\alpha_i + \hat{\mathbf{K}}_0(\mathbf{u}) + \sum_{i=1}^{N} \hat{\mathbf{K}}_i(\mathbf{u})\alpha_i \tag{24}$$

Taking into account this hypothesis, Eq. (5) can be rewritten as follows:

$$\dot{\mathbf{x}}(t) = \left(\mathbf{D}_0 + \hat{\mathbf{D}}_0(\mathbf{x}) + \sum_{i=1}^{N} (\mathbf{D}_i + \hat{\mathbf{D}}_i(\mathbf{x}))\alpha_i\right)\mathbf{x}(t) + \mathbf{V}\mathbf{f}(t) \tag{25}$$

where:

$$\mathbf{D}_0 = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ -\mathbf{M}^{-1}\mathbf{K}_0 & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix}; \quad \mathbf{D}_i = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ -\mathbf{M}^{-1}\mathbf{K}_i & \mathbf{0}_{n\times n} \end{pmatrix}$$
$$\hat{\mathbf{D}}_0(\mathbf{x}) = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ -\mathbf{M}^{-1}\hat{\mathbf{K}}_0(\mathbf{x}) & \mathbf{0}_{n\times n} \end{pmatrix}; \quad \mathbf{D}_i(\mathbf{x}) = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ -\mathbf{M}^{-1}\mathbf{K}_i(\mathbf{x}) & \mathbf{0}_{n\times n} \end{pmatrix} \tag{26}$$

The response vector $\mathbf{x}(t)$ is a vector stochastic process that can be characterized, from a probabilistic point of view, by the knowledge of the corresponding moments of various orders. Each of these moments is governed by a first order differential equation that can be found starting from the motion Eq. (25).

4.1 Mean response

The equation governing the mean of $\mathbf{x}(t)$ is obtained by applying the mean operator $E(\cdot)$ to both the members of Eq. (25), that is:

$$\dot{E}[\mathbf{x}(t)] = \mathbf{D}_0 E[\mathbf{x}(t)] + E[\hat{\mathbf{D}}_0(\mathbf{x})\mathbf{x}(t)] + \sum_{i=1}^{N} (\mathbf{D}_i E[\alpha_i \mathbf{x}(t)] + E[\alpha_i \hat{\mathbf{D}}_i(\mathbf{x})\mathbf{x}(t)]) + \mathbf{V}\mathbf{f}(t) \quad (27)$$

Besides of the moments $E[\hat{\mathbf{D}}_0(\mathbf{x})\mathbf{x}(t)]$ and $E[\alpha_i \hat{\mathbf{D}}_i(\mathbf{x})\mathbf{x}(t)]$, the solution of Eq. (27) requires the knowledge of the second order cross-moments $E[\alpha_i \mathbf{x}(t)]$ that are governed by first order differential equations whose expression can be easily found to be:

$$\dot{E}[\alpha_i \mathbf{x}(t)] = \mathbf{D}_0 E[\alpha_i \mathbf{x}(t)] + E[\alpha_i \hat{\mathbf{D}}_0(\mathbf{x})\mathbf{x}(t)] + \sum_{j=1}^{N} (\mathbf{D}_j E[\alpha_i \alpha_j \mathbf{x}(t)] + E[\alpha_i \alpha_j \hat{\mathbf{D}}_j(\mathbf{x})\mathbf{x}(t)])$$
$$+ \mathbf{V}\mathbf{f}(t) E[\alpha_i] \quad (28)$$

Eqs. (27) and (28) show that the exact evaluation of the response mean requires the solution of an infinite hierarchy of differential equations, besides of the evaluation of the moments including the nonlinear dynamical matrices $\hat{\mathbf{D}}_0(\mathbf{x})$ and $\hat{\mathbf{D}}_i(\mathbf{x})$. However, an approximate solution can be obtained by applying a closure scheme even for the nonlinear structural systems. The difference with respect to the linear case lies on the fact that here the closure schemes are necessary for both the evaluation of the above cited nonlinear moments and closing the equation hierarchy. The type of closure scheme to choose depends on the nonlinearity type of the dynamical matrices $\hat{\mathbf{D}}_0(\mathbf{x})$ and $\hat{\mathbf{D}}_i(\mathbf{x})$. For example, if they show polynomial type nonlinearity, a cumulant neglect closure can be suitable for the nonlinear case, too.

