DOI: http://dx.doi.org/10.12989/sem.2006.23.2.209
Structural Engineering and Mechanics, Vol. 23, No. 2 (2006) 209-216

Technical Note

Free and forced vibrations of a tapered cantilever beam carrying multiple point masses

Der-Wei Chen† and Tsung-Lung Liu‡

Department of Naval Architecture and Marine Engineering, Chung Cheng Institute of Technology, National Defense University, Yuansulin, Dashi, Taoyuan, Taiwan 335, Republic of China

(Received December 10, 2004, Accepted December 6, 2005)

1. Introduction

The literature relating to the vibration analysis of the non-uniform beams without any attachments (or the *unconstrained* non-uniform beams) is plenty (Abrate 1995, Laura *et al.* 1996, Datta and Sil 1996, Hoffmann and Wertheimer 2000), however, the information regarding the dynamic behaviors of the non-uniform beams carrying multiple various concentrated elements (or the *constrained* non-uniform beams) are relatively fewer (Auciello 1996, De Rosa and Auciello 1996, Auciello and Nole 1998, Auciello and Maurizi 1997, Wu and Hsieh 2000).

For the natural frequencies and mode shapes of the non-uniform (*constrained*) beams carrying concentrated attachments at either end or both ends (Auciello 1996, De Rosa and Auciello 1996, Auciello and Nole 1998), the solution procedures are exactly the same as those for the non-uniform *unconstrained* beams (Abrate 1995, Laura *et al.* 1996, Datta and Sil 1996, Hoffmann and Wertheimer 2000). The only difference is to change the boundary conditions for the non-uniform *unconstrained* beams to accommodate the effects of the attachments at either end or both ends of the *constrained* beams, such as the restoring force due to translational spring, restoring bending moment due to rotational spring, inertial force due to lumped mass and/or inertia mass moment due to concentrated mass moment of inertia. Because the problem becomes much complicated and intractable if the attachments are located at the arbitrary positions along the length of the beam (Auciello and Maurizi 1997, Wu and Hsieh 2000), the literature in this aspect is fewer particularly for the cases with more than two intermediate attachments.

In 1990, Wu and Lin determined the natural frequencies and mode shapes of an Euler-Bernoulli beam carrying any number of concentrated masses located at arbitrary points along the beam with the analytical-and-numerical-combined method (ANCM). The purpose of this paper is to use the last method to perform the free and forced vibration analyses of a tapered cantilever beam carrying any number of point masses located at arbitrary points along the beam and subjected to external load. To validate the numerical results of the ANCM, the traditional finite element method (FEM) was also used to solve the same problem and good agreement between the corresponding results was achieved.

† Professor, Corresponding author, E-mail: dwchen@ccit.edu.tw
‡ Associate Professor, E-mail: tlliu@ccit.edu.tw

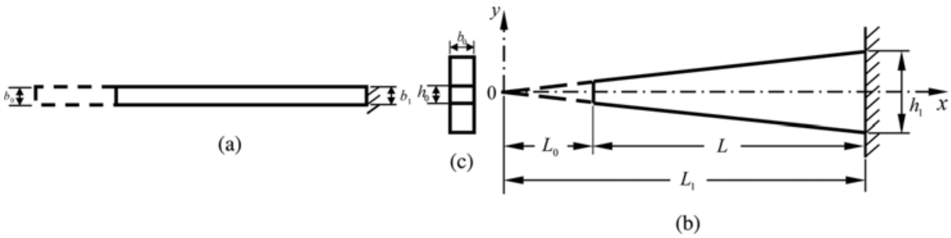

Fig. 1 Sketch for the tapered beam studied: (a) top view, (b) front view, (c) left side view

2. Natural frequencies and normal mode shapes of an unconstrained tapered beam

For the *unconstrained* tapered Euler-Bernoulli beam as shown in Fig. 1, its equation of motion is given by Goel (1976)

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right) + \rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2} = 0 \tag{1}$$

where x is the axial coordinate, y is the transverse deflection, E is the Young's modulus, ρ is the mass density of material, $A(x)$ is the cross-sectional area of beam, $I(x)$ is the moment of inertia of $A(x)$, and t is time.

