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Stability of a slender beam-column with locally varying Young's modulus

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Abstract. A locally varying temperature field or a mixture of two or more different materials can cause local variation of elasticity properties of a beam. In this paper, a new Euler-Bernoulli beam element with varying Young's modulus along its longitudinal axis is presented. The influence of axial forces according to the linearized 2nd order beam theory is considered, as well. The stiffness matrix of this element contains the transfer constants which depend on Young's modulus variation and on axial forces. Occurrence of the polynomial variation of Young's modulus has been assumed. Such approach can be also used for smooth local variation of Young's modulus. The critical loads of the straight slender columns were studied using the new beam element. The influence of position of the local Young's modulus variation and its type (such as linear, quadratic, etc.) on the critical load value and rate of convergence was investigated. The obtained results based on the new beam element were compared with ANSYS solutions, where the number of elements gradually increased. Our results show significant influence of the locally varying Young's modulus on the critical load value and the convergence rate.

Keywords: FEM; beam-column; variation of Young's modulus; FGM.

1. Introduction

The critical load of a slender column depends (after Euler buckling theory) on stiffness, column length and on boundary conditions. The buckling stiffness of the column is given by the cross-section moment of inertia and by the Young's modulus. In most cases a constant Young's modulus of the beam along its longitudinal axis is considered. The cross-sectional characteristics could be either constant or variable (Banerjee and Williams 1986, Chugh and Biggers 1976, Lee and Oh 2000, Li 2001, Murín and Kutiš 2002, Rubin 1996, Sapountzakis and Mokos 2004). Recently, new beam elements with functionally graded materials have been developed. Here the material properties change over the beam thickness (Chakraborty, Gopalakrishnan and Reddy 2003, Librescu, Oh and Song 2004, Sankar 2001). Less attention has been paid to variation of the elasticity modulus along the beam length.

Numerical analysis results show a strong influence of the cross-sectional characteristics variation on the slender beam-column stability. Therefore, it can be assumed, that the local variation of

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Young's modulus along the length will have a significant influence too (Murín and Kutiš 2004).

Varying temperature field or composition of two or more materials mixed together can bring about the elastic property variation. Such beams with varying elastic properties could be found not only in the special mechatronics and micromechanical systems, but also in the classical frame structures (Su and Banerjee 2004).

The Hermite or isoparametric beam elements are usually used for statical analysis of a frame structure containing beams with the stiffness variation. The stiffness matrix of this beam element contains an average value of Young's modulus. The average Young's modulus is derived from its nodal point values. Because these elements do not fulfil the equilibrium equations (in the local sense), a very fine mesh of elements is needed to get the correct results. To overcome this problem, a new beam element with varying Young's modulus along its longitudinal axis, under assumption of Euler-Bernoulli beam theory – the 1st order (geometric linear, small elastic displacements) and linearized 2nd order (small but finite elastic displacements, the equations of equilibrium are written in terms of geometry of deformed systems, the internal axial force is assumed as static determined – this theory is geometric linear too, (Rubin and Schneider 1996, Mehlhorn 1995)), is proposed in this paper. A polynomial variation of the Young's modulus has been assumed. The influence of local variation of Young's modulus, as well as the type of this variation, on the critical load value will be shown in several numerical examples. The results of these analyses will be compared with the ones obtained using the above mentioned classical beam elements.

2. Stiffness matrix of the beam element with varying Young's modulus

2.1 Variation of the beam stiffness

The 2-nodal straight 2D-beam element in the initial configuration is depicted in Fig. 1. Its axial stiffness at point x is defined as $B_A(x) = E(x)A$, where $E(x)$ is the varying Young's modulus and A is the cross-sectional area. The flexural stiffness about y-axis is $B_I(x) = E(x)I_y$, where I_y is the area moment of inertia about the y-axis. The variation of the Young's modulus is described by the polynomials

$$E(x) = E_i\left(1 + \sum_{k=1}^{t} \eta_{Ek} x^k\right) \tag{1}$$

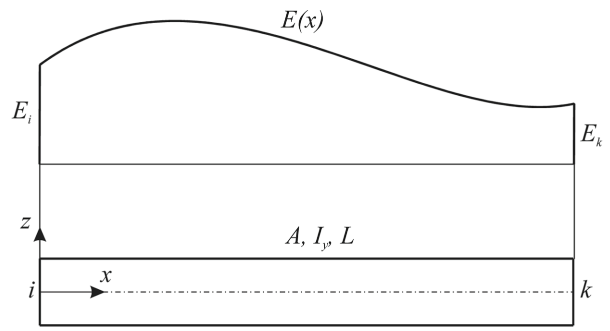

Fig. 1 Beam element with variation of Young's modulus

where E_i is the value of the Young's modulus at the nodal point i, the parameters t and η_{Ek} depend on variation of the Young's modulus along the longitudinal axis of the beam element.

