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Fuzzy reliability analysis of laminated composites

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Abstract. The strength behaviors of Fiber Reinforced Plastics (FRP) Composites can be greatly influenced by the properties of constitutive materials, the laminate structures, and load conditions etc, accompanied by many uncertainty factors. So the reliability study on FRP is an important subject of research. Many achievements have been made in reliability studies based on the probability theory, but little has been done on the roles played by fuzzy variables. In this paper, a fuzzy reliability model for FRP laminates is established first, in which the loads are considered as random variables and the strengths as fuzzy variables. Then a numerical model is developed to assess the fuzzy reliability. The Monte Carlo simulation method is utilized to compute the reliability of laminas under the maximum stress criterion. In the second part of this paper, a generalized fuzzy reliability model (GFRM) is proposed. By virtue of the fact that there may exist a series of states between the failure state and the function state, a fuzzy assumption for the structure state together with the probabilistic assumption for strength parameters is adopted to construct the GFRM of composite materials. By defining a generalized limit state function, the problem is converted to the conventional reliability formula that enables the first-order reliability method (FORM) applicable in calculating the reliability index. Several examples are worked out to show the validity of the models and the efficiency of the methods proposed in this paper. The parameter sensitivity analysis shows that some of the mean values of the strength parameters have great influence on the laminated composites' reliability. The differences resulting from the application of different failure criteria and different fuzzy assumptions are also discussed. It is concluded that the GFRM is feasible to use, and can provide an effective and synthetic method to evaluate the reliability of a system with different types of uncertainty factors.

Keywords: laminated composites; fuzzy reliability; Monte Carlo simulation; generalized limit state function.

1. Introduction

By virtue of its excellent properties, such as the high specific strength and stiffness, anti-fatigue, vibration absorption and corrosion resistance, the Fiber Reinforced Plastics (FRP) is widely used in automobile, ship, pipe, aircraft, space vehicle etc. In general, these structures are in service under special and severe circumstances, so high reliability is required of them. On the other hand, the inherent anisotropic behavior of composite materials leads to the high sensitivity of strength to load conditions and other factors, and composite materials generally exhibit large statistical variations in

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their mechanical properties. So the probabilistic analysis should play an important role in the structural assessment (Cruse 1994, Ben 1994) and the reliability study of composite materials is of great importance.

Several methods have been developed recently to probabilistically assess the static strength of fibrous composites (Lin and Kam 2000, Murotsu and Miki 1994, Gurlich and Pipes 1995). Zhao and Gao (1995), Mark and Gurlich (1998) researched the reliability of composite laminates by enumerating significant failure modes, and three kinds of load are considered, in-plane tension load, shear load and general in-plane load. Philippidis and Lekou (1998) considered strength parameters as random variables of two-parameter Weibull distribution under in-plane loads. These random parameters were transformed into standard normal distribution parameters to calculate the probability of failure. Gurvich and Pipes (1995) analyzed the reliability of laminated composite in bending by using the multi-step failure method, and the Monte Carlo method was utilized in the reliability computation. Jeong and Shenoi (1998) studied the reliability of composite laminate under transverse loads by considering the distributed loads, the strength parameters and the material properties as random variables. To evaluate the system reliability, the Tsai-Hill failure criterion and the direct simulation method were used.

Most of these studies used the first ply failure (FPF) assumption, that is, if any one of the plies in a laminate fails, the entire laminate is considered a failure. In fact, a composite laminate consists of plies that join and work together to resist external loads. If only one ply or several plies in a laminate has failed, the external loads will be redistributed among the remaining plies, and the laminate is capable of resisting the loads continually until it is completely fractured. A progressive damage process can characterize the failure process. It is generally too conservative to ignore the residual strength of a laminate after the first ply failure.

Some researchers (Mahadevan *et al*. 1997, Chen *et al*. 2002a, 2002b) considered the composite laminate as a structural system. It was assumed that the system will not fail until all the plies have failed. The system failure was estimated by identifying possible failure sequences, which lead to system failure. As the number of the possible failure sequences is very large for complicated laminates, efficient search techniques are needed to identify important failure sequences, and the system failure can be approximated as the union of these identified important failure sequences.

In a classical reliability model, some of the variables such as loads and strengths are considered as random and the rest deterministic. The binary state assumption is adopted for the structure system. That is, a system is either in function or failure state. The system reliability is evaluated based on the probability theory (Melchers 1999, Kogiso *et al*. 1997, Jeong and Shenoi 2000, Sciuva and Lomario 2003). However, in certain cases, consideration of the random uncertainty alone cannot help satisfactorily evaluate the reliability of a structural system. For example, the binary state assumption that a system is either in function or failure state is not rational since there are a series of states between the two limit states. So the fuzzy concept is necessary in analyzing the reliability.

The fuzzy concept has been introduced and developed in the reliability theory recently (Cai *et al*. 1993, 1995, Wei and Chen 2004, Lev *et al*. 1996, John *et al*. 1995). Several forms of fuzzy reliability theories are studied by making new assumptions to replace the probability assumption or the binary-state assumption.

In this paper, first we consider the strength parameters as fuzzy variables and the loads as random variables, and construct a fuzzy reliability model for composite materials. A simulation method of calculating failure probability is developed. Numerical examples are worked out to show the validity and the necessity of the method proposed. Then the fuzzy state assumption together with the probability assumption is employed to establish a generalized fuzzy reliability model (GFRM) for laminated composites, and the corresponding computation method by introducing a generalized performance function is developed. Examples are given to demonstrate the validity and efficiency of the model proposed. The influence of the strength parameters, the failure criteria, and the fuzzy parameters are also discussed.

2. A fuzzy reliability model

2.1 Basic theorem of fuzzy mathematics (Zimmermann 1991)

Based on the nature of fuzzy human thinking, L.A. Zadeh originated the fuzzy logic theory. Set A is a common set in the universe of discourse U. For any element u in U, $u \in A$ or $u \notin A$, a map from U to {0, 1} can be defined as

$$U \to \{0, 1\}, u \to A(u) = \begin{cases} 1 & u \in A \\ 0 & u \notin A \end{cases} \tag{1}$$

Here $A(u)$ is the characteristic function of the common set A. It can be rewritten as $C_A(u)$.

Suppose the map $\mu_{\tilde{A}}$ from U to [0, 1] defined in the universe of discourse U is

$$\mu_{\tilde{A}}\colon U \to [0, 1], u \to \tilde{A}(u) \in [0, 1] \tag{2}$$

$\tilde{A}$ is called the fuzzy set in U. $\tilde{A}(u)$ is the membership function of u to $\tilde{A}$ and can be expressed as $\mu_{\tilde{A}}(u)$. When $\mu_{\tilde{A}}(u)$ is 0 or 1, the fuzzy set $\tilde{A}$ degenerates to a common set A and $\mu_{\tilde{A}}(u)$ changes into the characteristic function $C_A(u)$.