The solution of Eq. (1) is given by Karman and Biot (1940)

$$W(\xi) = L_1^{-1/2}\xi^{-1/2}[c_1J_1(z) + c_2Y_1(z) + c_3I_1(z) + c_4K_1(z)] \tag{2}$$

with

$$\xi = x/L_1 \tag{3}$$

where L_1 is the length of the tapered beam from the sharp end (i.e., the origin of the axial coordinate x) to the large end, and

$$A_1 = b_1h_1, \quad I_1 = \frac{b_1h_1^3}{12} \tag{4a,b}$$

are respectively the cross-sectional area and moment of inertia of the beam at the large end (cf. Fig. 1). In Eq. (2), J_1 and Y_1 are the 1st order Bessel functions of first kind and second kind, while I_1 and K_1 are the 1st order modified Bessel functions of first kind and second kind, respectively, $c_i \sim c_4$ are integration constants determined by the boundary conditions, and

$$z = 2\beta\xi^{1/2} \tag{5}$$

with

$$\beta^4 = \omega^2 L_1^4\left(\frac{\rho A_1}{EI_1}\right) \tag{6}$$

For the tapered cantilever beam shown in Fig. 1, one has the following boundary conditions:

$$\frac{d^2W}{d\xi^2} = \frac{d}{d\xi}\left(EI(\xi)\frac{d^2W}{d\xi^2}\right) = 0 \quad \text{at} \quad \xi = \xi_0 = L_0/L_1 \tag{7a,b}$$

$$W = \frac{dW}{d\xi} = 0 \quad \text{at} \quad \xi = 1.0 \tag{8a,b}$$

The substitution of Eq. (2) into Eqs. (7a,b) and (8a,b) gives

$$c_1J_2(z_0) + c_2Y_2(z_0) + c_3I_2(z_0) - c_4K_2(z_0) = 0 \tag{9a}$$

$$c_1J_3(z_0) + c_2Y_3(z_0) + c_3I_3(z_0) + c_4K_3(z_0) = 0 \tag{9b}$$

$$c_1J_1(2\beta) + c_2Y_1(2\beta) + c_3I_1(2\beta) + c_4K_1(2\beta) = 0 \tag{9c}$$

$$c_1J_2(2\beta) + c_2Y_2(2\beta) - c_3I_2(2\beta) + c_4K_2(2\beta) = 0 \tag{9d}$$

where

$$z_0 = 2\beta\xi_0^{-1/2} = 2\beta(L_0/L_1)^{-1/2} \tag{10}$$

Non-trivial solution of Eq. (9) requires that

$$\begin{vmatrix} J_2(z_0) & Y_2(z_0) & I_2(z_0) & -K_2(z_0) \\ J_3(z_0) & Y_3(z_0) & I_3(z_0) & K_3(z_0) \\ J_1(2\beta) & Y_1(2\beta) & I_1(2\beta) & K_1(2\beta) \\ J_2(2\beta) & Y_2(2\beta) & -I_2(2\beta) & K_2(2\beta) \end{vmatrix} = 0 \tag{11}$$

From the last equation one may obtain the values of $\beta = \beta_r$ (r = 1, 2, 3, ...) using the half-interval technique (Faires and Burden 1993), and the associated values of $\omega = \omega_r$ obtained from Eq. (6) will be the corresponding natural frequencies, i.e.,

$$\omega_r = \left(\frac{\beta_r}{L_1}\right)^2 \sqrt{\frac{EI_1}{\rho A_1}} \quad (r = 1, 2, 3, \ldots) \tag{12}$$

The corresponding mode shapes may be obtained from Eq. (2), i.e.,

$$W_r(\xi) = L_1^{-1/2}\xi^{-1/2}[c_1J_1(z_r) + c_2Y_1(z_r) + c_3I_1(z_r) + c_4K_1(z_r)] \tag{13}$$

where

$$z_r = 2\beta_r\xi^{1/2} \tag{14}$$

The normal mode shapes $\overline{W}_r(\xi)$ is given by

$$\overline{W}_r(\xi) = C_rW_r(\xi) \tag{15}$$

where

$$C_r = \left(\frac{1}{\rho A_1}\right)^{1/2} \sqrt{\frac{1}{B_r}} \tag{16}$$

$$B_r = \int_{\xi_0}^{1} [c_{1r}J_1(z_r) + c_{2r}Y_1(z_r) + c_{3r}I_1(z_r) + c_{4r}K_1(z_r)]^2 d\xi \tag{17}$$