The variation of Young's modulus can be defined directly through the parameters η_{Ek} – Eq. (1) or by defining Young's modulus in equidistant points.

The stiffness of the beam using Eq. (1) is defined as:

- axial stiffness $B_A(x)$

$$B_A(x) = AE_i\left(1 + \sum_{k=1}^{t} \eta_{Ek} x^k\right) \tag{2}$$

- bending stiffness about the y-axis $B_I(x)$

$$B_I(x) = I_y E_i\left(1 + \sum_{k=1}^{t} \eta_{Ek} x^k\right) \tag{3}$$

2.2 Axial loading

In the case of axial loading, the situation has not changed if compared with the 1st order theory. The differential equation for the axial loading is

$$\frac{du(x)}{dx} = \frac{N(x)}{B_A(x)} \tag{4}$$

The displacement $u(x)$ and internal axial force $N(x)$ is shown in Fig. 2.

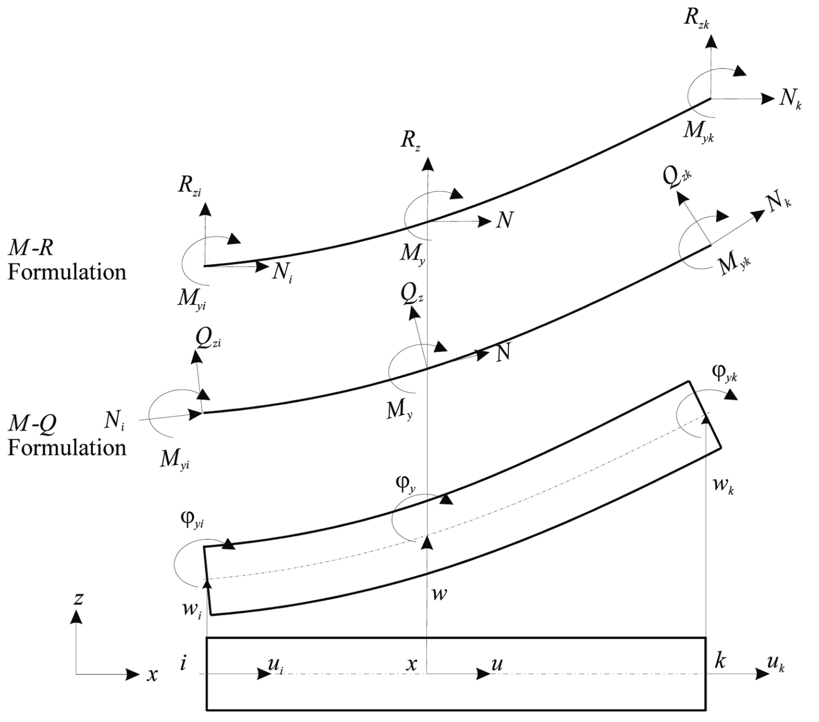

Fig. 2 Initial and deformed configuration – linearized 2nd order beam theory

The solution of differential equation for the axial loading is

$$u(x) = u_i - \frac{N_i}{B_{Ai}} d'_{N2}(x) \tag{5}$$

where u_i, N_i and B_{Ai} are the displacement, axial force and axial stiffness at the node i respectively, and $d'_{N2}(x)$ is the transfer function for the 1st order beam theory, dealt with in section 3. The sign of second term on the right side in Eq. (5) depends on positive orientation of force N_i. According to Fig. 2, this sign is minus.