Below we introduce an important decomposition theorem in fuzzy mathematics.

2.1.1 λ Truncated subset

Let $\tilde{A}$ be a fuzzy set in the universe of discourse U. For a given threshold $\lambda \in [0, 1]$, define

$$A_\lambda = \{x | x \in U, \mu_{\tilde{A}}(x) \geq \lambda\} \tag{3}$$

We call A_λ the λ truncated subset to $\tilde{A}$. From the definition, we know that A_λ is a common set.

2.1.2 Decomposition theorem

If $\tilde{A}$ is a fuzzy set in the universe of discourse U, then $\lambda\tilde{A}$ is also a fuzzy set, and its membership function is

$$\mu_{\lambda\tilde{A}}(x) = \lambda \wedge \mu_{\tilde{A}}(x) \tag{4}$$

The symbol " $\wedge$ " means an operation of taking the smaller value. With these definitions, we can decompose the fuzzy set into a series of common sets as follows:

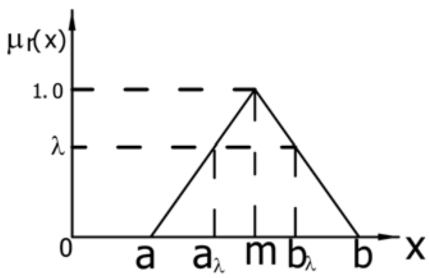


Fig. 1 The membership function

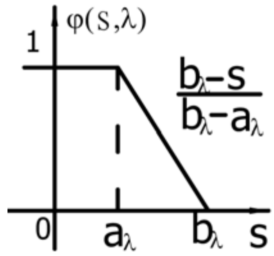


Fig. 2 A schematic illustration of $\varphi(s, \lambda)$ function

$$\tilde{A} = \bigcup_{\lambda \in [0,1]} \lambda A_{\lambda}$$

$$\mu_{\tilde{A}}(x) = \bigvee_{\lambda \in [0,1]} (\lambda \wedge C_{A_{\lambda}}(x)) \tag{5}$$

The symbol " $\vee$ " denotes an operation of taking the larger value.

2.2 Formulation of fuzzy reliability

The stress-strength interference model is utilized and expanded to analyze the fuzzy reliability of laminated composites. The strengths and stresses are modeled as fuzzy variables and random variables, respectively. A fuzzy reliability model is constructed as follows.

The fuzzy strength is expressed as

$$\tilde{r} = \int_{r \in V} \mu_{\tilde{r}}(r)/r \tag{6}$$

in which $\mu_{\tilde{r}}(r)$ is the membership function of the fuzzy strength. A symmetrical triangular distribution type membership function as shown in Fig. 1 is utilized to describe $\mu_{\tilde{r}}(r)$ in this paper (Jiang and Chen 2003). Generally, there exists much difficulty in directly calculating the fuzzy random reliability (Liu *et al.* 1997). Here a transferring process is adopted to calculate the fuzzy reliability. Let the probability density function of stress be $f_S(s)$, the fuzzy strength be $\tilde{r}$, and its membership function be $\mu_{\tilde{r}}(r)$. For the fuzzy variable $\tilde{r}$, given a threshold $\lambda \in [0, 1]$, then the corresponding λ truncated subset r_λ whose region is $[a_\lambda, b_\lambda]$ can be obtained. Suppose the fuzzy variable submits to a uniform distribution on the region $[a_\lambda, b_\lambda]$, i.e., its probability density function can be expressed as

$$f_{r_\lambda}(r) = 1/(b_\lambda - a_\lambda) \tag{7}$$

According to the stress-strength interference model, the safe probability at the threshold λ is computed as:

$$\begin{aligned} \tilde{R}_\lambda = P(s \le \tilde{r}_\lambda) &= \int_{-\infty}^{+\infty} f_{r_\lambda}(r)\left[\int_{-\infty}^{r} f_s(s)ds\right]dr \\ &= \int_{-\infty}^{\infty} f_s(s)\left[\int_{s}^{\infty} f_{r_\lambda}(r)\right]ds \end{aligned}$$

$$= \int_{-\infty}^{\infty} f_s(s) \times \max[\min((b_\lambda - s)/(b_\lambda - a_\lambda), 1), 0] ds$$

$$= \int_{-\infty}^{\infty} f_s(s) \varphi(s, \lambda) ds \tag{8}$$

The function $\varphi(s, \lambda)$ is defined as (Fig. 2)

$$\varphi(s, \lambda) = \max[\min((b_\lambda - s)/(b_\lambda - a_\lambda), 1), 0] \tag{9}$$

in which a_λ, b_λ are dependent on the threshold parameter λ, but are independent of the stress distribution. Integrating Eq. (8) in the extent [0, 1], one obtains the safe probability as follows (see Appendix):

$$\tilde{R} = P(s \le \tilde{r}) = \int_0^1 R_\lambda d\lambda = \int_0^1 \int_{-\infty}^{\infty} f_s(s) \varphi(s, \lambda) ds d\lambda \tag{10}$$

For a constant stress s_0, $f_s(s)$ is reduced to a Dirac's delta function $f_s(s) = \delta(s - s_0)$. By virtue of this, Eqs. (8) and (10) become

$$\tilde{R}_\lambda = \varphi(s_0, \lambda) \tag{11}$$

$$\tilde{R} = P(s_0 \le \tilde{r}) = \int_0^1 R_\lambda d\lambda = \int_0^1 \varphi(s_0, \lambda) d\lambda \tag{12}$$

2.3 Monte Carlo simulation procedure

Integration of Eq. (12) to get the fuzzy reliability is not straightforward. In this paper, the Monte Carlo Method is used to compute the fuzzy reliability of composite materials (Sciuva and Lomario 2003, Dong and Zhu 2000). The computing steps of the lamina reliability are as follows:

(1) Generating the stress specimen $s_i (i = 1, 2, \ldots, N)$ according to the stress distribution;

(2) Calculating the reliability R_i according to Eq. (12), i.e.,

$$\tilde{R}_i = P(s_i \le \tilde{r}) = \int_0^1 \varphi(s_i, \lambda) d\lambda \tag{13}$$

(3) The reliability of the lamina is obtained as

$$\tilde{R} \approx \hat{R} = \frac{1}{N} \sum_{i=1}^{N} \tilde{R}_i \tag{14}$$

The step (1) can be realized by a simulating method when the stress distribution $f_S(s)$ is known. The step (2) is further decomposed as follows:

(1) Generating the uniform distribution $\lambda_j (j = 1, 2, \ldots, n)$ in the region (0, 1).

