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Large deflection behavior of a flexible circular cantilever arc device subjected to inward or outward polar force

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Abstract. The problem of very large deflection of a circular cantilever arc device subjected to inward or outward polar force is studied. An exact elliptic integral solution is derived for the two cases and the results are checked using large displacement finite element analysis via the ANSYS package by performing a new novel modeling simulation technique for this problem. Excellent agreements have been obtained between the exact analytical solution and the numerical approach. From this study, a design chart for engineers is developed to predict the required value for the inward polar force for the device to switch on for a given angle forming the circular arc (θ_o). This study has several interesting applications in mechanical engineering, integrated circuit technology, nanotechnology and especially in microelectromechanical systems (MEMs) such as a MEM circular device switch subjected to attractive or repulsive magnetic forces due to the attachments of two magnetic poles at the fixed and at the free end of the circular cantilever arc switch device.

Keywords: ANSYS; circular cantilever arc; elastica; elliptic integral; polar force.

1. Introduction

Very large deflection problems of circular cantilever arc devices have several interesting mechanical engineering applications, such as precision mechanics in aerospace products, arms of robotics, scientific instrumentation and in the new emerging fields of precision engineering, such as integrated circuit technology, nanotechnology and especially microelectromechanical systems (MEMS). From mechanical point of view it becomes very important to handle accurately the *very large deflection* behavior of these mechanical devices. The most accurate method (analytical method) for predicting *very large deflection* behavior of highly flexible beams is the elastica analysis (Barten 1945, Bisshopp and Drucker 1945, Timoshenko and Gere 1961, Frisch-Fay 1962, Lau 1974, Mattiasson 1981, Chucheepsakul *et al.* 1994, 1995, 1996, Ohtsuki and Tsurumi 1996, Hartono 1997, Wang *et al.* 1998, Chucheepsakul *et al.* 1999, Hartono 2000, Tse and Lung 2000, Hartono 2001, Aristizabal-Ochoa 2004, Chucheepsakul and Phungpaigram 2004), which is based on the solution of the highly nonlinear governing equation using elliptic integrals.

Apparently the first one who studied the problem of large deflection of beams subjected to polar forces was (Willems 1966) as well as (Anderson and Done 1971). They considered a cantilever beam subjected to a polar force as a way of partial simulation to a nonconservative system under a

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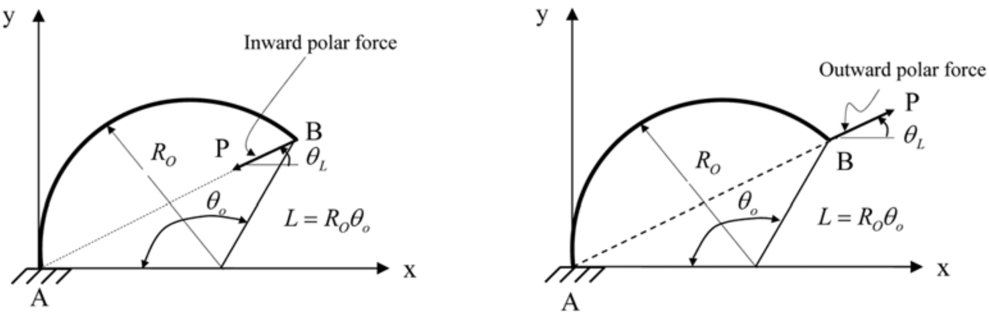


(a) Undeformed configuration of a circular cantilever arc subjected to inward Polar Force

(b) Undeformed configuration of a circular cantilever arc subjected to outward Polar Force

Fig. 1 Undeformed configuration of a circular cantilever arc subjected to inward or outward polar force

tangential follower force. Farshad (1973) has discussed the loading behavior of a polar force as a general conservative loading in contrast with a nonconservative loading. While Mladenov and Sugiyama (1983) have solved the problem of a straight cantilever beam subjected to inward or outward polar force directed toward a point located along the axis of the undeformed configuration of the beam. They solved the problem analytically using elliptic integrals.

From this literature survey, it can be seen that there are a little work on the very large deflection problems of beams subjected to polar forces. The aim of this paper is to study the behavior of a circular cantilever arc subjected to inward or outward polar force of constant magnitude directed always toward a pole A as shown in Fig. 1. An exact elliptic integral solution is derived for the two cases and the results are checked using large displacement finite element analysis of ANSYS package by performing a new novel modeling simulation technique for this problem. Excellent agreements have been obtained between the exact analytical solutions and the numerical approach.