2.3 Bending

The beam element with internal forces and with deformation variables in the deformed configuration (the linearized 2nd order beam theory) in the *xz*-plane is shown in Fig. 2. The physical representation of these parameters is shown in this figure. As we can see, there are two possible configurations, $M-R$ and $M-Q$ (R is the transversal and Q is the shear force, M is the bending moment). The basic beam equations are derived in the $M-Q$ configuration, but the final FEM equations are transformed to the $M-R$ configuration. The transformation equations between both formulations can be found in Rubin (1996). Both configurations coincide in the case of the 1st order beam theory.

In case of transversal bending around the *y*-axis we have

$$\frac{d^2 w(x)}{dx^2} = -\frac{M_y(x)}{B_I(x)} \tag{6}$$

$$\frac{dw(x)}{dx} = -\varphi_y(x) \tag{7}$$

and the differential equation of equilibrium is

$$\frac{dM_y(x)}{dx} = Q_z(x) \tag{8}$$

The relationship between $R_z(x)$ and $Q_z(x)$ is expressed by the following conditions: the angular displacement is small, i.e., $\sin\varphi_y = \varphi_y$ and $\cos\varphi_y = 1$. Using Eq. (7) this relationship can be written in the form

$$R_z(x) = Q_z(x) - N(x)\varphi_y(x) \tag{9}$$

The solution of these differential equations of the linearized 2nd order beam theory has been derived in Murín and Kutiš (2003) and it has the following form (under assumption of nodal loads only):

- deflection in the *z*-direction

$$w(x) = w_i - \varphi_{yi}x - \frac{1}{B_{Ii}}(-b_{y2}(x)M_{yi} - b_{y3}(x)Q_{zi}) \tag{10}$$

- angular displacement about the y-axis

$$\varphi_y(x) = \varphi_{yi} + \frac{1}{B_{Ii}}(-b'_{y2}(x) - b'_{y3}(x)Q_{zi}) \tag{11}$$

where w_i, φ_{yi}, M_{yi}, Q_{zi} and B_{Ii} is displacement, angle, moment, shear force and bending stiffness at node i respectively, and $b'_{y2}(x)$, $b_{y2}(x)$, $b'_{y3}(x)$ and $b_{y3}(x)$ are transfer functions for linearized 2nd order beam theory, which are dealt with in section 3.

2.4 Stiffness matrix for 2D beam element

The stiffness matrix of the beam element with varying Young's modulus can be derived by the direct stiffness method, or using new shape functions (Murín and Kutiš 2003).

The stiffness matrix has the form

$$\mathbf{K} = \begin{bmatrix} \frac{B_{Ai}}{d'_{N2}} & 0 & 0 & -\frac{B_{Ai}}{d'_{N2}} & 0 & 0 \\ & c_y b'_{y2} & -c_y b'_{y3} & 0 & -c_y b'_{y2} & c_y b_{y2} \\ & & c_y b_{y33} & 0 & c_y b'_{y3} & c_y b_{y3} \\ S & & & \frac{B_{Ai}}{d'_{N2}} & 0 & 0 \\ & Y & & & c_y b'_{y2} & c_y b_{y2} \\ & & M & & & c_y b_{y23} \end{bmatrix} \tag{12}$$

where $b_{y33} = Lb'_{y3} - b_{y3}$, $b_{y23} = Lb_{y2} - b_{y3}$ and $c_y = B_{Ii}(b_{y2}b'_{y3} - b_{y3}b'_{y2})$. All transfer functions in Eq. (12) are evaluated at $x = L$ and they are called transfer constants.

Displacement vector $\mathbf{u}$ and load vector $\mathbf{f}$ have the form

$$\mathbf{u}^T = [u_i \; w_i \; \varphi_{yi} \; u_k \; w_k \; \varphi_{yk}] \tag{13}$$

$$\mathbf{f}^T = [N_i \; R_{zi} \; M_{yi} \; N_k \; R_{zk} \; M_{yk}] \tag{14}$$

The final local relationship can be written in classical FEM form

$$\mathbf{Ku} = \mathbf{f} \tag{15}$$

3. Solution of the transfer constants

The transfer functions $b_{y2}(x)$, $b_{y3}(x)$ and their first derivatives (denoted by (')) depend on the variation of Young's modulus and on the internal axial force. It should be noted that a constant internal axial force along the length of the beam has been assumed. Meanwhile, the transfer function $d'_{N2}(x)$ depends on the Young's modulus variation only. For $x = L$ we get the transfer constants of the beam element – b_{y2}, b_{y3} and their derivatives b'_{y2} and b'_{y3} for linearized 2nd order beam theory and d'_{N2} for the 1st order beam theory. The transfer functions are derived from the analytical solution of the differential equations with non-constant parameters of the beam (Rubin 1996, Kutiš and Murín 2002).