(2) By taking λ_j as the threshold, and using Eqs. (11) and (12), the reliability under stress s_i is computed as

$$R_{\lambda j}(s_i) = \varphi(s_i, \lambda_j) \tag{15}$$

$$\tilde{R}_i = \sum_{j=1}^{n} R_{\lambda j}(s_i) = \sum_{j=1}^{n} \varphi(s_i, \lambda_j) \tag{16}$$

2.4 Numerical examples

The example shown in Fig. 3 deals with a square laminated plate subjected to bi-axis tension loads N_x and N_y. The stacking structure is $(0/\theta/-\theta/0)_s$, $a \times b = 20 \times 12.5$ cm^2, and the thickness of each ply is 0.125 mm. The composite material is a typical graphite/epoxy (T300/5208). Mechanical properties of the material and the load conditions are listed in Table 1. The tension loads are assumed to be normally distributed random variables. A symmetrical triangular distribution type membership function (Fig. 1) is proposed in which the left and right distribution parameters are taken as $a/m = 0.90$, $b/m = 1.10$ (Table 2).

The maximum-stress criterion is used in the example, i.e., $\sigma_1 \le X_t$, $\sigma_2 \le Y_t$, $\tau_{12} \le S$ for tension, $|\sigma_1| \le X_c$, $|\sigma_2| \le Y_c$, $\tau_{12} \le S$ for compression. Here X_t and X_C are the longitudinal tension and compression strengths, Y_t and Y_C the transverse tension and compression strengths, S the shear

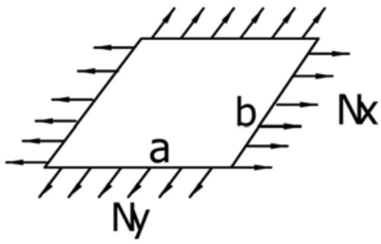


Fig. 3 A laminate subjected to bi-axis tension loads

Table 1 Loading conditions and the mechanical properties

	Units	Mean	Standard deviation
Nx	kN/m	1000	100
Ny	kN/m	100	10
E_1	GPa	181.0	
E_2	GPa	10.7	
G_{12}	GPa	7.17	
ν_{12}		0.28	

Table 2 Fuzzy strength parameters (MPa)

	Mean value m	Left distribution parameter a	Right distribution parameter b
Xt	1500	1350	1650
Xc	1500	1350	1650
Yt	40	36	44
Yc	246	221.4	270.6
S	68	61.2	74.8

Table 3 The fuzzy reliability and the conventional reliability of the FRP

Angle	No fuzzy	Fuzzy	Change (%)
20	0.1114	0.1195	7.27
22	0.7442	0.7346	−1.3
24	0.9993	0.9983	−0.1
26	0.9973	0.9942	−0.3
28	0.9897	0.9859	−0.4
30	0.9725	0.9642	−0.9
32	0.9353	0.9184	−1.8
34	0.8268	0.8134	−1.6
36	0.4553	0.4567	0.3
38	0.1952	0.2104	7.8

strength. With the previously mentioned Monte Carlo simulation method, the system fuzzy reliability (safe probability) is obtained and listed in Table 3. The conventional reliability results and the comparison are also shown in the table.

From Table 3, we can see that the fuzzy reliability and the conventional reliability have similar variation tendency as θ, and the maximum reliability is within the fiber degree range of 24-26 degrees. Within the range of 24 to 34 degrees, we have a relatively high reliability that is basically required in practice. The results show that the difference between the two reliability models within the range is very small. This is because the symmetrical triangle distribution pattern around the mean value (Fig. 1) was assumed for the fuzzy strengths, which consequently results in almost the same reliability as the case that strengths are deterministic and stresses are random. In fact, as long as the strength – stress interaction model is used in structural reliability analyses, the present fuzzy reliability model is nearly equivalent to the randomization of material strengths and stresses in the conventional way. The model developed above can provide an alternative method assessing structural reliability in the case where the distribution pattern of strength parameters is hard to determine. The other application of this model lies in that it can extend the concept of "safety". Treating the strengths as "fuzzy" variables substantially implies that a "fuzzy" judgment is made for the structure state. A structure may be in neither a completely safety situation, nor a failure state, rather it is in a safe status to a certain degree. The extension of this idea leads to a generalized fuzzy reliability model as will be treated in Section 3.

Figs. 4-7 show the influence of the mean values of the fuzzy strength parameters on reliability. The fiber orientation is fixed to 30 degrees, and the Maximum stress criterion is utilized in the calculation. These figures show that Xt makes positive contributions to fuzzy reliability of the system. But Xc, Yt and Yc exert little influence on the reliability in the case studied. This is because the layers in the laminate are either 0 degree or 30 degree, and the tension loading is primarily in the 0 degree direction. So the tension strength Xt has the most important effect on the reliability. For other combinations of stacking structures and loading conditions, different dependence features on the mean values of strengths may exist.