It is important to note that in this case the system of differential and algebraic equations resulting from the closure remains nonlinear, in contrast to the linear structural case in which it was linear.

It is obvious that the accuracy level of the approximate response depends on the closure order.

4.2 Second order response moments

The second order response moments are governed by a first order differential equation having the following form:

$$\dot{E}[\mathbf{x}(t) \otimes \mathbf{x}(t)] = (\mathbf{D}_0 \oplus \mathbf{D}_0) E[\mathbf{x}(t) \otimes \mathbf{x}(t)] + E[(\hat{\mathbf{D}}_0(\mathbf{x}) \oplus \hat{\mathbf{D}}_0(\mathbf{x}))(\mathbf{x}(t) \otimes \mathbf{x}(t))]$$
$$+ \sum_{i=1}^{N} ((\mathbf{D}_i \oplus \mathbf{D}_i) E[\alpha_i \mathbf{x}(t) \otimes \mathbf{x}(t)] + E[\alpha_i (\hat{\mathbf{D}}_i(\mathbf{x}) \oplus \hat{\mathbf{D}}_i(\mathbf{x}))(\mathbf{x}(t) \otimes \mathbf{x}(t))]) \quad (29)$$
$$+ (\mathbf{V}\mathbf{f}(t) \oplus \mathbf{V}\mathbf{f}(t)) E[\mathbf{x}(t)]$$

Besides of the moments depending on the nonlinear dynamical matrices, this equation requires the cross-moments $E[\alpha_i \mathbf{x}(t) \otimes \mathbf{x}(t)]$. They are governed by first order differential equations having the following form:

$$\dot{E}[\alpha_i \mathbf{x}(t) \otimes \mathbf{x}(t)] = (\mathbf{D}_0 \oplus \mathbf{D}_0) E[\alpha_i \mathbf{x}(t) \otimes \mathbf{x}(t)] + E[(\hat{\mathbf{D}}_0(\mathbf{x}) \oplus \hat{\mathbf{D}}_0(\mathbf{x}))(\alpha_i \mathbf{x}(t) \otimes \mathbf{x}(t))]$$
$$+ \sum_{j=1}^{N} ((\mathbf{D}_j \otimes \mathbf{D}_j) E[\alpha_i \alpha_j \mathbf{x}(t) \otimes \mathbf{x}(t)] + E[\alpha_i \alpha_j (\hat{\mathbf{D}}_j(\mathbf{x}) \oplus \hat{\mathbf{D}}_j(\mathbf{x}))(\mathbf{x}(t) \otimes \mathbf{x}(t))]) \quad (30)$$
$$+ (\mathbf{V}\mathbf{f}(t) \oplus \mathbf{V}\mathbf{f}(t)) E[\alpha_i \mathbf{x}(t)]$$

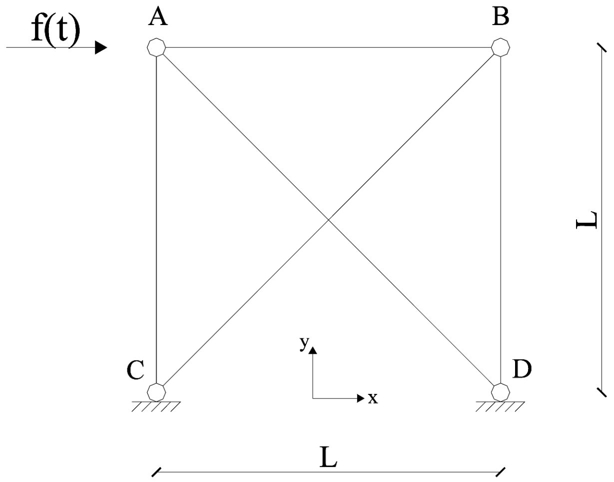

Fig. 1 Truss under examination

Even in this case it is necessary to apply a closure scheme both for closing the equation hierarchy and for evaluating the nonlinear moments.