The transfer functions $b_j(x)$ and $b'_j(x)$ for the linearized 2nd order beam theory can be rewritten as

$$b_j^{(n)}(x) = a_{j-n}(x)\sum_{t=0}^{\infty}\beta_{t,0}(x) \quad \text{for} \quad \begin{array}{l} n = 0 \text{ and } 1 \\ j = 2 \text{ and } 3 \end{array} \tag{16}$$

where $\beta_{t,0}(x)$ is given by the expression

$$\beta_{t,0}(x) = -\frac{N_i}{B_{1i}}\beta_{t,2}(x) - \sum_{k=1}^{t}\left[\eta_{Ek}\beta_{t,k}(x)\prod_{r=-k}^{-1}(s-1+r)\right] \tag{17}$$

and $a_{j-n}(x)$ is defined as

$$a_{j-n}(x) = \frac{x^{(j-n)}}{(j-n)!} \tag{18}$$

Single parameters are given by

$$s = j+t \qquad e = \frac{x}{s-n} \qquad \beta_{t,k} = e\beta_{t-1,k-1} \quad \text{for} \quad k = 1, \ldots, m$$

and initial values are

$$\beta_{0,0} = 1 \qquad \beta_{0,k} = 0 \qquad \text{for } k = 1, \ldots, m-1$$

The previous numerical procedure is developed for the linearized 2nd order beam theory, but if in Eq. (17) is put $N_i = 0$, the procedure can be also used for the 1st order beam theory.

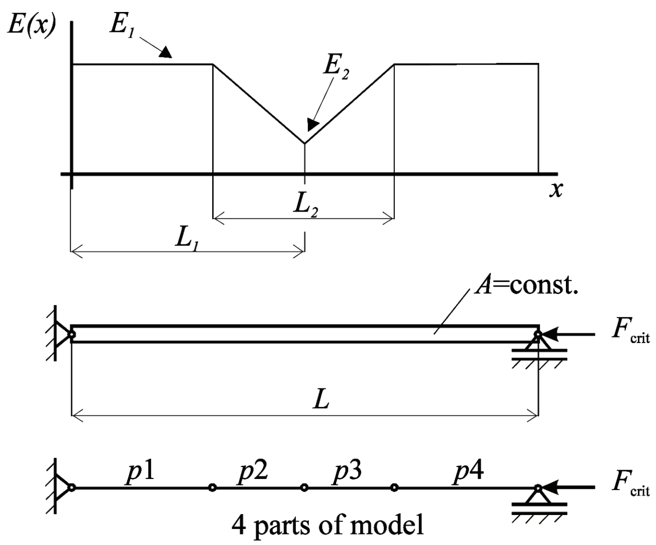

Fig. 3 Beam-column with non-constant Young’s modulus along its length

4. Numerical examples

Five numerical examples show the accuracy and effectiveness of the beam element. The beam cross-section is constant, but the material property – Young's modulus, is in some regions non-constant. In all examples, the goal is to determine the critical force F_{crit} using our beam element and to compare the obtained results with the ANSYS solution (ANSYS 6.1 2002), where the number of elements is gradually increased.

In all the examples, with the only exception of Example 5, abbreviation NOE represents the number of elements in each model part p - Fig. 3. In Example 5, NOE is the total number of elements.

In the first two examples, the Young's modulus in regions $p2$ and $p3$ varies linearly – Fig. 3. In these examples, influence of value of L_1 and L_2 on F_{crit} and NOE is examined. In Example 3 and 4, the influence of a different type of Young's modulus variation on F_{crit} and NOE is examined.

And finally, Example 5 investigates four basic types of Euler buckling, where the Young's modulus varies quadratically.