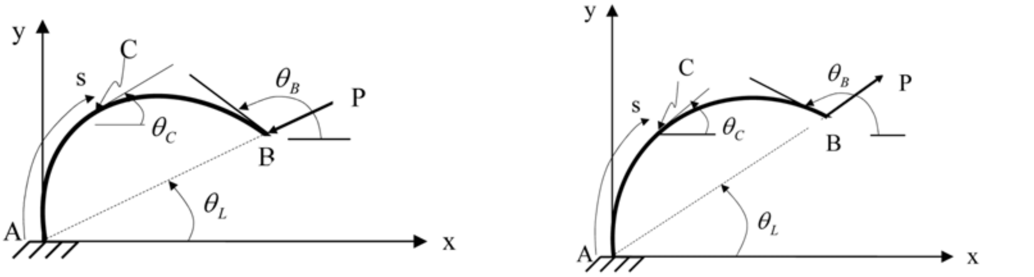
2. Exact analytical solution using elliptic integrals

A circular cantilever arc of constant radius (R_o), angle forming the circular arc (θ_o), length ($L = R_o\theta_o$) and flexural rigidity (EI) is subjected to two cases of loadings. In the first case, the cantilever device is subjected to inward polar force (P) as shown in Figs. 1(a) and 2(a). The direction of inward polar force is always towards point A.

In the second case, the cantilever device is subjected to outward polar force (P) as shown in Figs. 1(b) and 2(b). The outward polar force is always directed outwards from point A. Thus, in the two loading cases, bending moment at the fixed point of the cantilever device (point A) is always equal to zero.

Unless, specially stated otherwise, the following main assumptions consistent with elastica theory, will hold throughout the investigations of the problem at hand:

1. The structural material is linearly elastic.
2. The bending moment is proportional to the curvature according to Euler-Bernoulli law.
3. The cantilever beam is inextensible.



(a) Deformed configuration of a circular cantilever arc subjected to inward polar force

(b) Deformed configuration of a circular cantilever arc subjected to outward polar force

Fig. 2 Deformed configuration of a circular cantilever arc subjected to subjected to inward or outward polar force

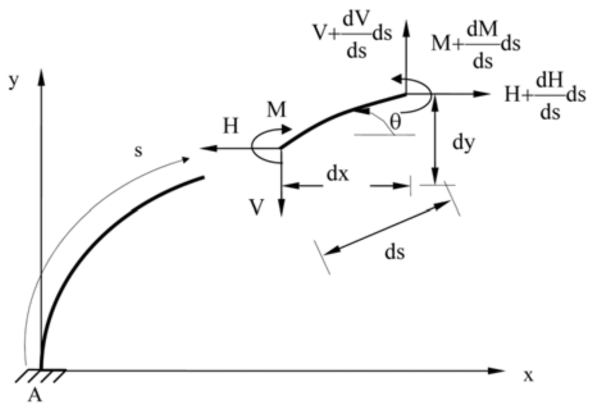


Fig. 3 Infinitesimal element (*ds*) of a circular cantilever arc subjected to internal forces

To this end, consider an infinitesimal element of length (*ds*) subjected to the forces and moments shown in Fig. 3. Based on geometrical relations, equilibrium of moment and forces and moment-curvature relationship (Euler-Bernoulli moment-curvature relationship), the four simultaneous nonlinear first-order ordinary differential equations governing the elastica behavior of the circular cantilever arc device subjected to inward or outward polar forces can be written as:

$$\frac{dx}{ds} = \cos\theta \tag{1}$$

$$\frac{dy}{ds} = \sin\theta \tag{2}$$

$$\frac{d\theta}{ds} = -\frac{1}{Ro} + \frac{M}{EI} \tag{3}$$

$$\frac{dM}{ds} = P\sin(\theta_L - \theta) \quad \text{due to inward polar force P} \tag{4a}$$

$$\frac{dM}{ds} = -P\sin(\theta_L - \theta) \quad \text{due to outward polar force P} \tag{4b}$$

It should be mentioned that the variables x, y, θ and M are functions of the deflected curved coordinate s.

2.1 Elliptic integral solution for case of inward polar force (P)

Now, differentiating Eq. (3) with respect to s, yields

$$\frac{d^2\theta}{ds^2} = \frac{1}{EI}\frac{dM}{ds} \tag{5}$$

Substituting Eq. (4a) into Eq. (5) results

$$\frac{d^2\theta}{ds^2} = \frac{\alpha^2}{L^2}\sin(\theta_L - \theta) \tag{6}$$

where $\alpha^2 = \dfrac{PL^2}{EI}$

Integrating with respect to θ results in

$$\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2 = \frac{\alpha^2}{L^2}\cos(\theta_L - \theta) + C_1 \tag{7}$$

The constant of integration C_1 can be obtained by applying the boundary condition using the fact that bending moment at the fixed point (point A) of the cantilever is always equal to zero (at $s = 0$, $\theta = \pi/2$ and $d\theta/ds = -1/R_o$) and after substituting this boundary condition into Eq. (7), then

$$\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2 = \frac{\alpha^2}{L^2}\cos(\theta_L - \theta) + \frac{1}{2R_o^2} - \frac{\alpha^2}{L^2}\sin\theta_L \tag{8}$$

Let $\phi = \theta_L - \theta$, then $\dfrac{d\phi}{ds} = -\dfrac{d\theta}{ds}$. Eq. (8) becomes

$$\left(\frac{d\phi}{ds}\right)^2 = \frac{2\alpha^2}{L^2}\left[\cos\phi + \frac{L^2}{2\alpha^2R_o^2} - \sin\theta_L\right] \tag{9}$$

or

$$\left(\frac{d\phi}{ds}\right)^2 = \frac{2\alpha^2}{L^2}[\cos\phi - \cos\psi] \tag{10}$$

where

$$\cos\psi = -\left[\frac{L^2}{2\alpha^2R_o^2} - \sin\theta_L\right] \tag{11}$$

Thus, separating the variables, taking the appropriate sign and integrating between 0 and s:

$$s = \int_{\phi_A}^{\phi_C} \frac{L d\phi}{\sqrt{2}\alpha\sqrt{\cos\phi - \cos\psi}} \tag{12}$$