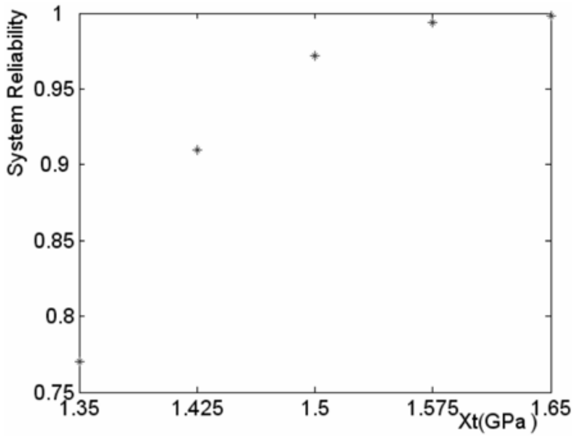


Fig. 4 Influence of the mean value of *Xt*

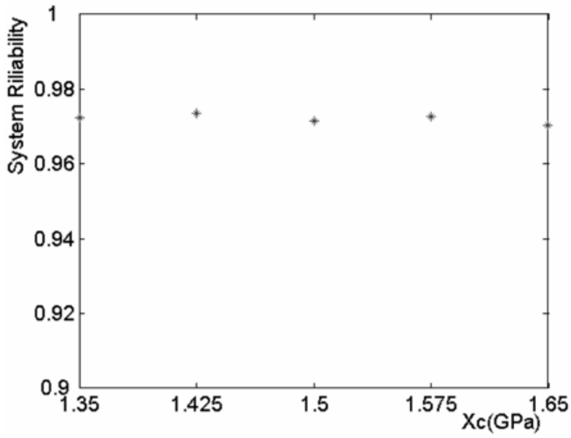


Fig. 5 Influence of the mean value of *Xc*

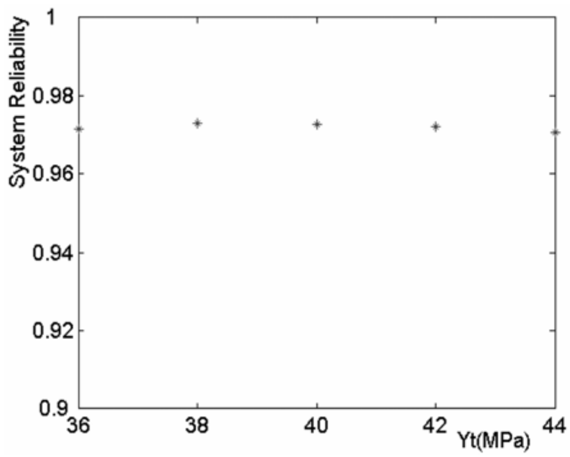


Fig. 6 Influence of the mean value of *Yt*

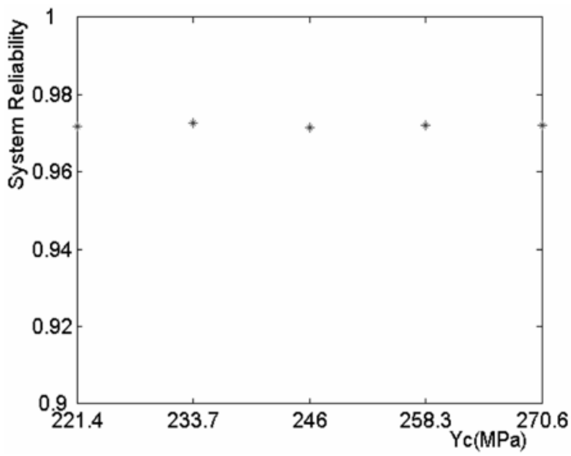


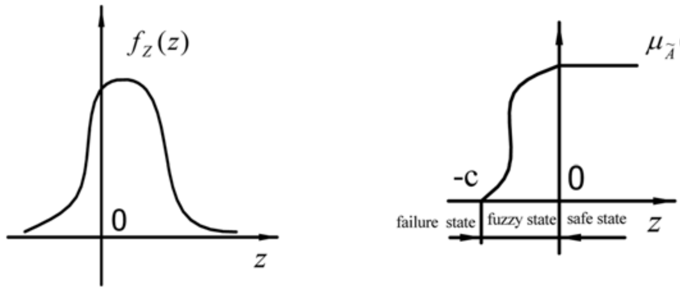
Fig. 7 Influence of the mean value of *Yc*

3. A generalized fuzzy reliability model

3.1 Theoretical consideration

In the previous section, a fuzzy reliability model in which strengths and stresses are modeled as fuzzy variables and random ones respectively, and the binary-state assumption for a system was adopted in the formulation. In the light of the safety degree of a system, a generalized fuzzy reliability model of composite materials can be constructed by making new assumptions to replace the binary-state assumption. By virtue of the fact that there may exist a series of states between the failure state and the function state, a fuzzy assumption for the structure state together with the probabilistic assumption for strength parameters is made in the following analysis.

Let the basic random variables of a structure be $X = \{X_1, X_2, \ldots, X_m\}^T$, the corresponding probability density function $f_X(x)$, and the performance function $Z = g(X)$. According to the reliability theory, the failure probability can be expressed as Melchers (1999)



(a) The probability density function of Z (b) The membership function of a safety state

Fig. 8 Fuzzy safety state of a structure and the membership function

$$p_f = P(Z = g(X) < 0) = \int_{g(X) < 0} f_X(x)dx \tag{17}$$

From the probability density function $f_X(x)$, we can get the density function of Z, $f_Z(z) = f_X(x)$ $|dx/dz|$, so the failure probability expression can be changed to

$$P_f = \int_{z < 0} f_Z(z)dz = \int_{-\infty}^{0} f_Z(z)dz \tag{18}$$

Define an event $A:\{Z = g(X) \geq 0\}$, and then the structure reliability takes the form

$$R = \int_{g(X) \geq 0} f_X(x)dx = \int_{0}^{\infty} f_Z(z)dz = \int_{\infty}^{-\infty} f_Z(z)C_A(z)dz \tag{19}$$

in which $C_A(z)$ is the characteristic function of the common event A.

$$C_A(z) = \begin{cases} 1, z \in A \\ 0, z \notin A \end{cases} = \begin{cases} 1 \quad z \geq 0 \\ 0 \quad z < 0 \end{cases} \tag{20}$$

Now we introduce a fuzzy state assumption in the reliability analysis of structures. Let $\tilde{A}$ represent the fuzzy safety event, $\tilde{A}:\{Z = g(x) \tilde{\geq} 0\}$, and the membership function of $\tilde{A}$ is $\mu_{\tilde{A}}(z)$ (as shown in Fig. 8). From the analogy to Eq. (19), the structure reliability is computed as follows:

$$R = \int_{g(X) \tilde{\geq} 0} f_X(x)dx = \int_{-\infty}^{\infty} f_Z(z)\mu_{\tilde{A}}(z)dz \tag{21}$$

The above formulation is based on the probabilistic assumption together with the fuzzy state assumption, called a generalized fuzzy reliability model. Let the membership function $\mu_{\tilde{A}}(z)$ be expressed as

$$\mu_{\tilde{A}}(z) = \begin{cases} 1 & z \geq 0 \\ \mu'(z) & -c < z < 0 \\ 0 & z \leq -c \end{cases} \tag{22}$$

in which $\mu'(z)$ is utilized to describe the fuzziness, and the parameter $c \geq 0$. For the case $\mu_{\tilde{A}}(z) = 1$, the structure is in a safe state. If $\mu_{\tilde{A}}(z) = 0$, the structure is in a failure state. When $\mu_{\tilde{A}}(z)$ follows in the region [0, 1], the structure is in a safe state at degree $\mu_{\tilde{A}}(z)$. Let $c = 0$, then $\mu_{\tilde{A}}(z)$ reduces to $C_A(z)$, the characteristic function of the common event A. The assumption (22) means that the completely fail region is decreased as compared with Eqs. (19), (20). For $z < 0$, a structure may not be necessarily in a failure status. By using (22), Eq. (21) becomes

$$\tilde{R} = P(Z \tilde{\geq} 0) = \int_{-\infty}^{\infty} f_Z(z)\mu_{\tilde{A}}(z)dz$$
$$= \int_0^{\infty} f_Z(z)dz + \int_{-c}^{0} f_Z(z)\mu'(z)dz \tag{23}$$

The first term in the above equation gives the conventional reliability formula, and the second one is the added part by introducing the fuzzy assumption Eq. (22) in the region [$-c$, 0] (Dong 2000). Instead reducing the failure region as Eq. (22) does, an assumption that decreases the safe region can be made. In that case, the reliability becomes smaller as compared with the conventional one.

3.2 FORM - Computation procedure of reliability

The decomposition theorem indicates that any fuzzy set can be represented by a number of common sets. By applying this theorem, we can transfer a fuzzy set problem into a common set problem. For the computation of reliability, the λ truncated subset concept is utilized to transform the fuzzy reliability problem into conventional reliability problems.

