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An optimal regularization for structural parameter estimation from modal response

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Abstract. Solutions to the problems of structural parameter estimation from modal response using least-squares minimization of force or displacement residuals are generally sensitive to noise in the response measurements. The sensitivity of the parameter estimates is governed by the physical characteristics of the structure and certain features of the noisy measurements. It has been shown that the regularization method can be used to reduce effects of the measurement noise on the estimation error through adding a regularization function to the parameter estimation objective function. In this paper, we adopt the regularization function as the Euclidean norm of the difference between the values of the currently estimated parameters and the *a priori* parameter estimates. The effect of the regularization function on the outcome of parameter estimation is determined by a regularization factor. Based on a singular value decomposition of the sensitivity matrix of the structural response, it is shown that the optimal regularization factor is obtained by using the maximum singular value of the sensitivity matrix. This selection exhibits the condition where the effect of the *a priori* estimates on the solutions to the parameter estimation problem is minimal. The performance of the proposed algorithm is investigated in comparison with certain algorithms selected from the literature by using a numerical example.

Keywords: parameter estimation; measurement noise; estimation errors; regularization; singular value decomposition.

1. Introduction

Construction of a suitable mathematical model of the structural system is necessary for the prediction of the structural responses under different specified loadings. Building a mathematical model of a structure usually requires the laws of physics and structural mechanics, for example, the Newton's laws of motion, the geometry of deformation, and the material constitutive laws. Generally, the largest portion of assumptions used in composing a structural model concerns the material constitutive properties for each structural member since this information can be obtained only through structural testing.

Various algorithms for identification of structural constitutive properties have been proposed in the literature over the past few decades (Astrom and Eykhoff 1971, Hajela and Soeiro 1990, Mottershead and Friswell 1993). These methods generally determine, for a given structural model, a

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set of constitutive parameters yielding the best-fit response with respect to the measured response obtained from testing the actual structure. Both static and dynamic responses of the structure have been adopted as the measurement data.

Parameter estimation algorithms based on the constrained minimization of the least-squares errors between the measured and the computed response of the structure suffers the problems of non-uniqueness and sensitivity of solutions. These problems become severe when the measured data are polluted with noise or when the constructed structural model fails to characterize the essential components of the actual structural system (Bui 1994, Hjelmstad 1996, Hansen 1998).

It is well known that the regularization technique can be used to overcome the problems of non-uniqueness and discontinuity of solutions (Golub *et al.* 1978, Hansen 1992, 1998). This method applies a regularization function, which acts as a penalty term, to the initial error function. The consistency of the regularization effect on the parameter estimation results is controlled by a regularization factor which is included in the regularization function. By selecting an optimal regularization factor, the regularization effect is well-balanced and a numerically stable solution to the parameter estimation problem is obtained. Some intuitive schemes have been proposed for the selection of optimal regularization factors based on some a priori information of the system parameters (Lee *et al.* 1999, 2000, Yeo *et al.* 2000, Park *et al.* 2001). However, the accuracy of the outcome of a parameter estimation problem cannot by any means be guaranteed by using the presumed *a priori* parameter estimates since in many cases these estimates are not reliable.

This paper adopts the framework of the regularization method to improve the performance of a parameter estimation algorithm from the measured modal response of a structure in the presence of the measurement noise. A new technique is proposed, based on the singular value decomposition of the sensitivity matrix of the structural response, to select an optimal regularization factor for a least-squares optimization problem associated with the structural parameter estimation method. In the present method, the *a priori* estimates of structural parameters are not required in advance of the estimation process. Hence, the problem of unreliability of the *a priori* estimates is eliminated.

A numerical simulation study is carried out using a nine-story shear building as the model problem. The results from the simulation study are presented to demonstrate the effectiveness of the present algorithm in comparison with some proposed algorithms in the literature.