Using the usual substitutions of the elastica method are now introduced:

$$k = \sin\frac{\psi}{2} \tag{13}$$

$$k\sin\gamma = \sin\frac{\phi}{2} \tag{14}$$

Hence

$$\frac{\alpha s}{L} = F(k, \gamma_C) - F(k, \gamma_A) \tag{15}$$

Where $F(k, \gamma)$ is incomplete elliptic integral of the first kind. Now, if the loading conditions and the mechanical characteristics of the circular cantilever arc are known, then the load parameter α can easily be determined, but the values of s, k, γ_A and γ_C are still unknown. To this end, Eq. (15) has to be applied at the free end of the cantilever (point B in Fig. 2a) and the following expression is obtained:

$$\alpha = F(k, \gamma_B) - F(k, \gamma_A) \tag{16}$$

Eq. (16) can be solved for the load parameter α by assuming a value for the angle of rotation at the free end of the cantilever θ_B. Before calculating α, one must get the value of the inclination angle θ_L of the inward polar force P from the boundary condition at point B using the fact that bending moment at the free point (point B) of the cantilever is always equal to zero, then at $s = L$, $\theta = \theta_B$ and $d\theta/ds = -1/R_o$, and after substituting this end condition into Eq. (8), then

$$\cos(\theta_L - \theta_B) = \sin\theta_L \tag{17}$$

or

$$\cos\phi_B = \sin\theta_L \tag{18}$$

Now, after calculating the load parameter α, it is now possible to obtain the expressions suitable for the calculation of the x and y coordinates of elastic curve of the circular cantilever arc using Eqs. (1) and (2):

$$x_C(\gamma_C) = \frac{L}{\alpha}\{[F(k, \gamma_C) - F(k, \gamma_A) - 2E(k, \gamma_C) + 2E(k, \gamma_A)]\cos\theta_L + [2k\cos\gamma_C - 2k\cos\gamma_A]\sin\theta_L\} \tag{19}$$

$$y_C(\gamma_C) = \frac{L}{\alpha}\{[F(k, \gamma_C) - F(k, \gamma_A) - 2E(k, \gamma_C) + 2E(k, \gamma_A)]\sin\theta_L - [2k\cos\gamma_C - 2k\cos\gamma_A]\cos\theta_L\} \tag{20}$$

Where $E(k, \gamma)$ is incomplete elliptic integral of the second kind. It should be mentioned that in the general case in this work, the absolute value of the right-hand side of Eq. (11) is greater than 1 if

$$\left[\frac{L^2}{2\alpha^2 R_o^2}\right] \geq 1 + \sin\theta_L \tag{21}$$

In this case k must be obtained directly. In fact, with a simple trigonometric substitutions from Eqs. (11) and (13) it is possible to obtain

$$k = \sqrt{\frac{1}{2}}\sqrt{1 + \left[\frac{L^2}{2\alpha^2 R_o^2} - \sin\theta_L\right]} \tag{22}$$

From Eq. (22) it follows that k is always a positive quantity, but is greater than 1 if condition (Eq. 21) holds true. In this case, as the elliptic integrals are defined only for values of k smaller than 1, it is necessary to introduce the following substitutions as follows:

$$k^* = \frac{1}{k}, \quad k\sin\gamma = \sin\gamma^* \tag{23}$$

The new expressions of Eqs. (15), (16), (19) and (20) can be easily obtained, it follows:

$$\frac{\alpha s}{L} = k^*[F(k^*, \gamma_C^*) - F(k^*, \gamma_A^*)] \tag{24}$$

$$\alpha = k^*[F(k^*, \gamma_B^*) - F(k^*, \gamma_A^*)] \tag{25}$$

$$x_C(\gamma_C^*) = \frac{L}{\alpha}\left\{\left[k^*F(k^*, \gamma_C^*) - k^*F(k^*, \gamma_A^*) - 2\frac{E(k*, \gamma_C^*)}{k^*} + 2\frac{1-k^{*2}}{k^*}F(k^*, \gamma_a^*) + 2\frac{E(k*, \gamma_C^*)}{k^*} - 2\frac{1-k^{*2}}{k^*}F(k^*, \gamma_A^*)\right]\cos\theta_L + \left[2\frac{1}{k^*}\cos\gamma_C - 2\frac{1}{k^*}\cos\gamma_A\right]\sin\theta_L\right\} \tag{26}$$

$$y_C(\gamma_C^*) = \frac{L}{\alpha}\left\{\left[k^*F(k^*, \gamma_C^*) - k^*F(k^*, \gamma_A^*) - 2\frac{E(k*, \gamma_C^*)}{k^*} + 2\frac{1-k^{*2}}{k^*}F(k^*, \gamma_A^*) + 2\frac{E(k*, \gamma_C^*)}{k^*} - 2\frac{1-k^{*2}}{k^*}F(k^*, \gamma_A^*)\right]\sin\theta_L + \left[2\frac{1}{k^*}\cos\gamma_C - 2\frac{1}{k^*}\cos\gamma_A\right]\cos\theta_L\right\} \tag{27}$$

Once α is determined, the horizontal and vertical coordinates x and y along the curved coordinate can be obtained from Eqs. (19) to (20) for k less than or equal to 1 and Eqs. (26) and (27) for k greater than to 1.