4.1 Example 1

Fig. 3 shows the simple supported beam-column with a constant circular cross-section. The diameter of the cross-section is d = 0.02 [m]. The beam-column length is L = 1 [m]. The material properties are non-constant along the beam length; the variation of Young's modulus is also shown in this figure. As presented in Fig. 3, the non-constant Young's modulus is characterized by its two values: $E_1 = 2.1 \times 10^{11} [\text{Pa}]$ and $E_2 = 1 \times 10^{11} [\text{Pa}]$. The position of E_2 is located at the mid-point of length L_2. Lengths L_1 and L_2 characterize the regions with the non-constant Young's modulus. In this example, the length L_2 is constant with the value $L_2 = L/3$. The length L_1 gradually decreases according to the second row of Table 1. The objective is to determine the critical force of the beam-column.

Table 1 The critical force in Example 1

New beam element – F_{crit} [N]				
NOE	$L_1 = 12L/24$	$L_1 = 10L/24$	$L_1 = 8L/24$	$L_1 = 7L/24$
1	12782	12932	13387	13741
ANSYS Beam3 element – F_{crit} [N]				
1	13362	13495	13885	14165
2	12933	13084	13532	13859
3	12847	13000	13459	13795
4	12814	12971	13432	13772
6	12793	12949	13413	13755
8	12786	12941	13407	13750
10	12782	12938	13404	13747
20	12782	12933	13399	13743

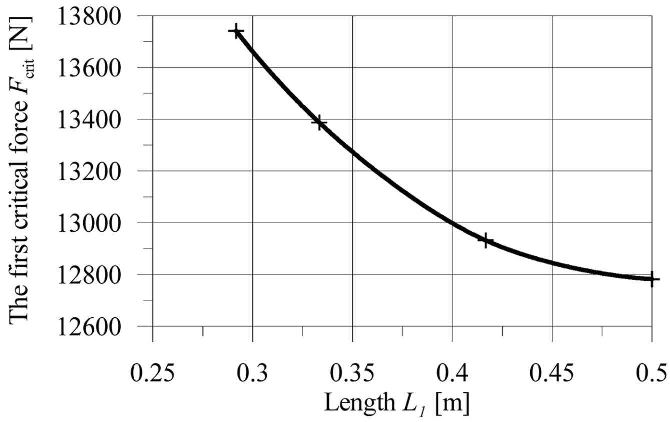

Fig. 4 The critical force dependence of length L_1 – Example 1 (symmetric about the midspan)

The problem has been solved by our beam element with transfer constant, as well as with FEM ANSYS program. The model is divided into four parts – $p1$ to $p4$ (Fig. 3). Parts $p1$ and $p4$ have constant material properties; the variation of Young's modulus along the parts $p2$ and $p3$ is linear. Using our beam element, each part was modeled by only one element. In the ANSYS program, the number of elements was gradually increased. The number of elements – NOE – in Table 1 represents the number of elements in each part, i.e., when the NOE in a single part is 3, the total number of elements of the system is 12. The results are shown in Table 1.

Assuming Young's modulus constant along the length (equal to E_1), the critical force is $F_{crit,const}$ = 16278 [N].

As it can be seen from Fig. 4, the minimal value of the critical force is reached, when the position (L_1) of local variation of Young's modulus is in the midspan. Conversely, if a non-constant Young's modulus region is moved closer to the left or right end of the beam, the critical force is increased.

Comparing to the case with the constant Young's modulus in case of $E_1 > E_2$, the value of the critical force is considerably decreased by the local variation of the Young's modulus.

Table 2 The critical force – Example 2

New beam element – F_{crit} [N]						
NOE	$L_2 = L/2$	$L_2 = L/3$	$L_2 = L/4$	$L_2 = L/5$	$L_2 = L/6$	$L_2 = 8L/100$
1	11693	12782	13465	13935	14273	15250
ANSYS Beam3 element – F_{crit} [N]						
1	12609	13362	13904	14289	14572	15404
2	11929	12933	13586	14033	14357	15293
3	11799	12847	13520	13980	14311	15270
4	11753	12814	13496	13961	14295	15262
6	11720	12793	13479	13947	14283	15256
8	11709	12786	13473	13942	14279	15254
10	11703	12782	13470	13939	14277	15253
20	11696	12782	13467	13936	14275	15251