2. Structural parameter estimation scheme

Consider a parameterized finite-element model of a structure with N_d degrees of freedom. The undamped free vibration of the structure is governed by the generalized eigenvalue problem

$$\mathbf{K}(\mathbf{x})\psi_i = \lambda_i \mathbf{M}\psi_i \tag{1}$$

where $\mathbf{K}(\mathbf{x})$ is the linear stiffness matrix, $\mathbf{M}$ is the structural mass matrix, λ_i and ψ_i are, respectively, the eigenvalue (the square of the natural frequency) and the eigenvector (mode shape) for the ith mode. Each element in the structural model is designated to one of the parameter groups Ω_1, Ω_2, …, Ω_{N_p} where N_p is the total number of parameter groups in the finite-element model. For the present study it is assumed that an element m belonging to group Ω_k is characterized by a single constitutive parameter x_k. This constitutive parameter can be, for example, the axial stiffness of a truss member or the bending stiffness of a Bernoulli beam element. A more generalized version of

the present parameterization scheme may be adopted for structural members having multiple stiffness parameters (Hjelmstad and Shin 1996). With the present parameterization scheme, the linear stiffness matrix $\mathbf{K}(\mathbf{x})$ is parameterized by N_p parameters $\mathbf{x} = \{x_1, x_2, \ldots, x_{N_p}\}^T$. The stiffness matrix of the structural model can be expressed as

$$\mathbf{K}(\mathbf{x}) = \sum_{k=1}^{N_p} \sum_{m=\Omega_k} x_k \mathbf{G}_m \tag{2}$$

where $\mathbf{G}_m$ is the kernel matrix containing the geometrical information of element m.

Banan and Hjelmstad (1993) proposed the output error estimator (OEE) that is cast as a constrained minimization of the least-squares error function

$$\begin{aligned} &\text{Minimize} \quad J_E(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{N_m} \delta_i \|\mathbf{e}_i(\mathbf{x})\|^2 \\ &\mathbf{x} \in \Re^{N_p} \\ &\text{Subject to } \mathbf{c}(\mathbf{x}) \le \mathbf{0} \end{aligned} \tag{3}$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector, δ_i is the significance factor of the ith mode, N_m is the number of modes with measurement data and $\mathbf{c}(\mathbf{x})$ is the vector for the constraints on parameter values. The output error $\mathbf{e}_i(\mathbf{x})$ in Eq. (3) is defined as

$$\mathbf{e}_i(\mathbf{x}) \equiv \psi_i - \lambda_i \mathbf{Q}\mathbf{B}_i^{-1}(\mathbf{x})\hat{\mathbf{M}}\hat{\psi}_i = \psi_i - \psi_i^c \tag{4}$$

where $\mathbf{Q}$ is the boolean matrix that extracts the components of the response vector associated with measured degrees of freedom from the eigenvector by the relationship $\hat{\psi}_i = \mathbf{Q}\psi_i$, and $\mathbf{B}_i(\mathbf{x}) \equiv \mathbf{K}(\mathbf{x}) - \lambda_i[\mathbf{O}|\overline{\mathbf{M}}]$ is defined using a partitioning of the mass matrix $\mathbf{M} = [\hat{\mathbf{M}}|\overline{\mathbf{M}}]$ into a part $\hat{\mathbf{M}}$ associated with the measured degrees of freedom and a part $\overline{\mathbf{M}}$ associated with unmeasured degrees of freedom, whereas $\mathbf{O}$ is a zero matrix. Detailed derivation of these matrices can be found in the work by Pothisiri and Hjelmstad (2002).

For simplicity of the following mathematical derivation, we shall rewrite the objective function in Eq. (3) in a single vector form

$$J_E(\mathbf{x}) = \frac{1}{2}\|\hat{\Phi} - \hat{\Phi}^c(\mathbf{x})\|^2 \tag{5}$$

in which $\hat{\Phi} = \{\sqrt{\delta_1}\hat{\psi}_1, \sqrt{\delta_2}\hat{\psi}_2, \ldots, \sqrt{\delta_{N_m}}\hat{\psi}_{N_m}\}^T$ and $\hat{\Phi}^c(\mathbf{x}) = \{\sqrt{\delta_1}\hat{\psi}_1^c, \sqrt{\delta_2}\hat{\psi}_2^c, \ldots, \sqrt{\delta_{N_m}}\hat{\psi}_{N_m}^c\}^T$.