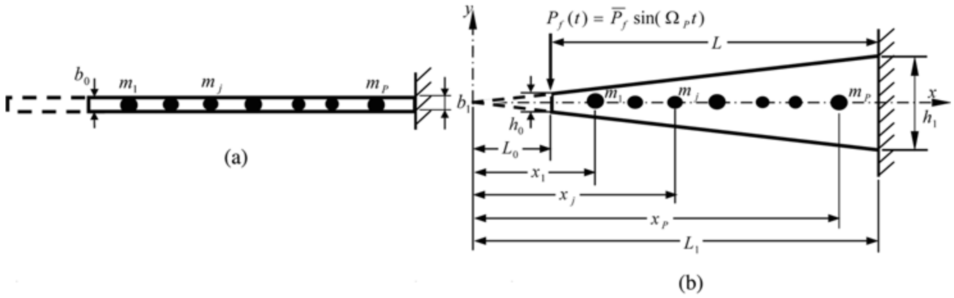

Fig. 2 A tapered cantilever beam carrying P point masses m_j located at $\xi_j = x_j/L_1$ ($j = 1\sim P$) and subjected to an external concentrated force $P_f(t) = \bar{P}_f \sin(\Omega_P t)$ at the free end: (a) top view, (b) front view

3. Equation of motion and eigenvalue equation of the constrained beam

For the tapered Euler-Bernoulli beam carrying P point masses with magnitudes m_j ($j = 1\sim P$) located at x_j ($j = 1\sim P$) and subjected to an external concentrated force $P_f(t) = \bar{P}_f \sin(\Omega_P t)$ at the free end, as shown in Fig. 2, its equation of motion is given by

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right) + \rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2} = -\sum_{j=1}^{P} m_j \frac{\partial^2 y(x,t)}{\partial t^2}\delta(x - x_j) + \bar{P}_f \sin(\Omega_P t)\delta(x - L_0) \quad (18)$$

where $\bar{P}_f$ and Ω_p are the amplitude and exciting frequency of the external load, respectively, $\delta(\cdot)$ is the Dirac delta function and the meanings of the other symbols are exactly the same as those appearing in Eq. (1).

According to the expansion theorem (Meirovitch 1967), we set

$$y(x,t) = \sum_{s=1}^{\bar{n}} \bar{W}_s(x)\eta_s(t) \quad (19)$$

where $\bar{W}_s(x)$ is the s-th normal mode shape of the unconstrained beam obtained in the last section, $\eta_s(t)$ is the associated generalized coordinate, and $\bar{n}$ is the total number of mode considered (or superposed).

By means of the procedures of the ANCM given by Wu and Lin (1990), one obtains

$$\ddot{\eta}_r(t) + \omega_r^2\eta_r(t) = -\sum_{j=1}^{P}\sum_{s=1}^{\bar{n}} m_j \bar{W}_r(\xi_j)\bar{W}_s(\xi_j)\ddot{\eta}_r(t) + \bar{P}_f \sin(\Omega_P t)\bar{W}_r(\xi_0) \quad (r = 1\sim\bar{n}) \quad (20)$$

where ω_r is the r-th natural frequency of the *unconstrained* beam.

Rewriting Eq. (20) in matrix form gives

$$[\tilde{M}]_{\bar{n}\times\bar{n}}\{\ddot{\eta}_r(t)\}_{\bar{n}\times 1} + [\tilde{K}]_{\bar{n}\times\bar{n}}\{\eta_r(t)\}_{\bar{n}\times 1} = \{F(t)\}_{\bar{n}\times 1} \quad (21)$$

where

$$\{F(t)\}_{\bar{n}\times 1} = \{\bar{P}_f \sin(\Omega_P t)\bar{W}_r(L_0/L_1)0\ldots 0\}_{\bar{n}\times 1} \quad (22a)$$