2.2 Elliptic integral solution for case of outward polar force (P)

Following the same procedure of elliptic formulations described above for the case of cantilever arc subjected to inward polar force, the elliptic integral solutions for the case of outward polar force are:

$$\alpha^2 = \frac{PL^2}{EI} \tag{28}$$

$$\phi = \theta_L - \theta - \pi \tag{29}$$

$$\cos\psi = -\left[\frac{L^2}{2\alpha^2 R_o^2} + \sin\theta_L\right] \tag{30}$$

$$s = \int_{\phi_A}^{\phi_C} \frac{L\,d\phi}{\sqrt{2}\,\alpha\sqrt{\cos\phi - \cos\psi}} \tag{31}$$

$$k = \sin\frac{\psi}{2} \tag{32}$$

$$k\sin\gamma = \sin\frac{\phi}{2} \tag{33}$$

$$\frac{\alpha s}{L} = F(k, \gamma_C) - F(k, \gamma_A) \tag{34}$$

$$\alpha = F(k, \gamma_B) - F(k, \gamma_A) \tag{35}$$

$$\cos(\phi_B - \pi) = \sin\theta_L \tag{36}$$

or

$$-\cos\phi_B = \sin\theta_L \tag{37}$$

$$x_C(\gamma_C) = \frac{L}{\alpha}\{[F(k,\gamma_C) - F(k,\gamma_A) - 2E(k,\gamma_C) + 2E(k,\gamma_A)]\cos\theta_L + [2k\cos\gamma_C - 2k\cos\gamma_A]\sin\theta_L\} \tag{38}$$

$$y_C(\gamma_C) = \frac{L}{\alpha}\{[F(k,\gamma_C) - F(k,\gamma_A) - 2E(k,\gamma_C) + 2E(k,\gamma_A)]\sin\theta_L - [2k\cos\gamma_C - 2k\cos\gamma_A]\cos\theta_L\} \tag{39}$$

It should be mentioned that in the general case in this work, the absolute value of the right-hand side of Eq. (30) is greater than 1 if

$$\left[\frac{L^2}{2\alpha^2 R_o^2}\right] \geq 1 - \sin\theta_L \tag{40}$$

In this case k must be obtained directly. In fact, with a simple trigonometric substitutions from Eqs. (30) and (32) it is possible to obtain

$$k = \sqrt{\frac{1}{2}}\sqrt{1 + \left[\frac{L^2}{2\alpha^2 R_o^2} + \sin\theta_L\right]} \tag{41}$$

From Eq. (41) it follows that k is always real, but is greater than 1 if condition (Eq. 41) holds true. In this case, as the elliptic integrals are defined only for values of k smaller than 1, it is necessary to introduce the following substitutions as follows:

$$k^* = \frac{1}{k^*}, \quad k\sin\gamma = \sin\gamma^* \tag{42}$$

The new expressions of Eqs. (34), (35), (38) and (39) can be easily obtained, it follows:

$$\frac{\alpha s}{L} = k^*[F(k^*, \gamma_C^*) - F(k^*, \gamma_A^*)] \tag{43}$$

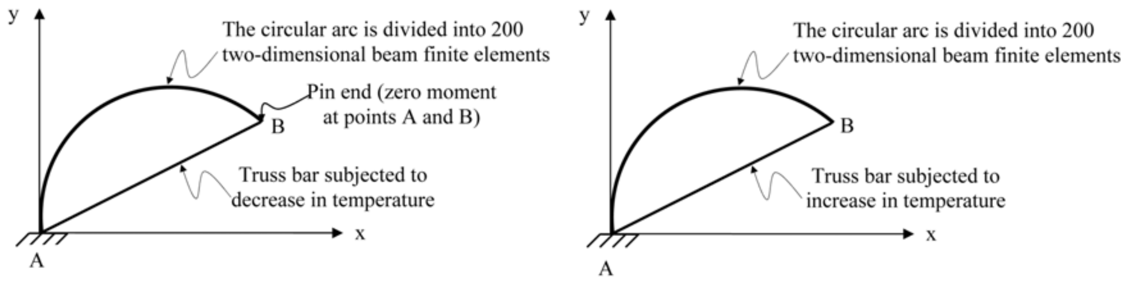
$$\alpha = k^*[F(k^*, \gamma_B^*) - F(k^*, \gamma_A^*)] \tag{44}$$