3. Regularization

3.1 Analysis of regularization scheme

The least-squares minimization of the output error defined in Eq. (3) suffers the problems of non-uniqueness and sensitivity of solutions arising from using incomplete and noisy measurement data. The present study adopted the Tikhonov regularization technique (Groetsch 1984) to resolve the

problem of sensitivity of solutions to the parameter estimation problem. The regularization function proposed herein is expressed as

$$J_R(\mathbf{x}) = \frac{1}{2}\alpha^2 \left\| \mathbf{L}(\mathbf{x} - \mathbf{x}_0) \right\|^2 \tag{6}$$

where α denotes the regularization factor, $\mathbf{L}$ denotes the scaled finite difference approximation to the ith derivative of $(\mathbf{x} - \mathbf{x}_0)$, often chosen as the identity matrix $\mathbf{I}$ ($i = 0$), and $\mathbf{x}_0$ denotes the *a priori* estimates of the system parameters. For parameter estimation problems in general, accurate information of the *a priori* parameter estimates are usually not known in advance.

Note that the scaling matrix $\mathbf{L}$ can generally be selected such that the size of the solution vector is properly controlled. Various choices of this matrix may be adopted using different-order finite difference operators to improve the performance of regularization (see for example, Fierro *et al.* 1997). Nonetheless, we select $\mathbf{L} = \mathbf{I}$ in the current study and we set the values of the *a priori* parameter estimates as zero, that is $\mathbf{x}_0 = \mathbf{0}$, to convert the problem to the standard form (Hansen 1992) since our main concern is on the selection of the regularization factor α. The above selection of *a priori* estimates represents the situation in which these values are not known. By assuming no *a priori* knowledge is available, instabilities of the parameter estimation algorithm suffered from a poor selection of $\mathbf{x}_0$ can be avoided. We define the regularized output error estimator (ROEE) as

$$\begin{aligned} &\text{Minimize} \quad J(\mathbf{x}) = J_E(\mathbf{x}) + J_R(\mathbf{x}) = \frac{1}{2}\left\| \hat{\mathbf{\Phi}} - \hat{\mathbf{\Phi}}^c(\mathbf{x}) \right\|^2 + \frac{1}{2}\alpha^2 \left\| \mathbf{L}(\mathbf{x} - \mathbf{x}_0) \right\|^2 \\ &\mathbf{x} \in \Re^{N_p} \\ &\text{Subject to} \quad \mathbf{c}(\mathbf{x}) \le \mathbf{0} \end{aligned} \tag{7}$$

The solutions to the ROEE can be obtained by using the recursive quadratic programming technique with Fletcher active sets (Luenberger 1989). The implementation of the technique is described in detail by Banan and Hjelmstad (1993). In this method, the non-linear regularized objective function $J(\mathbf{x})$ and the constraints $\mathbf{c}(\mathbf{x})$ in Eq. (7) are approximated by the quadratic function and the linearized constraints, respectively, as

$$J(\mathbf{x}) = \frac{1}{2}\mathbf{d}_k^T \mathbf{H}_k \mathbf{d}_k - \mathbf{d}_k^T \mathbf{S}_k^T \mathbf{e}_k + \alpha^2 \left[\frac{1}{2}\mathbf{d}_k^T \mathbf{L}_k^T \mathbf{L}_k \mathbf{d}_k - \mathbf{d}_k^T \mathbf{L}_k^T \mathbf{L}_k (\mathbf{x}_0 - \mathbf{x}_k) \right] \tag{8}$$

and

$$\mathbf{c}(\mathbf{x}_k) + \nabla \mathbf{c}(\mathbf{x}_k)^T \mathbf{d}_k \le \mathbf{0} \tag{9}$$