$$[\tilde{M}]_{\bar{n}\times\bar{n}} = [1]_{\bar{n}\times\bar{n}} + [\bar{B}]_{\bar{n}\times\bar{n}} \quad (22b)$$

$$[\tilde{K}]_{\bar{n}\times\bar{n}} = [\omega^2]_{\bar{n}\times\bar{n}} \tag{22c}$$

$$\{\ddot{\eta}_r(t)\}_{\bar{n}\times 1} = \{\ddot{\eta}_1(t) \;\; \ddot{\eta}_2(t) \;\; \dots \;\; \ddot{\eta}_{\bar{n}}(t)\}_{\bar{n}\times 1} \tag{22d}$$

$$\{\eta_r(t)\}_{\bar{n}\times 1} = \{\eta_1(t) \;\; \eta_2(t) \;\; \dots \;\; \eta_{\bar{n}}(t)\}_{\bar{n}\times 1} \tag{22e}$$

$$[\bar{B}] = \sum_{j=1}^{P} m_j \{\overline{W}(\xi_j)\}\{\overline{W}(\xi_j)\}^T \tag{22f}$$

$$\{\overline{W}(\xi_j)\} = \{\overline{W}_1(\xi_j) \;\; \overline{W}_2(\xi_j) \;\; \dots \;\; \overline{W}_{\bar{n}}(\xi_j)\} \tag{22g}$$

Eq. (21) is the equations of motion of an undamped vibration system. By using the Newmark direct integration method (Bathe 1996), one may obtain the generalized co-ordinates $\eta_r(t)$, $r = 1 \sim \bar{n}$, and substituting the values of $\eta_r(t)$ into Eq. (19) will determine the forced vibration response of the *constrained* tapered cantilever beam, the vertical displacements $y(x, t)$. In the above equations, the symbols $\ulcorner\ \lrcorner$, $[\ ]$ and $\{\ \}$ represent the diagonal matrix, square matrix and column vector, respectively.

For free vibration of the *constrained* beam, one has

$$\eta_r(t) = \bar{\eta}_r e^{i\bar{\omega}t} \tag{23}$$

where $\bar{\eta}_r$ is the amplitude of $\eta_r(t)$ and $\bar{\omega}$ is the natural frequency of the *constrained* beam.

From Eqs. (21) and (23), by letting the external load $\{F(t)\} = 0$, one obtains the eigenvalue equation of the *constrained* beam to be

$$\ulcorner\omega^2\lrcorner\{\bar{\eta}\} - (\ulcorner 1 \lrcorner + [\bar{B}])\bar{\omega}^2\{\bar{\eta}\} = 0 \tag{24}$$

and the associated mode shapes to be

$$\tilde{W}_s(\xi) = \{\overline{W}(\xi)\}^T\{\bar{\eta}\}^{(s)} \quad (s = 1 \sim \bar{n}) \tag{25}$$

4. Numerical results and discussions

The physical properties and dimensions of the tapered cantilever beam studied are (cf. Fig. 1): Young's modulus $E = 2.051 \times 10^{11}$ N/m^2, mass density $\rho = 7850$ kg/m^3, beam width $b_1 = b_0 = 0.1$ m, beam depth at large end $h_1 = 0.4$ m, distance from origin to large end of beam $L_1 = 2.0$ m, distance from origin to small end of beam $L_0 = 0.4$ m. The numerical results of this paper are based on the total number of beam elements $N_e = 80$ for FEM and the total number of modes superposed $\bar{n} = 6$ for ANCM.

4.1 Natural frequencies and mode shapes of the tapered cantilever beam

To show the effectiveness of the presented theory, the tapered cantilever beam shown in Fig. 2 with $P = 0$ (no attachment), $P = 1$ (one point mass attached) and $P = 5$ (five point masses attached) are investigated. The magnitudes and locations of the point masses are shown in Table 1. From

Table 1 Comparison between the lowest five natural frequencies obtained from ANCM and FEM ($m_1 = m_b/5 = 60.288$ kg located at $\xi_1 = x_1/L_1 = L_0/L_1 = 0.2$ for $P = 1$; $m_j = m_b/5 = 60.288$ kg located at $\xi_j = x_j/L_1 = 0.3, 0.45, 0.6, 0.75, 0.9$, $j = 1\sim5$, for $P = 5$)