For a fuzzy safety event $\tilde{A}:\{Z = g(x) \tilde{\geq} 0\}$, given threshold $\lambda \in [0, 1]$, the truncated subset corresponding to λ is defined as

$$A_\lambda = \{\mu_{\tilde{A}}(z) \geq \lambda\} = \{\mu_{\tilde{A}}(g(x)) \geq \lambda\} \tag{24}$$

Based on the conventional reliability theory, the safety probability of a system at the threshold λ can be expressed as:

$$\tilde{R}_\lambda = \int_{A_\lambda} f_X(x)dx \tag{25}$$

The system reliability is computed as (see Appendix)

$$\tilde{R} = \int_0^1 \tilde{R}_\lambda d\lambda \tag{26}$$

Eq. (25) is the point crucial to the reliability computation. If the limit state function is nonlinear, it is very difficult to directly calculate the integral of Eq. (25). Here the FORM (first-order reliability method) is used to solve this problem.

Define a generalized limit state function at the threshold λ as

$$G(X, \lambda) = g(X) - \mu_{\tilde{A}}^{-1}(\lambda) \tag{27}$$

The second term on the right side represents the inverse function of $\lambda = \mu_{\tilde{A}}(z)$. From Eqs. (24) and (27), the common set A_λ is equivalent to $G(x, \lambda) \geq 0$. Then the reliability expression (25) can be written as

$$R_\lambda = \int_{G(X,\lambda) \geq 0} f_X(x)dx \tag{28}$$

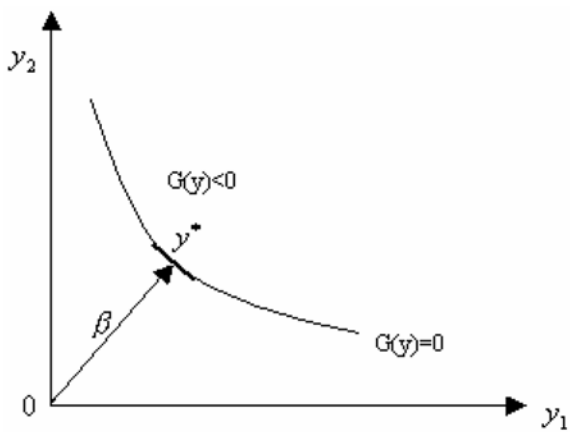


Fig. 9 Geometrical illustration of the reliability index β

Since the above equation has the same form as the conventional reliability computation formula (19), we can use the methods available in the reliability theory to assess the system reliability (Melchers 1999, Mahadevan 1997). Here the first-order reliability method (FORM) is used for calculating the reliability.

In the first-order reliability method, the random variable vector X is firstly transformed to Y, the vector of equivalent uncorrelated standard normal variables. The component reliability index is computed as $\beta = (y^{*T} \cdot y^*)^{1/2}$ in which y^* is the point in the limit state $G(Y) = 0$ at a minimum distance from the origin. A geometrical illustration for a limit state involving only two random variables is shown in Fig. 9. The failure probability is computed as $P(G \le 0) = \Phi(-\beta)$, where Φ is the cumulative distribution function of the standard normal variable.

In Fig. 9, y^* is referred to as the design point, or the most probable failure point (MPP), which can be found using the following iterative formula.

$$y_{i+1} = \left[y_i^T \alpha_i - \frac{G(y_i)}{|\nabla G(y_i)|} \right] \alpha_i \tag{29}$$

where $\nabla G(y_i)$ is the gradient vector of the limit state function at y_i, and α_i the unit vector normal to the limit state surface away from the origin.

By using FORM the reliability index $\tilde{\beta}_\lambda$ corresponding to the threshold λ is obtained and the reliability $\tilde{R}_\lambda$ is computed as $\tilde{R}_\lambda = \Phi(\tilde{\beta}_\lambda)$. A simulation method is utilized to complete the fuzzy reliability assessment procedure as follows:

(1) Generate the uniform distribution $\lambda_i (i = 1, 2, \ldots, n)$ in the region (0, 1).

(2) Compute the reliability $\tilde{R}_{\lambda_i}$ corresponding to the threshold λ_i by using the FORM.

(3) The system reliability is obtained as

$$\tilde{R} = \frac{1}{n} \sum_{i=1}^{n} \tilde{R}_{\lambda_i} \tag{30}$$

3.3 Numerical examples

A structure shown in Fig. 3 is used as an example again. We assume that the strength parameters

Table 4 Random variables

	Units	Mean	Standard deviation
X_t	MPa	1500	150
X_C	MPa	1500	150
Y_t	MPa	40	4.0
Y_C	MPa	246	24.6
S	MPa	68	6.8

are independent normally distributed random variables. Their distribution properties are shown in Table 4. The loads are $Nx = 300$ kN/m, $Ny = 250$ kN/m.

The Tsai-Wu failure criterion is used in the example, i.e.,

$$G = 1 - (F_{LL}\sigma_L^2 + F_{TT}\sigma_T^2 + F_{SS}\sigma_S^2 + 2F_{LT}\sigma_L\sigma_T + F_L\sigma_L + F_T\sigma_T) = 0 \tag{31}$$

$$F_{LL} = 1/(X_t X_C), F_{TT} = 1/(Y_t Y_C), F_{SS} = 1/S^2$$
$$F_L = 1/X_t - 1/X_C,\ F_T = 1/Y_t - 1/Y_C,\ F_{LT} = (-1/2)\sqrt{F_{LL}F_{TT}} \tag{32}$$

in which X_t, X_C represent the longitudinal tension and compression strengths, Y_t and Y_C the transverse tension and compression strengths, S the shear strength of a ply.

The following expression is used for the membership function of the fuzzy state

$$\mu_{\tilde{A}}(z) = \begin{cases} 1 & z \ge 0 \\ \dfrac{z+c}{c} & -c < z < 0 \\ 0 & z \le -c \end{cases} \tag{33}$$

$$c = |0.1\overline{G}| \tag{34}$$

in which $\overline{G}$ is computed from Eq. (31) by letting all the random variables take their mean values.