$$x_C(\gamma_C^*) = \frac{L}{\alpha}\Bigg\{\Bigg[k^*F(k^*, \gamma_C^*) - k^*F(k^*, \gamma_A^*) - 2\frac{E(k_*, \gamma_C^*)}{k^*} + 2\frac{1-k^{*2}}{k^*}F(k^*, \gamma_a^*) + 2\frac{E(k_*, \gamma_C^*)}{k^*} - 2\frac{1-k^{*2}}{k^*}F(k^*, \gamma_A^*)\Bigg]\cos\theta_L + \Bigg[2\frac{1}{k^*}\cos\gamma_C - 2\frac{1}{k^*}\cos\gamma_A\Bigg]\sin\theta_L\Bigg\} \tag{45}$$

$$y_C(\gamma_C^*) = \frac{L}{\alpha}\Bigg\{\Bigg[k^*F(k^*, \gamma_C^*) - k^*F(k^*, \gamma_A^*) - 2\frac{E(k_*, \gamma_C^*)}{k^*} + 2\frac{1-k^{*2}}{k^*}F(k^*, \gamma_A^*) + 2\frac{E(k_*, \gamma_C^*)}{k^*} - 2\frac{1-k^{*2}}{k^*}F(k^*, \gamma_A^*)\Bigg]\sin\theta_L + \Bigg[2\frac{1}{k^*}\cos\gamma_C - 2\frac{1}{k^*}\cos\gamma_A\Bigg]\cos\theta_L\Bigg\} \tag{46}$$

3. Large displacement finite element analysis using ANSYS Package

As a check for this problem, a large displacement finite element analysis using the multi-purpose computer program **ANSYS** is performed by dividing the circular cantilever arc into two-hundred beam finite elements with (201) nodes. A numerical instability problem is developed while performing the first run for this problem, due to the fact that bending moment at the fixed point of the circular cantilever arc is always equal to zero (the inward or outward polar force is always pass through this point). To overcome this instability problem, a new novel simulation is developed by using a truss element connecting the fixed point and the tip point of the circular cantilever arc as shown in Fig. 4.



(a) ANSYS modeling for a circular cantilever arc subjected to inward polar force

(b) ANSYS modeling for a circular cantilever arc subjected to outward polar force

Fig. 4 ANSYS modeling for a circular cantilever arc subjected to inward or outward polar force using fictitious truss bar under increase or decrease in temperature

The function of the truss element is to simulate the presence of the inward or outward polar force and to insure that the polar force is always directed toward or outward the fixed point of the cantilever (point A). It should be mentioned that another function of using the truss element is to insure zero bending moment at points A and B. Now, if the truss element is subjected to a cooling process (drop in temperature), then it simulates the presence of inward polar force and if the truss is subjected to a heating process (increase in temperature), then it simulates the presence of outward polar force.

To this end, the following fundamental parameters are introduced to the program:

1) Number of loading increments = 100
2) Maximum number of iterations during each increment = 50
3) Number of iterations before updating tangent stiffness matrix = 25
4) Method of updating tangent stiffness matrix = iterative
5) Tolerance for displacement convergence criterion = $1*10^{-5}$
6) Solution strategy = Modified Newton-Raphson iteration method

4. Results and discussion

Fig. 5 shows the deformed configurations for a straight vertical cantilever subjected to inward polar force (P). From this figure it can be seen that the results of the elliptic integral solution are in excellent agreements with finite elements using ANSYS. It can be seen from Fig. 5 that simulation of the truss element used in finite element fails to give results when the free end of the straight

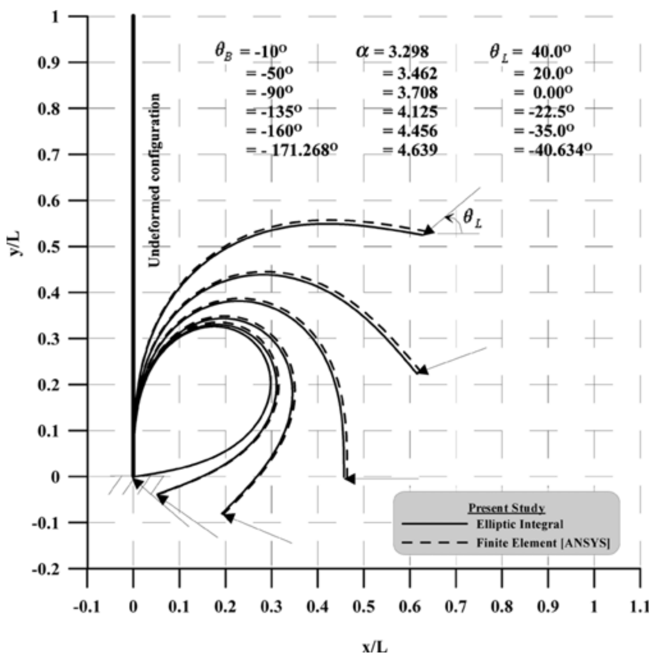


Fig. 5 Deformed configurations of a straight vertical cantilever subjected to inward polar force with different θ_B values