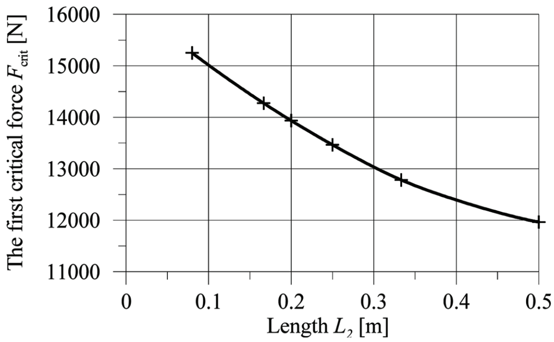

Fig. 5 The critical force dependence on length L_2 (symmetric about the midspan)

4.2 Example 2

The second example analyses the effect of length L_2 on the critical force in the same system as in Example 1. The only parameter changed is the length L_2 – its value is gradually decreased according to the second row in Table 2. The length L_1 has been chosen as $L_1 = L/2$.

From Fig. 5 can be seen, that with increasing the length L_2 of the local Young's modulus variation, the first critical force decreases, as expected. It is clear, that the critical force is closer to $F_{crit,const}$, when the length L_2 is smaller.

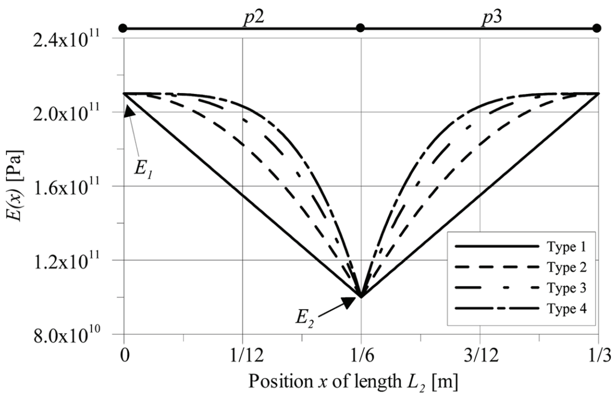

Fig. 6 Variations of Young's modulus along the length L_2

Table 3 The critical force – Example 3

New beam element – F_{crit} [N]				
NOE	Type 1	Type 2	Type 3	Type 4
1	12782	13844	14409	14763
ANSYS Beam3 element – F_{crit} [N]				
2	12933	14181	14855	15273
6	12793	13880	14468	14842
10	12782	13854	14426	14784
20	12782	13840	14407	14762

4.3 Example 3

The third example analyses the effect of Young's modulus type variation in parts $p2$ and $p3$ on F_{crit}. The geometry is identical with Example 1, i.e., length $L_1 = L/2$ and $L_2 = L/3$. The Young's modulus is $E_1 = 2.1 \times 10^{11}$ [Pa] and $E_2 = 1 \times 10^{11}$ [Pa], still, the variation is different in parts $p2$ and $p3$ – the length L_2. For part $p2$, variations are as follows.

Type 1: $E(x) = 2.1 \times 10^{11} - 6.6 \times 10^{11} x [\mathrm{Pa}]$

Type 2: $E(x) = 2.1 \times 10^{11} - 3.96 \times 10^{12} x^2 [\mathrm{Pa}]$

Type 3: $E(x) = 2.1 \times 10^{11} - 2.376 \times 10^{13} x^3 [\mathrm{Pa}]$

Type 4: $E(x) = 2.1 \times 10^{11} - 1.4256 \times 10^{14} x^4 [\mathrm{Pa}]$

For part $p3$, the Young's modulus is symmetric about the central point of beam - Fig. 6.

The results for individual types are shown in Table 3.

As we can see from Table 3, the beam with linear Young's modulus variation has minimal critical force. The critical force increases, when the order of polynomial, which describes the Young's modulus variation becomes higher, as expected.