where the subscript k denotes the iteration count, ∇ is the gradient operator with respect to the system parameters $\mathbf{x}$, $\mathbf{d}_k$ is the direction vector of system parameters at iteration k. Note that the matrix $\mathbf{L}_k$ is set as the identity matrix $\mathbf{I}$ for the current study, but is shown in its general form herein to accommodate the possible variation. In Eq. (8), $\mathbf{S}_k$ and $\mathbf{H}_k$ are the sensitivity matrix and the Gauss-Newton Hessian matrix of the error function $\mathbf{e}(\mathbf{x})$, respectively, which are given by the following expressions

$$\mathbf{S}_k = \nabla \hat{\mathbf{\Phi}}^c(\mathbf{x}) \tag{10}$$

and $$\mathbf{H}_k = \mathbf{S}_k^T \mathbf{S}_k \tag{11}$$

Notice that the Gauss-Newton approximation which neglects the second derivatives of the error function is used in the computation of the Hessian matrix of Eq. (11).

Based on the work by Park *et al.* (2001), the sensitivity matrix $\mathbf{S}_k$ can be decomposed using the singular value decomposition (SVD) method

$$\mathbf{S}_k = \mathbf{Z}_k \mathbf{\Omega}_k \mathbf{V}_k^T \tag{12}$$

In the above equation, $\mathbf{S}_k \in \Re^{(\hat{N}_d N_m) \times N_p}$, $\mathbf{Z}_k \in \Re^{(\hat{N}_d N_m) \times N_p}$ and $\mathbf{V}_k \in \Re^{(\hat{N}_d N_m) \times N_p}$ are orthogonal matrices whose dimensions are constrained by the inequality of identifiability $\hat{N}_d N_m \geq N_p$ in which $\hat{N}_d$ and N_m, respectively, are the number of degrees of freedom and the number of modes with measurement information, and N_p is the number of unknown system parameters. The singular value matrix $\mathbf{\Omega}_k \in \Re^{N_p \times N_p}$ is defined as $\mathbf{\Omega}_k \equiv \text{diag}\,[\sigma_1, \sigma_2, \ldots, \sigma_{N_p}]$ in which $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{N_p} \geq 0$. Note that the singular values can be zero if the sensitivity matrix $\mathbf{S}_k$ is rank-deficient. The rank-deficiency of the sensitivity matrix can be avoided by ensuring that the inequality of identifiability is always satisfied (i.e., $\hat{N}_d N_m \geq N_p$). By using the present singular value decomposition scheme, the solution of the ROEE takes the final form

$$\mathbf{V}_k^T \mathbf{L}_k \mathbf{x}_{k+1} = \mathbf{diag}(1 - \omega_j) \mathbf{V}_k^T \mathbf{L}_k \mathbf{x}_{k+1}^E + \mathbf{diag}(\omega_j) \mathbf{V}_k^T \mathbf{L}_k \mathbf{x}_0 \tag{13}$$

where ω_j is the filter factor (Hansen 1992) which is defined as $\omega_j = \alpha^2/(\sigma_j^2 + \alpha^2)$, $\mathbf{x}_{k+1}$ and $\mathbf{x}_{k+1}^E$ are the solutions at the $(k+1)$th iteration that minimize the regularized and the original objective function, $J(\mathbf{x})$ and $J_E(\mathbf{x})$, respectively. The solution $\mathbf{x}_{k+1}^E$, which corresponds to the original objective function, is usually referred to as the so-called *a posteriori* solution.

3.2 Optimal regularization factor

Some of the well-known algorithms for finding an optimal regularization factor in linear inverse problems assume a fixed regularization factor throughout the optimization process (Golub *et al.* 1978, Hansen 1992). For many applications in non-linear inverse problems, this assumption is not tenable since the error function may be overwhelmed by the effect of the regularization function for some computation steps during the course of the iterative solution-finding algorithm.

Lee *et al.* (1999) proposed the variable regularization factor scheme (VRFS) that adjusts the regularization factor by a pre-determined reduction factor such that the error function is always larger than the regularization function during the optimization process. Park *et al.* (2001) proposed the geometric mean scheme (GMS) in which the optimal regularization factor is selected from the smallest and largest singular values of the sensitivity matrix to balance the maximum and minimum effect of the *a priori* estimates and the *a posteriori* solution simultaneously. The GMS method performs well only when the *a priori* estimates of the structural parameters are close to the actual parameters.