Table 5 The fuzzy reliability and the conventional reliability

Angle	Fuzzy	No fuzzy	Change
40	−9.2859	−9.5916	3.2%
45	−6.3891	−6.8533	6.8%
50	−3.1231	−3.3087	5.6%
55	−0.9109	−0.9630	5.4%
60	0.5483	0.5241	5.0%
65	1.4989	1.4348	4.2%
70	2.0609	1.9834	3.9%
75	2.0034	1.9288	3.9%
80	1.8083	1.7381	4.0%
85	1.6717	1.6106	3.8%
90	1.6324	1.5663	4.2%

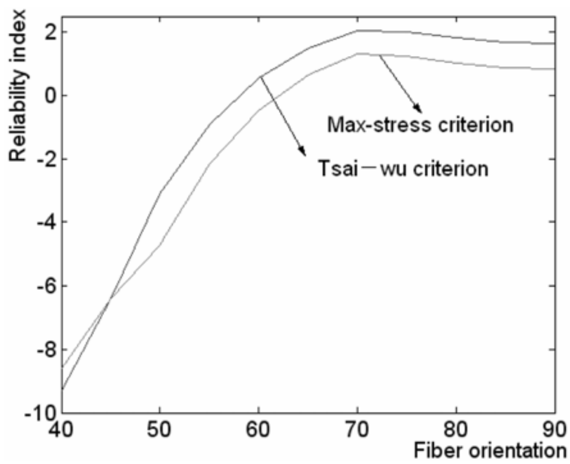


Fig. 10 Influence of different failure criteria on the reliability

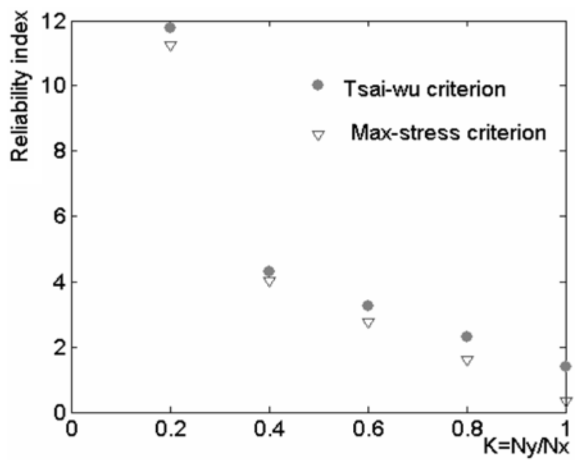


Fig. 11 Influence of load ratio on the reliability

3.4 Results and discussion

Using the generalized fuzzy reliability model and computation method mentioned above, the reliability index of the laminate is computed and listed in Table 5. The conventional reliability index is also listed in the Table. The fuzzy reliability and the conventional reliability have similar change tendency as θ. The comparison shows that the former is higher than the conventional one as expected from Eq. (23)

Fig. 10 shows the difference in the reliability by applying different failure criteria. The following integrated expression for the maximum stress failure criterion is used (Jeong and Shenoi 1998, Ochoa and Reddy 1992):

$$G = 1-(F_{LL}\sigma_L^2 + F_{TT}\sigma_T^2 + F_{SS}\sigma_S^2 + 2F_{LT}\sigma_L\sigma_T + F_L\sigma_L + F_T\sigma_T) = 0 \tag{35}$$

$$F_{LL} = 1/(X_t X_C), F_{TT} = 1/(Y_t Y_C),\ F_{SS} = 1/S^2$$
$$F_L = 1/X_t - 1/X_C,\ \ F_T = 1/Y_t - 1/Y_C,\ \ F_{LT} = (-1/2)F_L F_T \tag{36}$$

From Fig. 10 we can see that the selection of failure criteria greatly influences the reliability of composite materials. When the fiber orientation is less than 45 degrees, $\beta < -6$ and $P_f \approx 1$. So what we are concerned about is the region of $\theta > 45$ degrees in this case. We see that the results of Tsai-Wu failure criterion are larger than the maximum stress criterion. In the maximum stress criterion, stresses in the principal axis directions of a composite material must be less than the strengths in the same directions simultaneously, or the structure will fail. The maximum stress failure criterion is somewhat conservative, and the result of Tsai-Wu failure criterion is closer to the reality in general (Zhang *et al.* 1993). We also find from Fig. 10 that the reliability index reaches maximum when $\theta \approx 70^\circ$ regardless of the failure criterion used.

Let $\theta \approx 70^\circ$, and the fuzzy region parameter c in Eq. (34) be $c/\overline{G} = 0$, 0.1, 0.2 respectively, then the reliability index takes the values of 1.98, 2.06, and 2.14 correspondingly. The reliability index

Table 6 The optimum fiber orientation and reliability index for different load ratios K

Load ratio K	Optimum fiber orientation		Reliability index	
	Tsai-Wu criterion	Max-stress criterion	Tsai-Wu criterion	Max-stress criterion
0.0	5.564	5.175	29.288	29.360
0.2	41.000	42.088	11.783	11.252
0.4	50.051	49.945	4.319	4.038
0.6	57.958	58.367	3.240	2.767
0.8	69.415	69.402	2.318	1.599
1.0	89.248	89.731	1.406	0.318

increases almost proportionally as the parameter c becomes greater, owing to the fact that a larger parameter c means a smaller definitely failure region. Let a parabolic function $[(z+c)/c]^2$ replace the linear membership function $(z+c)/c$ in Eq. (33), a value of 2.03 for the reliability index was obtained, implying that the distribution form of the membership function has weak influence on the result.

By using the optimum toolbox in Matlab, we get the reliability index of the optimum structure at different load ratios $K = Ny/Nx$. The results are shown in Table 6 and Fig. 11.

It's found that under a certain load ratio, the optimum fiber orientation takes almost the same value whenever the Tsai-Wu failure criterion or the Maximum stress failure criterion is used, but quite different reliabilities are obtained. As the load ratio increases, the reliability index of the optimum structure drops sharply, and the difference between the two criteria becomes more significant. The reliability index from Tsai-Wu theory is larger than that of maximum stress failure criterion as mentioned above. For deterministic analyses, the optimum fiber orientation for $K = 0$ and $K = 1$ should be 0 degree and 90 degrees, respectively. Table 6 shows that the optimum fiber orientations from the present model are different from the results of deterministic cases for $K = 0$ and $K = 1$.

Figs. 12-15 show the influence of the mean values of strength parameters on reliability. The fiber