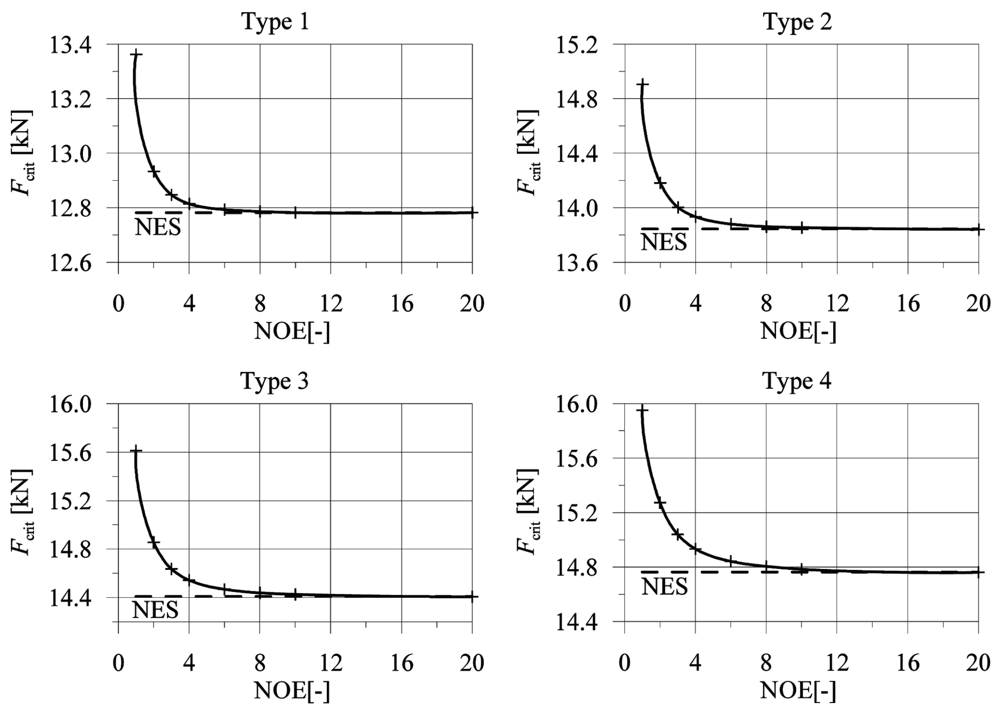

Fig. 7 Convergence of results of the Hermite beam element to our new element results – Example 3 (NES – new element solution)

Fig. 7 shows a very good effectiveness and accuracy of our new beam element. As it can be seen from the figure, the convergence of a classical beam element to our solution is very rapid mainly in the first two types of the Young's modulus variation. In Types 3 and 4, where the Young's modulus is described by higher order polynomial, the convergence rate is not so rapid and it is necessary to use more elements to obtain relevant results.

Table 4 The critical force – Example 4

New beam element – F_{crit} [N]						
NOE	K = 1/10	K = 1/5	K = 1/2	K = 2	K = 5	K = 10
1	12077	13181	14858	17803	19919	21539
ANSYS Beam3 element – F_{crit} [N]						
2	14371	14628	15324	17665	19874	21430
6	12906	13565	14928	17794	19918	21545
10	12509	13345	14879	17804	19926	21551
20	12207	13212	14857	17808	19929	21553