For most parameter estimation problems the accurate values of the *a priori* estimates of the model parameters are usually not known prior to the estimation. Hence, these values cannot be assumed known in advance of actually finding the solution to the parameter estimation problems. Let us consider the solution to the ROEE algorithm as shown in Eq. (13) with respect to the definition of

the filter factor ω_j for a fixed value of the regularization factor α. It can be seen that by substituting the maximum value of the sensitivity matrix $\sigma_{\max}$ into the definition expression of the filter factor ω_j, the value of ω_j tends to decrease to a certain value $\omega_{\sigma_{\max}}$ such that $1 - \omega_{\sigma_{\max}}$ approaches closer to a unit value, giving the lowest and the highest weight, respectively, to the first and the second term on the right hand side of Eq. (13). This condition corresponds with the situation in which the effect of the *a priori* estimates on the solution for the ROEE algorithm is minimal. Consequently, the *a priori* estimates of system parameters are not required to initiate the solution algorithm. The optimal regularization factor α_{opt} is obtained, using a similar procedure to that proposed by Park *et al.* (2001), by forcing the minimum effect of the *a priori* estimates and the maximum effect of the *a posteriori* solution to occur simultaneously, that is

$$(1 - \omega_{\sigma_{\max}}) = \omega_{\sigma_{\max}} \quad \text{or} \quad \omega_{\sigma_{\max}} = \frac{1}{2} \tag{14}$$

Both sides of the above equation are the weight factors on the right hand side of Eq. (13) that correspond to the maximum singular value of the sensitivity matrix $\sigma_{\max}$. This equation yields the optimal regularization factor as

$$\alpha_{\text{opt}} = \sqrt{\sigma_{\max}^2} \tag{15}$$

By using the present method, we implicitly maximize the effect of the *a posteriori* solution to the parameter estimation problem. However, there is still chance where the selected regularization factor overweighs the effect of the *a priori* estimates on the regularization function. To resolve this problem, a limit on the value of the regularization function is set for each of the iterations during the optimization process by the following expression

$$\alpha^2 \left\| \mathbf{x} - \mathbf{x}_0 \right\|^2 \leq \sum_{i=1}^{N_m} \delta_i \left\| \mathbf{e}_i(\mathbf{x}) \right\|^2 \tag{16}$$

In case where the value of the regularization function is larger than the right-hand side Eq. (16), the regularization factor is reduced by

$$\alpha_{l+1} = \beta \alpha_l \tag{17}$$

where the subscript l denotes the iteration count for the optimization process and β is a predefined reduction factor ranging from 0 to 1. Since the optimal regularization factor in Eq. (15) can vary during the optimization process in accord with Eq. (17), the present method will be called the variable maximum singular value (VMSV) scheme.

4. Numerical simulation study

The performance of the ROEE algorithm is examined in comparison with the OEE in the current simulation study by using the VRFS, GMS and VMSV methods for the selection of the optimal regularization factor α_{opt} to be used during the minimization of Eq. (7). The example structure is the nine-story shear building with fixed base shown in Fig. 1. The structural model consists of nine degrees of freedom which characterizes the horizontal translations at the story levels. The structural