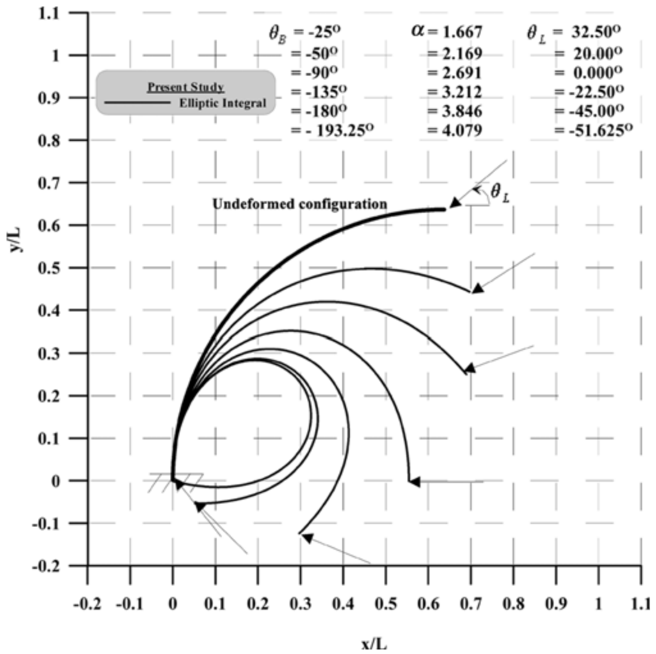


Fig. 6 Deformed configurations of a quarter circular cantilever arc subjected to inward polar force with different θ_B values

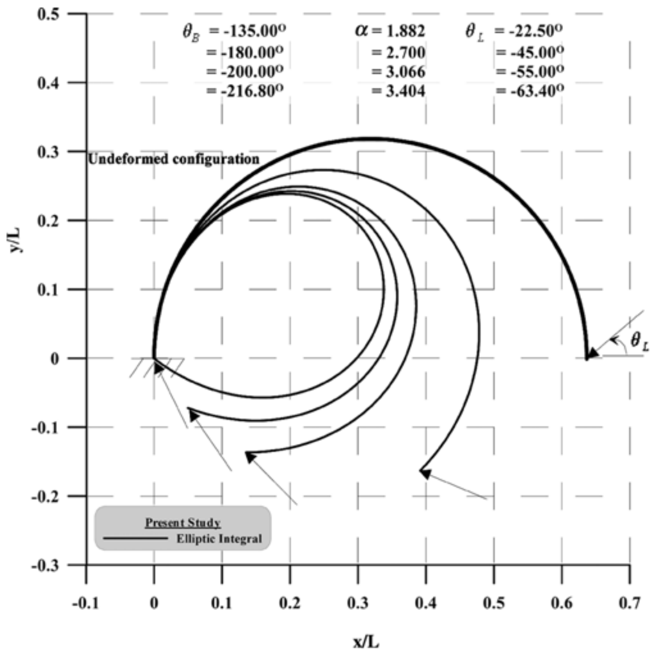


Fig. 7 Deformed configurations of a semi-circular cantilever arc subjected to inward polar force with different θ_B values

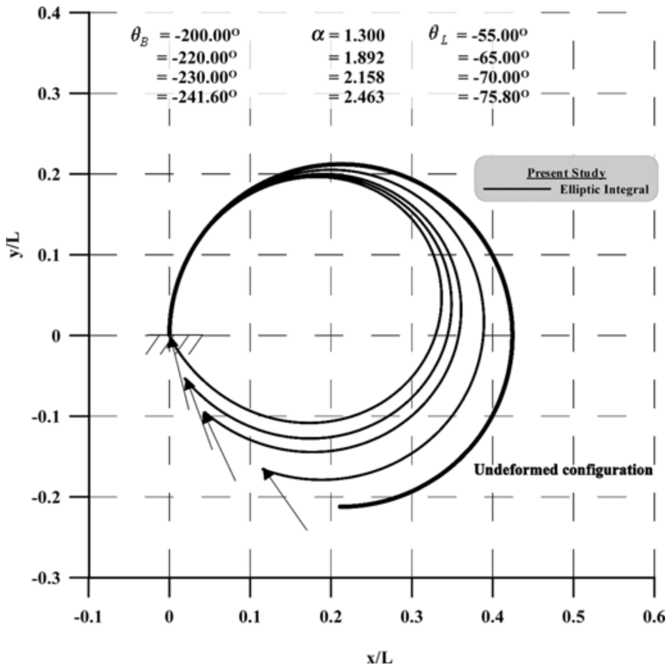


Fig. 8 Deformed configurations of a three-quarter circular cantilever arc subjected to inward polar force with different θ_B values