5. Numerical applications

5.1 Application 1

As an example, the truss represented in Fig. 1 is taken into account. It is characterized by the following deterministic geometrical and physical parameters: $L = 10$ m; the cross element area is $A = 0.04$ m^2; the mass is considered as lumped in the nodes A and B and its value is $m = 10000$ kg; the damping factor is $\xi = 0.05$. The only uncertain parameter is chosen to be the Young modulus in each bar. It is given by the following relationship:

$$E(\alpha_i) = \bar{E}(1 + \alpha_i); \quad i = 1, 2, \ldots, 5 \tag{31}$$

where $\bar{E} = 210 \cdot 10^9$ N/m^2 is the mean value of the Young modulus that is assumed equal for all the bars; α_i are independent random variables having a probability density function uniformly distributed in the range [−0.2;0.2].

This truss can be considered as a bar type FE uncertain structure. As a consequence, the treatment shown in the section 3 can be applied for finding the means and the second order moments of the response. In particular, if the second order cumulant neglect closure is applied for a first order approximate evaluation of the response statistics, then Eqs. (10), (11) and (17) have to be considered. These equations must be coupled with the closure relationships expressed in Eqs. (13) and (20), that, taking into account the zero mean of the random variables α_i, become:

$$\begin{aligned} E[\alpha_i \alpha_j x_m(t)] &= E[\alpha_i \alpha_j]E[x_m(t)]; \\ E[\alpha_i x_m(t) x_p(t)] &= E[x_m(t)]E[\alpha_i x_p(t)] + E[x_p(t)]E[\alpha_i x_m(t)] \end{aligned} \tag{32}$$

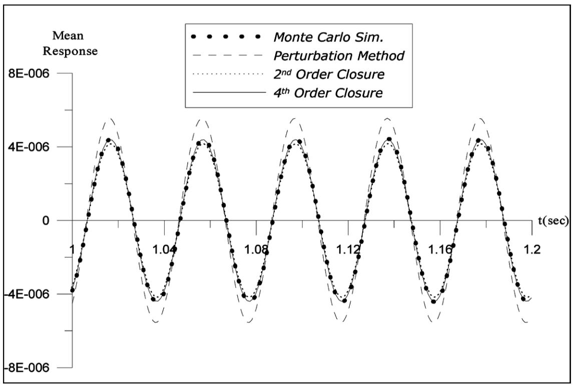

Fig. 2 Mean response of the horizontal displacement of the node A

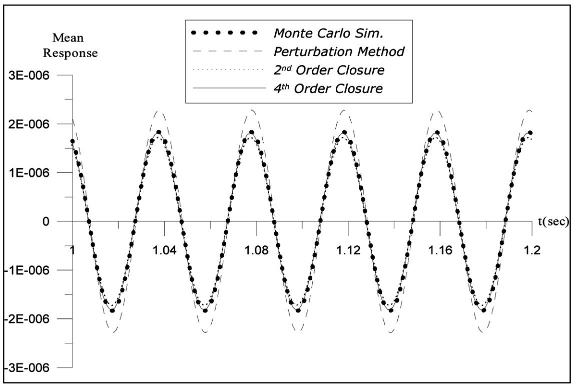

Fig. 3 Mean response of the vertical displacement of the node B

More accurate results are obtained if the fourth order cumulant neglect closure is adopted. With this aim, Eqs. (10), (11), (14) and (15) have to be considered for the response mean evaluation. Eqs. (17), (21) and (22) must be considered, together with the previous ones, for the evaluation of the response second order moments. All these equations must be coupled with the closure relationships expressed into Eqs. (16) and (23), that, in this case of zero-mean random variables α_i, become:

$$\begin{aligned} E[\alpha_i\alpha_j\alpha_k\alpha_l x_m(t)] &= E[x_m(t)]E[\alpha_i\alpha_j\alpha_k\alpha_l] - 2E[x_m(t)]\sum_3 E[\alpha_i\alpha_j]E[\alpha_k\alpha_l] \\ &\quad + \sum_{10} E[\alpha_i\alpha_j]E[\alpha_k\alpha_l x_m(t)]; \\ E[\alpha_i\alpha_j\alpha_k x_m(t)x_p(t)] &= E[x_m(t)]E[\alpha_i\alpha_j\alpha_k x_p(t)] + E[x_p(t)]E[\alpha_i\alpha_j\alpha_k x_m(t)] \\ &\quad - 2E[x_m(t)]E[x_p(t)]E[\alpha_i\alpha_j\alpha_k] - 2E[x_m(t)]\sum_3 E[\alpha_i\alpha_j]E[\alpha_k x_p(t)] \\ &\quad - 2E[x_p(t)]\sum_3 E[\alpha_i\alpha_j]E[\alpha_k x_m(t)] + \sum_3 E[\alpha_i\alpha_j]E[\alpha_k x_m(t)x_p(t)] \end{aligned} \tag{33}$$