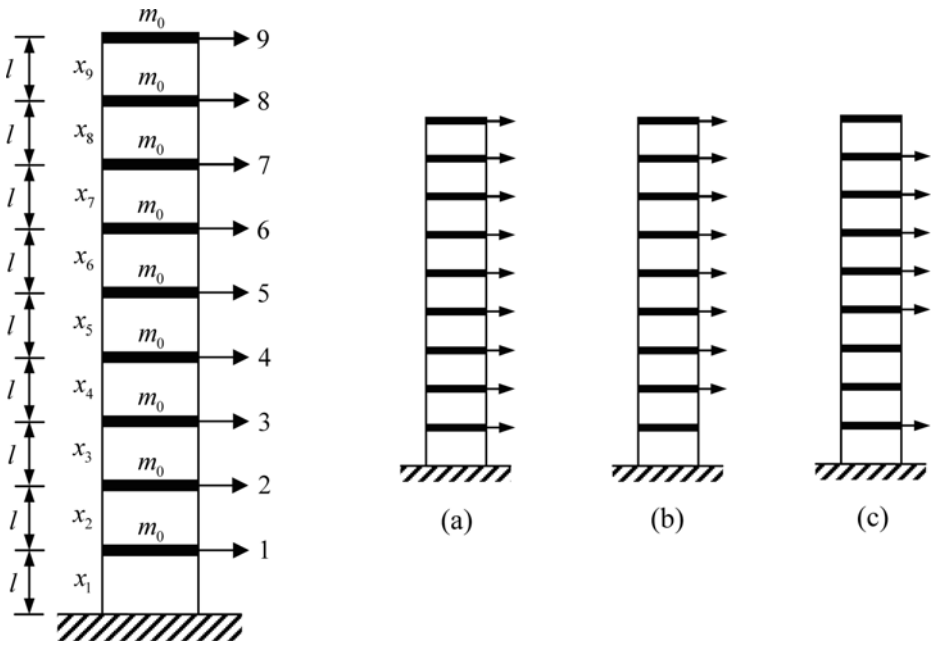

Fig. 1 The nine-parameter shear building with specified patterns of measurement locations: (a) Case (A); (b) Case (B); and (c) Case (C)

model is parameterized with nine parameters $\mathbf{x} = \{x_1, x_2, x_3, \ldots, x_9\}^T$. The bending stiffness of the columns located on the ith story is given by $k_i = x_i k_0$. The nominal properties of the structure are chosen such that $k_0/m_0 = 1.0\text{s}^2$. We assume that the actual parameters associated with the baseline structural model are given as $\hat{\mathbf{x}} = \{2.0, 2.0, 2.0, 1.0, 1.0, 1.0, 1.0, 1.0, 1.0\}^T$.

The noisy measurements are simulated by adding the uniform random variates with known statistical properties to the noise-free data that are obtained from a free-vibration analysis of the specified baseline structural model. The jth noisy measurements $\tilde{\mathbf{\Phi}}_j$ are simulated from the noise-free response vector $\hat{\mathbf{\Phi}}$ as

$$\tilde{\mathbf{\Phi}}_j = [\mathbf{I} + \xi_j]\hat{\mathbf{\Phi}} \tag{18}$$

where $\xi_j \equiv \text{diag}[\xi_1, \xi_2, \ldots, \xi_{\hat{N}_d N_m}]$ is a diagonal matrix of the uniform random variates in the range $[-\varepsilon, \varepsilon]$ in which ε is the amplitude of noise in the measurements.

The influence of certain features of the noisy measurements on the performance of the parameter estimation algorithms under consideration is examined through numerical simulations. Each of the simulated experiments in the present study are listed in Table 1. The numerical simulations in Case I are carried out to study the distribution of solutions to the parameter estimation problem obtained by using the VMSV and the VRFS methods through variation of the reduction factor β defined in Eq. (17) and the number of modes with measurement information N_m. Notice that the level of noise amplitude is fixed at 20% for all simulated measurements of Case I and that N_m lowest modes (modes with the lowest frequencies) are used for all simulation cases in the current study. In Case II the effect of variation in the number of measured modes N_m on the performance of the parameter estimation algorithms is examined by using different levels of noise amplitude ranging between

Table 1 Summary of the numerical experiments

Case	Measurement pattern	Number of measured modes	Measurement noise (%)	Reduction factor (β)	A priori estimates ($\mathbf{x}_0$)
I	(A)	1-9	20	0.1-0.9	$\mathbf{0}$
	(B)	1-9	20	0.1-0.9	$\mathbf{0}$
	(C)	1-9	20	0.1-0.9	$\mathbf{0}$
II	(A)	1-9	0-20	0.9	$\mathbf{0}$
	(B)	1-9	0-20	0.9	$\mathbf{0}$
	(C)	1-9	0-20	0.9	$\mathbf{0}$
III	(A)	9	20	0.9	$\mathbf{0} - 2\hat{\mathbf{x}}$
	(B)	9	20	0.9	$\mathbf{0} - 2\hat{\mathbf{x}}$
	(C)	9	20	0.9	$\mathbf{0} - 2\hat{\mathbf{x}}$