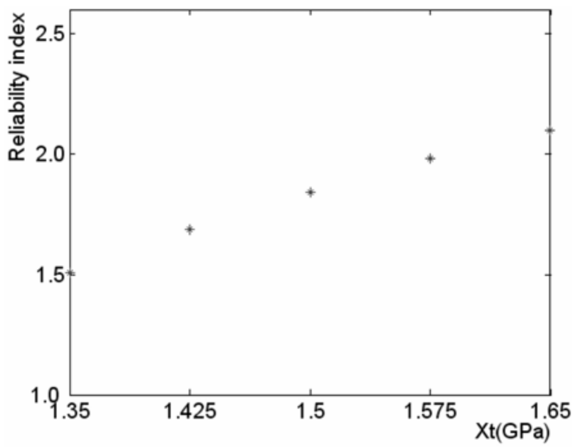


Fig. 12 Influence of the mean value of Xt

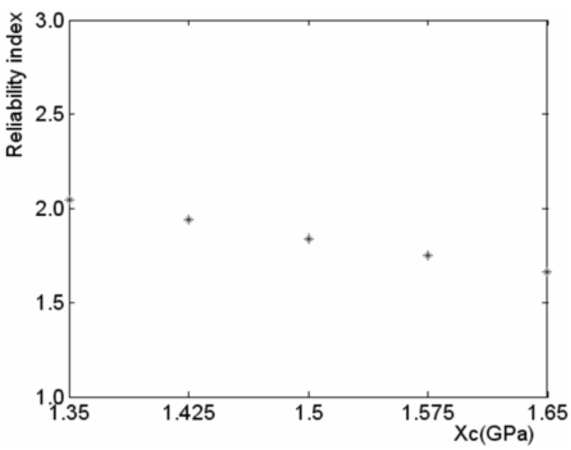


Fig. 13 Influence of the mean value of Xc

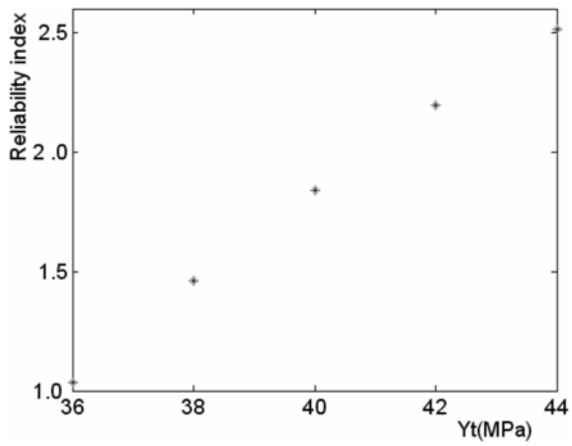


Fig. 14 Influence of the mean value of *Yt*

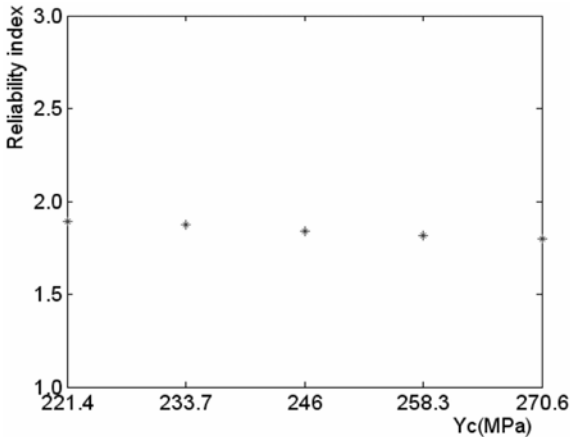


Fig. 15 Influence of the mean value of *Yc*

orientation is fixed at 75 degrees, and the Tsai-Wu criterion is utilized in the calculation. These figures show that *Xt* and *Yt* have positive contributions to fuzzy reliability index β. As *Xc* increases, β decreases gently, while *Yc* has little effect on the reliability. In a laminated composite, *Yt* has the smallest value, so a laminate's strength may greatly depend on *Yt*. Fig. 14 shows that the sensitivity of β to *Yt* is largest as expected.

The model proposed in Section 2 can also be regarded as a kind of generalization of the concept of "safety" as mentioned previously, with each strength fuzziness effect being treated individually. The generalized fuzzy reliability model developed in this section combines the effects in an integrated form. It not only extended the "safety" concept by introducing the fuzzy state assumption to replace the conventional binary-state one, it is feasible to treat other fuzzy information, thus providing an effective and synthetic method to evaluate the reliability of a system with different types of uncertainties.

4. Conclusions

The fuzzy concept is first introduced to describe the strength in the reliability computation of composite laminates. Then, the fuzzy state assumption is used to substitute the two-value state assumption to construct a generalized fuzzy reliability model. In the first model, the strength parameters are considered as fuzzy variables and the loads as random ones. The model can provide an alternative method assessing structural reliability in the case where the distribution pattern of strength parameters is hard to determine. Furthermore, treating the strengths as "fuzzy" variables means a "fuzzy" judgment is made for the structure state. This can be regarded as a special case of the fuzzy state assumption.

In the second model, a generalized fuzzy reliability model (GFRM) is established on the basis of the fuzzy state assumption together with the probabilistic assumption, which can effectively evaluate the reliability of a system with different types of uncertainty factors.

In the analysis of fuzzy reliability, how to convert the fuzzy reliability problem into a conventional reliability problem is the key point for computation. The truncated subset concept and the decomposition theorem in fuzzy mathematics are applied to transforming the fuzzy reliability problem into a set of conventional reliability ones. Numerical examples are given to illustrate the validity and efficiency of the method proposed.

The GFRM provides a method that can evaluate the reliability of composite materials more naturally and synthetically, and makes an offer to the theoretical support in fuzzy reliability optimum design analyses of composite materials that is under way by the authors.

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Notation

A	: common set, common event
a, b, m	: fuzzy strength parameters
$\tilde{A}$	: fuzzy set, fuzzy event
A_λ	: λ truncated subset to $\tilde{A}$
a_λ, b_λ	: end points of the region of λ truncated subset
$C_A(u)$	: characteristic function of A
$f_S(s)$	: probability density function of stress S
$f_{r_\lambda}(r)$	: probability density function of r_λ
$f_X(x)$	: probability density function of X
$f_Z(z)$	: probability density function of Z
$g(X)$	: performance function or limit state function
$G(X, \lambda)$	: generalized limit state function
P_f	: failure probability
$\tilde{r}$	: fuzzy strength variable
$\tilde{r}_\lambda$	: λ truncated subset to $\tilde{r}$
$\tilde{R}$	: the fuzzy safe probability of a structure
$\tilde{R}_\lambda$	: the safe probability at the threshold λ
S	: the shear strength of a ply
X	: a vector of basic random variables
X_t, X_C	: the longitudinal tension and compression strengths of a ply
Y_t, Y_C	: the transverse tension and compression strengths of a ply
$\mu_{\tilde{A}}(u)$	: the membership function of $\tilde{A}$
λ	: threshold
$\mu_{\tilde{r}}(r)$	: the membership function of $\tilde{r}$
$\varphi(s, \lambda)$	: a function related to the stress s and the threshold λ defined in 2.2
θ	: fiber orientation
β	: reliability index

Appendix

The principle based on which Eq. (10) holds is explained in this appendix.