In Figs. 2-3 the stationary mean response displacement corresponding to the nodes A (horizontal displacement) and B (vertical displacement) are reported, respectively, under the condition that the excitation is a horizontal force acting on the node A and having a sinusoidal form of frequency $\omega_f = 156$ rad/sec. This frequency corresponds to the first mode frequency of the deterministic truss characterized by a stiffness matrix equal to $\mathbf{K}_0$, for which, in other works (Falsone and Ferro 2004), the not high accuracy of many approximated approaches has been evidenced. In these figures the results obtained by the proposed approach are compared with those obtained by a first order classical perturbation and with those by a Monte Carlo simulation implemented with 20000 samples. In Figs. 4-5 the second order moments of the same response displacements are reported. From the analysis of these figures, the optimum level of accuracy of the proposed approach is evident.

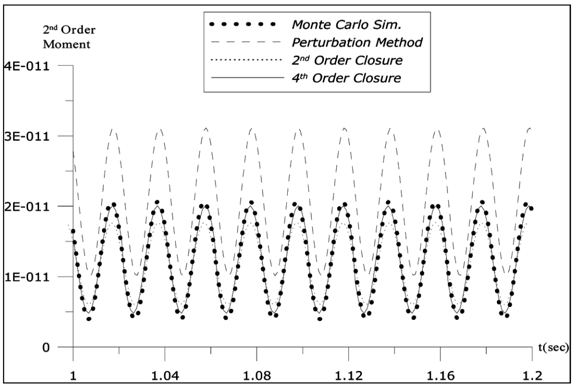

Fig. 4 Second order moment of the horizontal displacement of the node A

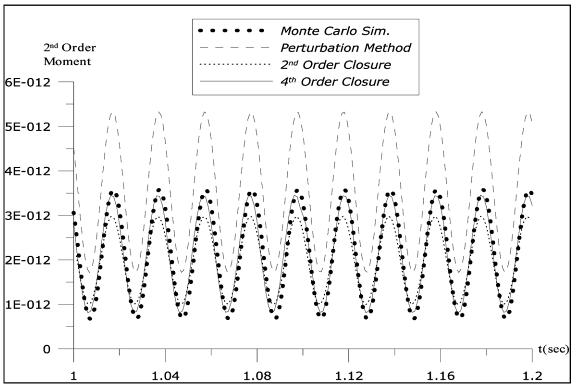

Fig. 5 Second order moment of the vertical displacement of the node B

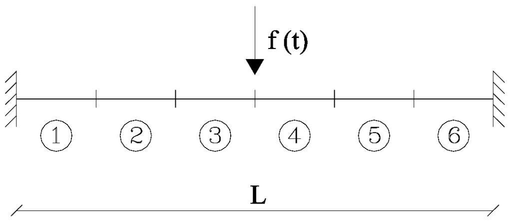

Fig. 6 Beam under examination

5.2 Application 2

As second example, the beam represented in Fig. 6 is taken into account. It is characterized by the following geometrical and physical parameters: $L = 6$ m; the cross section inertia moment is $I = 1.33 \times 10^{-4}$ m^4, the mass density factor is $\mu = 7850$ kg/m^3; the damping factor is $\xi = 0.05$. The beam is discretized into $N = 6$ FEs with length equal to $l = 1$ m.

An uncertain Young modulus is considered for each finite element and it is defined by the following relationship:

$$E(\alpha_i) = \bar{E}(1 + \alpha_i); \quad i = 1, 2, \ldots, 6 \tag{34}$$

where $\bar{E} = 210 \cdot 10^9$ N/m^2. The random variables α_i are assumed to be zero-mean gaussian correlated variables characterized by the following exponential correlation function:

$$\rho(\Delta x) = \exp\left(-\frac{|\Delta x|}{\lambda}\right) \tag{35}$$

with correlation length $\lambda = 0.4L$ and standard deviation $\sigma = 0.2$.