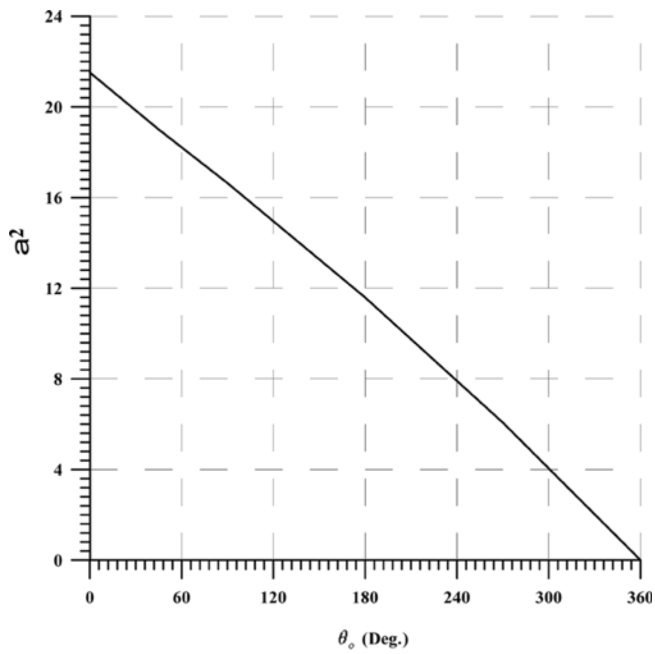


Fig. 9 Design chart for the required inward polar force to switch on the device for a given angle forming the arc

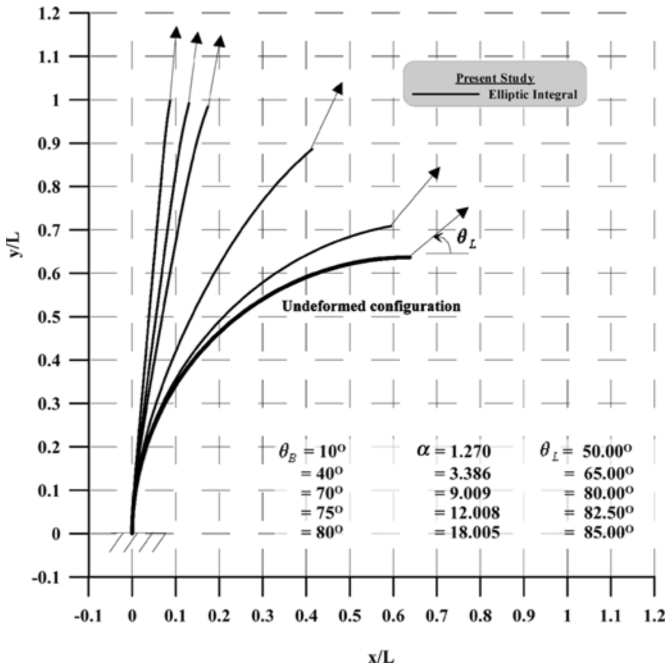


Fig. 10 Deformed configurations of a quarter circular cantilever arc subjected to outward polar force with different θ_B values

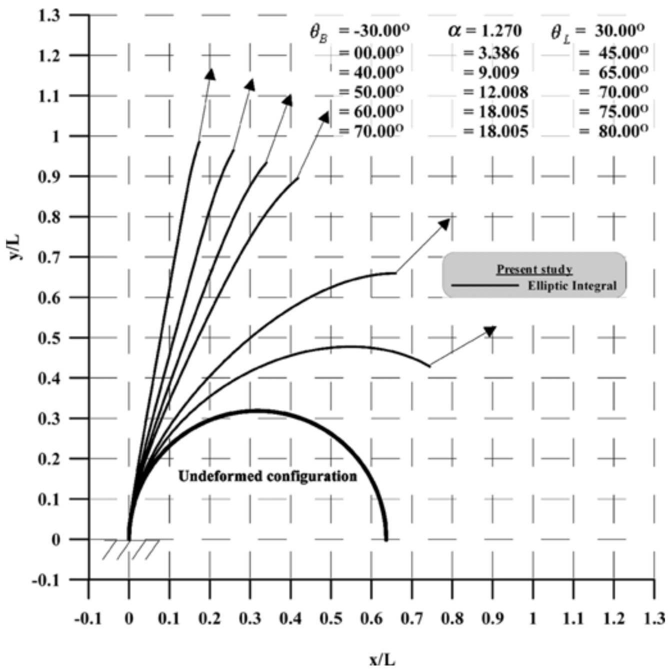


Fig. 11 Deformed configurations of a semi-circular cantilever arc subjected to outward polar force with different θ_B values

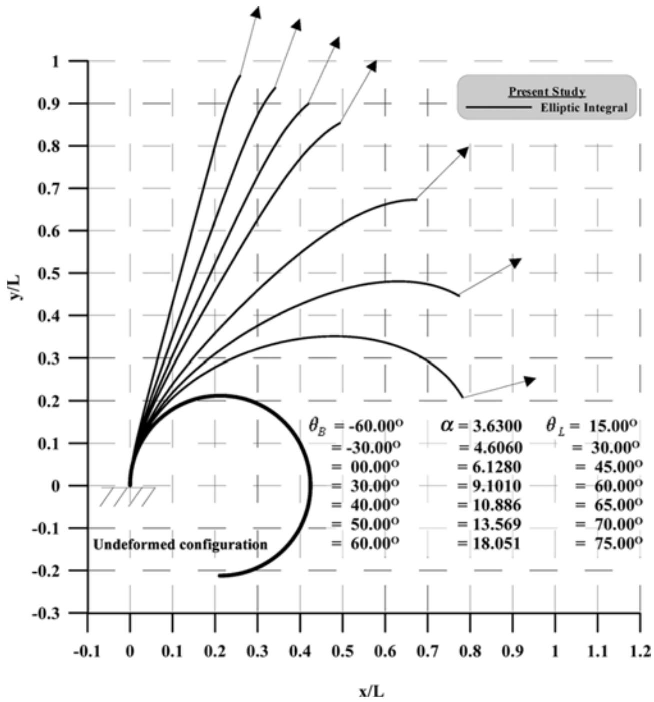


Fig. 12 Deformed configurations of a three-quarter circular cantilever arc subjected to outward polar force with different θ_B values