Number of point masses P	Methods	Natural frequencies, ω_i or $\overline{\omega}_i$ (rad/sec)					CPU time (sec)
		ω_1 or $\overline{\omega}_1$	ω_2 or $\overline{\omega}_2$	ω_3 or $\overline{\omega}_3$	ω_4 or $\overline{\omega}_4$	ω_5 or $\overline{\omega}_5$	
0	ANCM	989.6626	3629.5821	8503.9741	15704.6849	25267.5120	0.218
	FEM	989.5017	3628.6311	8501.3310	15699.4691	25258.8597	3.141
	*Difference	0.016%	0.026%	0.031%	0.033%	0.034%	-
1	ANCM	569.6273	2508.8947	6743.2318	13408.5313	22570.1693	0.355
	FEM	569.3039	2503.2976	6709.0487	13286.4974	22236.4917	3.203
	*Difference	0.057%	0.223%	0.507%	0.910%	1.478%	-
5	ANCM	613.2201	2525.5381	6366.4999	12184.0282	16089.9494	0.422
	FEM	613.1226	2524.3389	6353.4962	12105.9359	15885.0965	2.937
	*Difference	0.016%	0.047%	0.204%	0.641%	1.273%	-

*$Difference = (\omega_{i,ANCM} - \omega_{i,FEM}) \times 100\%/\omega_{i,ANCM}$ or $Difference = (\overline{\omega}_{i,ANCM} - \overline{\omega}_{i,FEM}) \times 100\%/\overline{\omega}_{i,ANCM}$

Table 1 one finds that the lowest five natural frequencies obtained from FEM are very close to the corresponding ones obtained from ANCM. The percentage differences shown in Table 1 are calculated with the formula: $Difference = (\omega_{i,ANCM} - \omega_{i,FEM}) \times 100\%/\omega_{i,ANCM}$ for $i = 1\sim5$, where $\omega_{i,ANCM}$ and $\omega_{i,FEM}$ denote the i-th natural frequencies obtained from the ANCM and the FEM, respectively. Although the magnitude of the single point mass for the case of $P = 1$ is only one fifth of the total point masses for the case of $P = 5$, the lowest two natural frequencies of the cantilever beam carrying a tip mass are lower than the corresponding ones of the same beam carrying five uniformly distributed point masses. This means that the effect of distribution of the attached point masses along the beam length must be considered in addition to the effect of the magnitudes of the point masses. From the final column of Table 1 one also finds that the CPU times required by the ANCM are much less than those required by the FEM.

4.2 Forced vibration responses

For the forced vibration system shown in Fig. 2, all physical properties for the cantilever beam carrying five point masses ($P = 5$) are the same as those shown in Table 1. The exciting force located at the free end ($x = L_0 = 0.4$ m) is $P_f(t) = 5.0 \times 10^4 \sin(\Omega_P t)$ N. The time interval is $\Delta t = 0.0035$ sec, and the initial conditions are $y(x,0) = \dot{y}(x,0) = \ddot{y}(x,0) = 0$. Besides, the two parameters required by the Newmark integration method are: $\delta = 0.5$ and $\alpha = 0.25$.

(i) Time histories: The time histories of the vertical displacements at the free end, $y(L_0, t)$, are shown in Fig. 3(a) for the case of exciting frequency $\Omega_P = 5.0$ rad/sec and in Fig. 3(b) for the case of $\Omega_P = 10.0$ rad/sec. From the two figures one finds that the time histories obtained from the ANCM (represented by the dashed lines) are in good agreement with those obtained from the FEM (represented by the solid lines). The CPU time required by the ANCM is 0.28 sec and that required by the FEM is 3.2 sec.