Using the same approach of the first example, it is possible to evaluate the mean value and the second order moment of the vertical displacement of the forced node under the action of a sinusoidal force.

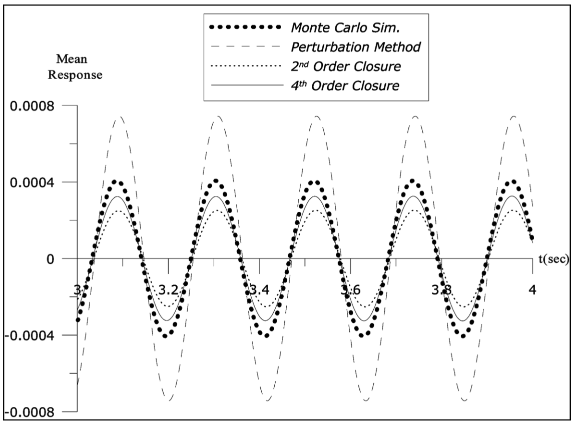

Fig. 7 Mean response of the middle node vertical displacement

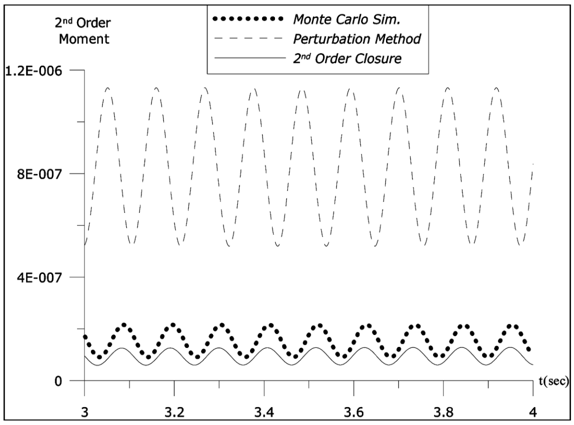

Fig. 8 Second order moment of the middle node vertical displacement

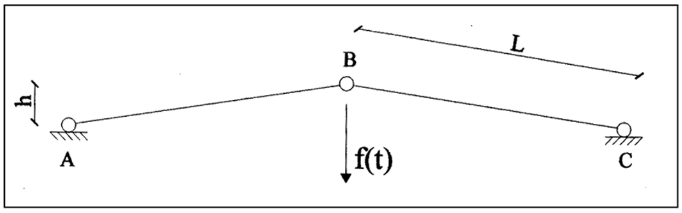

Fig. 9 Truss under examination

In Figs. 7-8 the stationary mean value and the second order moment of the vertical displacement of the forced node are reported. Even for this example, the results are compared with those obtained by a first order classical perturbation and with those obtained by a Monte Carlo simulation implemented with 20000 samples.

5.3 Application 3

In the third example the truss represented in Fig. 9 is analyzed. It is endowed with geometrical non-linearity and affected by an uncertain Young modulus. The system is characterized by the following geometrical and physical parameters: $L = 10$ m; the cross section area is $A = 0.04$ m^2; the mass density factor is μ =7850 kg/m^3; the damping factor is $\xi = 0.05$; the ratio $h/L = 0.15$.

The structure is composed by $N = 2$ bar type FEs with an uncertain Young modulus expressed as:

$$E(\alpha_i) = \overline{E}(1 + \alpha_i); \quad i = 1, 2, \dots, 5 \tag{36}$$

where $\overline{E} = 210 \cdot 10^9$ N/m^2 is the mean value of the Young modulus assumed equal for both the bars. α_i are independent random variables having a probability density function uniformly distributed in the range [–0.4;0.4]. The structure is subjected to a vertical constant force.