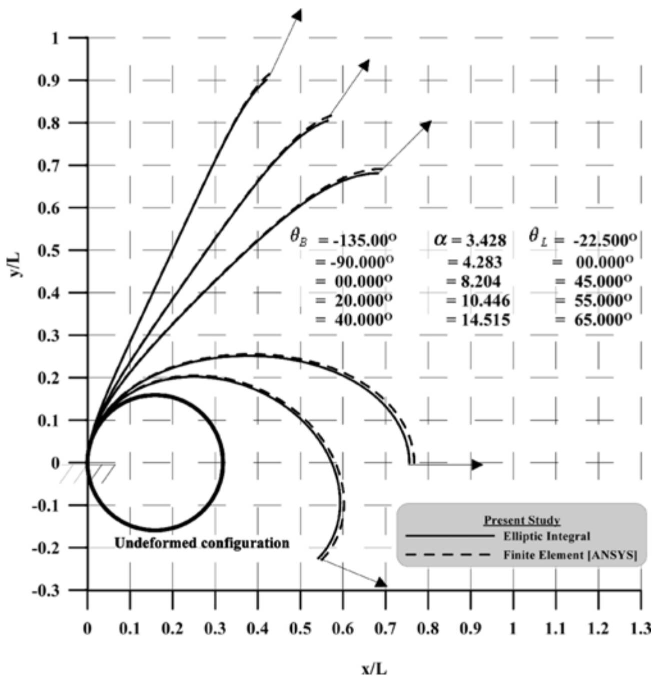


Fig. 13 Deformed configurations of a complete circle cantilever arc subjected to outward polar force with different θ_B values

vertical cantilever (point A in Fig. 1a or Fig. 2a) coincides with the fixed end (point B in Fig. 1a or Fig. 2a). This is due to the fact that the truss element is practically impossible to have a zero length when it is subjected to the cooling process (drop in temperature).

Figs. 6 to 8 show deformed configurations for the case of quarter-circular arc, semi-circular arc and three-quarter circular cantilever arc respectively using elliptic integral solution.

From this study, one can estimate the required value for the inward polar force for the case of cantilever arc device to switch on (point B coincides with point A as shown in Fig. 1a or Fig. 2a). Fig. 9 shows a graph that can be used as a design chart for engineers to determine the required design value for the inward polar force (α^2) to switch on the device (the circular cantilever arc) for a given angle forming the circular arc (θ_o).

Figs. 10 to 13 show the deformed configurations for the case of quarter-circular arc, semi-circular arc, three-quarter circular arc and complete circular cantilever arc subjected to outward polar force respectively. From Fig. 13 it can be seen that the results of the elliptic integral solution are in excellent agreements with the finite elements using **ANSYS**. It should be noted that a convergence problems occurs for case of using elliptic integral solution and finite elements when the outward polar force increases rapidly (theoretically α^2 approaches to infinity) to make the circular arc as a vertical straight cantilever. Also, this study did not give any solutions for the trivial cases such as a complete circular arc subjected to inward polar force or the case of a vertical straight cantilever subjected to outward polar force.

5. Conclusions

The problem of a circular cantilever arc device subjected to inward or outward polar force is solved in a closed-form solution using elliptic integral and then checked numerically by performing a large displacement finite element analysis using the ANSYS package. Excellent agreements have been obtained between the exact analytical solution and the numerical approach. A design chart for engineers for estimating the inward polar force required for switching on the device is obtained. This study has several applications in integrated circuit technology, nanotechnology and especially in microelectromechanical systems (MEMs) such as MEM circular device switch subjected to attraction or repulsive magnetic forces.

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Notation

$E(k, \gamma)$: elliptic integral of the second kind.
EI : Flexural rigidity of the circular cantilever arc.
$F(k, \gamma)$: Incomplete elliptic integral of the first kind.
L : Total length of the circular cantilever arc.
M : Bending moment at a distance s.
P : Inward or outward polar force at the tip of the cantilever beam.
R_o : Radius of the circular cantilever arc.
s : Curved coordinate along the deflected beam.
x : Horizontal coordinate at a distance s.
x_C : Horizontal displacement at the tip of the cantilever beam.
y : Vertical coordinate at a distance s.
y_C : Vertical displacement at the tip of the cantilever beam.
α^2 : Non-dimensional parameter for the polar force and equal to PL^2/EI
θ : Angle of rotation at a distance s.
θ_B : Angle of rotation at the tip of the cantilever beam.
θ_L : Inclination angle of the inward or outward polar force.
θ_o : Angle forming the circular cantilever.