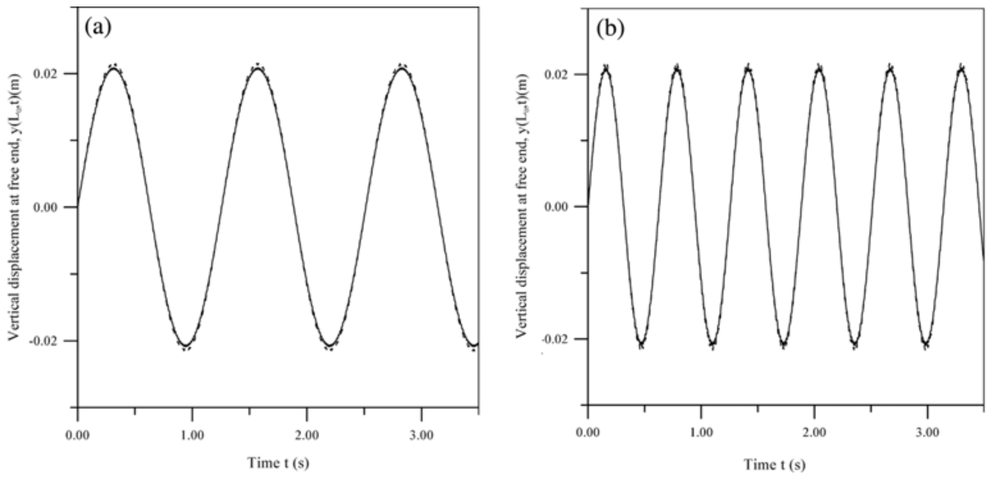

Fig. 3 Time histories of vertical displacements at the free end for the tapered cantilever beam carrying five point masses ($P = 5$) subjected to a tip concentrated force $P_f(t) = 5.0 \times 10^4 \sin(\Omega_P t)$ N: (a) $\Omega_P = 5.0$ rad/sec: (b) $\Omega_P = 10.0$ rad/sec; ——, by FEM; ········, by ANCM

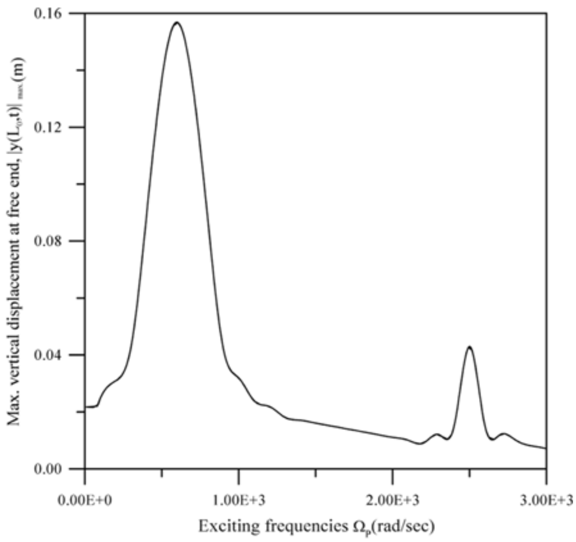

Fig. 4 The frequency-response curves for the tapered cantilever beam carrying five point masses ($P = 5$) subjected to a tip concentrated force $P_f(t) = 5.0 \times 10^4 \sin(\Omega_P t)$ N as shown in Fig. 2

(ii) Frequency-response curves: Fig. 4 shows the frequency-response curves for the free end of the *constrained* beam, where the ordinate represents the maximum vertical displacements of the free end, $|y(L_0, t)|_{\max}$, and the abscissa the exciting frequencies Ω_P of the external load. It is noted that the values of Ω_P corresponding to the first and second humps of each curve are approximately equal to the first and second natural frequencies of the *constrained* tapered cantilever beam, $\overline{\omega}_1 = 613.2201$ rad/sec and $\overline{\omega}_2 = 2525.5381$ rad/sec respectively. The CPU time required by ANCM is 840 sec and that required by the FEM is 9600 sec.

5. Conclusions

Although the finite element analysis has become one of the most popular and general numerical methods of structural analysis, the use of large finite element programs which are capable of handling virtually to any degree of complexity, is cumbersome, costly, and time consuming. It is preferable to use continuum methods, if closed-form solution methods for such a system are possible. Therefore, the ANCM introduced in this paper can provide not only a check against the computer finite element model, but also a means by which the effect of a parameter change on a system can be readily gauged, which is useful in the design process.

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