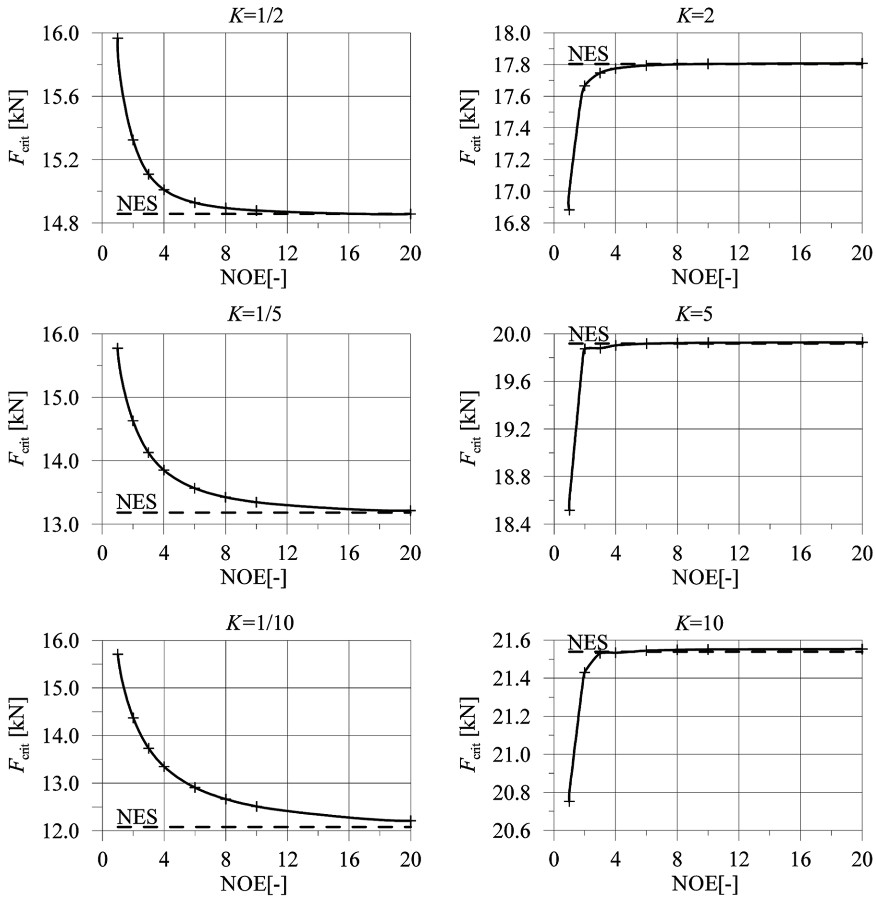

Fig. 8 Convergence of results of the Hermite beam element to our new element results – Example 4 (NES – new element solution)

4.4 Example 4

This example examines the influence of magnitude E_2 on F_{crit} and NOE. All parameters are identical with those of the previous example – type variation 4, but the Young's modulus E_2 can be written as $E_2 = K \cdot E_1$. The parameter K is changed from 1/10 to 10. The obtained results are shown in Table 4.

Fig. 8 shows a very good effectiveness and accuracy of our new beam element. As we can see from Table 4 and Fig. 8, the rate of convergence is faster in cases when $K > 1$ than it is in cases when $K < 1$. In cases when $K > 1$, the increase of K yields an increase of the convergence rate, and in cases when $K < 1$, the decrease of K yields a decrease of the convergence rate.

4.5 Example 5

This example investigates four basic types of Euler buckling where the cross-section is constant, but the Young's modulus varies quadratically. The diameter of the cross-section is $d = 0.02$ [m], the length is $L = 1$ [m] and the variation of Young's modulus is, as shown in Fig. 9,

$$E(x) = 2.1 \times 10^{11} - 2.2 \times 10^{11}x + 1.1 \times 10^{11}x^2[\mathrm{Pa}]$$

The obtained results for individual types of buckling are shown in Table 5.

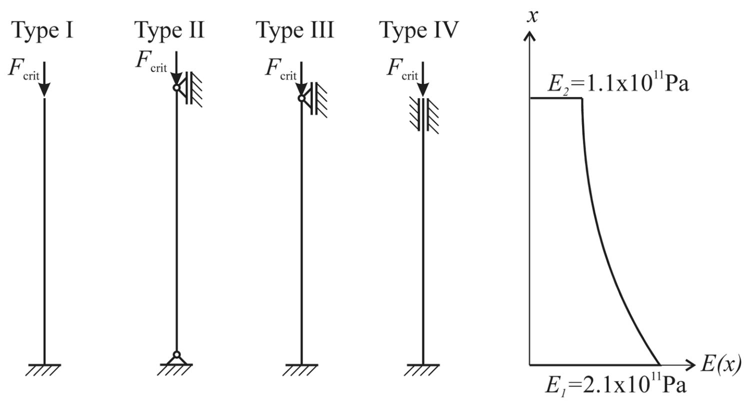

Fig. 9 Four basic types of buckling with non-constant Young's modulus

Table 5 The critical force – Example 5

New beam element – F_{crit} [N]				
NOE	I	II	III	IV
1	2955	9867	20649	41008
ANSYS Beam3 element – F_{crit} [N]				
2	2861	9963	20151	40271
6	2944	9874	20593	40954
10	2951	9869	20627	40970
20	2954	9867	20644	40997

5. Conclusions

The local Young's modulus variation exerts an influence on the beam stiffness. This paper deals with the new beam element with a continuous variation of Young's modulus which can be used in statical analysis of frame structures with such a varying stiffness. The influences of the position and width of local Young's modulus variation on the critical axial force of the straight column has been investigated using this new beam element. The results of the proposed numerical examples show that the local Young's modulus variation has a considerable influence on the critical load of the column. The same examples have been tacked using the Hermite beam element with an increasing number of elements. Comparison of both solutions shows a very good effectiveness and accuracy of our beam element.

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