Let X denote a random variable, the probability density function of X is $f(x)$, the membership function of X belonging to the fuzzy event $\tilde{A}$ is $\mu_{\tilde{A}}(x)$, then the probability of the fuzzy event $\tilde{A}$ can be expressed as

$$P(\tilde{A}) = \int_{-\infty}^{+\infty} \mu_{\tilde{A}}(x) f(x) dx \tag{A1}$$

If $\tilde{A}$ is a convex fuzzy set defined in the real number field, then its λ truncated subset A_λ, which is determined by the inequality $\mu_{\tilde{A}}(x) \geq \lambda$, is a common set:

$$A_\lambda = [a_\lambda, b_\lambda] \tag{A2}$$

A_λ is a region on a real number axis. The region $[a_\lambda, b_\lambda]$ may be an open one, i.e., $a_\lambda = -\infty$ or $b_\lambda = +\infty$. The feature function of the common event A_λ is written as

$$C_{A_\lambda}(x) = \begin{cases} 1 & x \in A_\lambda \\ 0 & x \notin A_\lambda \end{cases} \tag{A3}$$

So, the probability of the λ truncated subset A_λ of $\tilde{A}$ is computed as

$$P(A_\lambda) = \int_{-\infty}^{+\infty} C_{A_\lambda}(x) f(x) dx = \int_{a_\lambda}^{b_\lambda} f(x) dx \tag{A4}$$

Eq. (A4) is the probability of fuzzy event $\tilde{A}$ when the threshold is λ.

Suppose that there exists a point c at which $\mu_{\tilde{A}}(c) = 1$ (c is a real number), and $\mu_{\tilde{A}}(x)$ is strictly monotonic and derivable in the extents $[-\infty, c]$ and $[c, +\infty]$, then the probability of the fuzzy event $\tilde{A}$ expressed by Eq. (A1) can be rewritten as

$$P(\tilde{A}) = \int_0^1 P(A_\lambda) d\lambda = \int_0^1 [\int_{a_\lambda}^{b_\lambda} f(x) dx] d\lambda \tag{A5}$$

To prove Eq. (A5), the following two propositions need to be clarified first.

Proposition 1. If the strictly monotonic function $\mu(x)$ is derivable and $\mu'(x)$ is not equal to 0 in the region $[c, +\infty]$, and $\mu(c) = 1, \mu(+\infty) = 0$; function $f(x)$ is continuous in the region $[c, +\infty]$, and $\int_c^{+\infty} f(x) dx$ is convergent, then

$$\int_c^{+\infty} f(x) \mu(x) dx = \int_0^1 [\int_c^{\mu^{-1}(\lambda)} f(x) dx] d\lambda \tag{A6}$$

where $\mu^{-1}(\lambda)$ is the inverse function of $\lambda = \mu(x)$.

Proof: Since the strictly monotonic function $\mu(x)$ is derivable, the inverse function of $\lambda = \mu(x)$, $x = \mu^{-1}(\lambda)$, must exist and can be derived:

$$\frac{d}{d\lambda}[\mu^{-1}(\lambda)] = \frac{1}{\mu'(x)} = \frac{1}{\mu'[\mu^{-1}(\lambda)]} \tag{A7}$$

$$dx = \frac{1}{\mu'[\mu^{-1}(\lambda)]} d\lambda \tag{A8}$$

Noting that $\lambda = 1$ when $x = c$ and $\lambda = 0$ when $x = +\infty$, we have

$$\int_c^{+\infty} f(x) \mu(x) dx = \int_1^0 \lambda \cdot f[\mu^{-1}(\lambda)] \cdot \frac{1}{\mu'[\mu^{-1}(\lambda)]} d\lambda \tag{A9}$$

Consider that

$$\frac{d}{d\lambda}\left[\int_c^{\mu^{-1}(\lambda)} f(x)dx\right] = f[\mu^{-1}(\lambda)] \cdot \frac{d[\mu^{-1}(\lambda)]}{d\lambda} = \frac{f[\mu^{-1}(\lambda)]}{\mu'[\mu^{-1}(\lambda)]} \tag{A10}$$

Applying the method of integration by parts to Eq. (A9), one obtains

$$\begin{aligned}
\int_c^{+\infty} f(x)\mu(x)dx &= \int_1^0 \lambda \cdot d\left[\int_c^{\mu^{-1}(\lambda)} f(x)dx\right] \\
&= \lambda \cdot \int_c^{\mu^{-1}(\lambda)} f(x)dx \Big|_1^0 - \int_1^0 \left[\int_c^{\mu^{-1}(\lambda)} f(x)dx\right]d\lambda \\
&= 0 \cdot \int_c^{+\infty} f(x)dx - 1 \cdot \int_c^{\mu^{-1}(1)} f(x)dx + \int_0^1 \left[\int_c^{\mu^{-1}(\lambda)} f(x)dx\right]d\lambda \\
&= \int_0^1 \left[\int_c^{\mu^{-1}(\lambda)} f(x)dx\right]d\lambda
\end{aligned} \tag{A11}$$

Proposition 2. If the strictly monotonic function $\mu(x)$ is derivable and $\mu'(x)$ is not equal to 0 in the region $[-\infty, c]$, and $\mu(c) = 1, \mu(-\infty) = 0$; function $f(x)$ is continuous in the region $[-\infty, c]$ and $\int_{-\infty}^{c} f(x)dx$ is convergent, then

$$\int_{-\infty}^{c} f(x)\mu(x)dx = \int_0^1 \left[\int_{\mu^{-1}(\lambda)}^{c} f(x)dx\right]d\lambda \tag{A12}$$

Proof of proposition 2 is similar to that of proposition 1. Having clarified propositions 1 and 2, Eq. (A5) can be proved as follows:

$$\int_{-\infty}^{+\infty} \mu(x)f(x)dx = \int_{-\infty}^{c} \mu(x)f(x)dx + \int_c^{+\infty} \mu(x)f(x)dx \tag{A13}$$

For $a_\lambda \in [-\infty, c]$, it is always possible to find a λ so that $a_\lambda = \mu^{-1}(\lambda)$. By the use of proposition 2, one gets

$$\int_{-\infty}^{c} f(x)\mu(x)dx = \int_0^1 \left[\int_{a_\lambda}^{c} f(x)dx\right]d\lambda \tag{A14}$$

And for $b_\lambda \in [c, +\infty]$, we can find the relation $b_\lambda = \mu^{-1}(\lambda)$. From proposition 1, we have

$$\int_c^{+\infty} f(x)\mu(x)dx = \int_0^1 \left[\int_c^{b_\lambda} f(x)dx\right]d\lambda \tag{A15}$$

So

$$\begin{aligned}
\int_{-\infty}^{+\infty} \mu(x)f(x)dx &= \int_0^1 \left[\int_{a_\lambda}^{c} f(x)dx\right]d\lambda + \int_0^1 \left[\int_c^{b_\lambda} f(x)dx\right]d\lambda \\
&= \int_0^1 \left[\int_{a_\lambda}^{c} f(x)dx + \int_c^{b_\lambda} f(x)dx\right]d\lambda \\
&= \int_0^1 \left[\int_{a_\lambda}^{b_\lambda} f(x)dx\right]d\lambda
\end{aligned} \tag{A16}$$

Then:

$$P(\tilde{A}) = \int \mu_{\tilde{A}}(x)f(x)dx = \int_0^1 \left[\int_{a_\lambda}^{b_\lambda} f(x)dx\right]d\lambda \tag{A17}$$