Because of the geometrical non-linearity the proposed approach must be applied using the relations expressed in section 4. It is important to note that, for the present example, Eq. (24) can be simplified as follows:

$$\mathbf{K}_{NL}(\mathbf{u}, \boldsymbol{\alpha}) = \mathbf{K}_L(\boldsymbol{\alpha}) + \hat{\mathbf{K}}_{NL}(\mathbf{u}, \boldsymbol{\alpha}) = \mathbf{K}_0 + \sum_{i=1}^{N} \mathbf{K}_i \alpha_i + \hat{\mathbf{K}}_0(\mathbf{u}) + \sum_{i=1}^{N} \hat{\mathbf{K}}_i(\mathbf{u})\alpha_i \tag{37}$$

where:

$$\mathbf{K}_0 = \frac{2EA}{L}\begin{pmatrix} \cos^2\varphi_1 & 0 \\ 0 & \sin^2\varphi_1 \end{pmatrix}; \quad \mathbf{K}_i = \frac{2EA}{L}\begin{pmatrix} \cos^2\varphi_i & \sin\varphi_i\cos\varphi_i \\ \sin\varphi_i\cos\varphi_i & \sin^2\varphi_i \end{pmatrix};$$

$$\hat{\mathbf{K}}_0(\mathbf{u}) = \frac{EA[g_1(\mathbf{u}) + g_2(\mathbf{u})]}{L}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad g_i(\mathbf{u}) = \frac{1}{L}(\cos\varphi_i \ \sin\varphi_i)\mathbf{u} + \frac{1}{L^2}\mathbf{u}^T\mathbf{u};$$

$$\varphi_1 = \arcsin\frac{h}{L} = -\varphi_2 \tag{38}$$

If the second order closure scheme is considered, the evaluation of the response means and second order moments requires the solution of Eqs. (27-30) and the corresponding closure equations. It is important to evidence that these equations are not independent, so that they must be solved together.

The second and fourth order closure schemes have been applied and the results related to the mean value and to the second order moment of the vertical displacement are given in Fig. 10 and in Fig. 11, respectively. From the analysis of these figures, the accuracy of the proposed approach is evident in the non-linear cases, too. At last in Table 1 the CPU time percentages relate to the application of the proposed approach are compared with those necessary for the Monte Carlo simulation application.

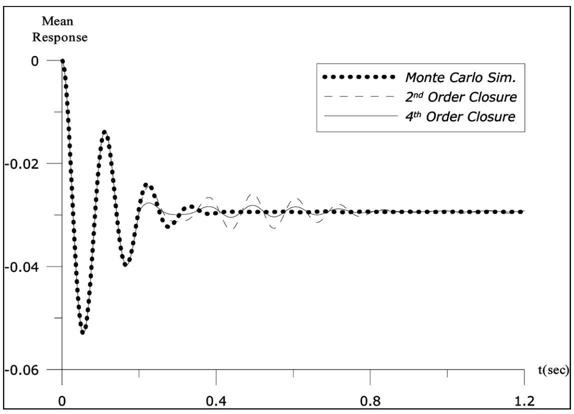

Fig. 10 Mean response of the middle node vertical displacement

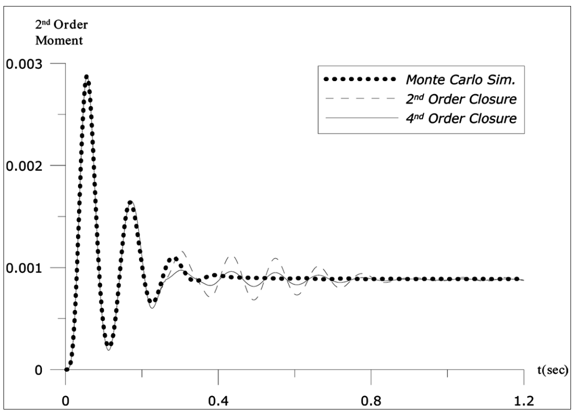

Fig. 11 Second order moment of the middle node vertical displacement

Table 1 CPU time comparison

	CPU time		
Application	1	2	2
t(sec)	%	%	%
SMC	100	100	100
2 Order Closure	2.2	11.8	15.2
4 Order Closure	60.0	47.4	55.0

6. Conclusions

An approach for the approximate evaluation of the response statistical moments of uncertain FE discretized structures has been presented. It is based on the solution of the differential equations governing these moments. They require the knowledge of the cross-moments between the response and the random variables characterizing the structural uncertainties. The infinite hierarchy of equations governing these cross-moments has been truncated by using a cumulant neglect closure of a fixed order. It has been shown that, for the case of linear structures, the set of differential equations resulting from the application of this procedure is linear. The application of the proposed approach to some simple examples of linear and nonlinear structures has revealed a good level of accuracy, even for low order of closure schemes (at maximum a fourth order closure has been applied).

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