$\varepsilon = 0$ and 20% at 2.5% intervals. In Case III we investigate the effect of using different *a priori* estimates on the outcome of the selected algorithms for 20% noisy measurements.

Algorithms that estimate structural parameters from modal response using least-squares minimization of the displacement residuals generally do not have unique solutions when the data are spatially sparse (Hjelmstad 1996). It has been observed that the number and character of the multiple solutions depend upon the locations of the response measurements. The multiplicity of solutions typically arises if incomplete measurement data in which modal displacements are not measured at all degrees of freedom of the structural model are used. In the current study we use three different patterns of the measurement locations as shown in Fig. 1 as input to the parameter estimation problem to illustrate possible variation in the number of solutions and the sensitivity of the parameter estimates to measurement noise.

For the incomplete measurement cases in Table 1 the random starting point scheme proposed by Pothisiri and Hjelmstad (2002) is used to identify the multiple solutions to the parameter estimation problem. A set of 100 random starting points is selected from a collection of points within a N_p-dimensional hyper-ellipsoid centered at the point associated with the actual parameter values $\hat{\mathbf{x}}$. Each of the random starting points will drive each of the algorithms to a collection of solutions which is a subset of the complete solution set to the minimization problems (3) and (7), respectively, for the OEE and ROEE algorithms. Among the candidate solutions within the collection, the best solution for each noisy measurement case is identified as the set of parameters $\mathbf{x}^*$ associated with the global minimum $J(\mathbf{x}^*)$.

For each measurement case that we investigate in Table 1, 100 different noisy data sets are generated from the noise-free data in accord with Eq. (18) using the specified values of noise amplitude ε. As in the earlier work of Pothisiri and Hjelmstad (Luenberger 1989), the simulated data sets are used as a common database to which each of the parameter estimation algorithms is applied to create population of the parameter estimates. The performance of each algorithm is assessed by using certain statistical indices for the parameter estimation results.

Several indices of identification error are used to compare the performance of the investigated algorithms. First, the average root quadratic bias (*RQB*) is computed as a measure of distance between the average of the parameter estimates $\bar{\mathbf{x}}^*$ for the simulated population and the actual parameters $\hat{\mathbf{x}}$. The average root quadratic bias is defined as

$$RQB = \frac{\left\| \bar{\mathbf{x}}^* - \hat{\mathbf{x}} \right\|}{N_p \|\hat{\mathbf{x}}\|} \tag{19}$$

where N_p is the number of estimated parameters in the model. The average $\bar{\mathbf{x}}^*$ is an approximation of the expected value of the estimated parameters (or centroid of the distribution of the parameter estimates in N_p-dimensional space), which is given by

$$\bar{\mathbf{x}}^* = \frac{1}{N_t} \sum_{t=1}^{N_t} \mathbf{x}_t^* \tag{20}$$

in which $\mathbf{x}_t^*$ denotes the parameter estimates associated with the global minimum for the tth noisy measurements of the N_t simulated data sets under consideration. The quadratic bias is a quantitative measure of the accuracy of the parameter estimation results. The precision of the parameter estimation outcome is indicated by the average standard deviation (*SD*), which is defined as

$$SD = \frac{1}{N_p \|\hat{\mathbf{x}}\|} \sqrt{\frac{1}{N_t - 1} \sum_{t=1}^{N_t} \left\| \mathbf{x}_t^* - \bar{\mathbf{x}}^* \right\|^2} \tag{21}$$

The computed average standard deviation is a measure of the scatter of the parameter estimates with respect to the expected value. The overall performance of the investigated algorithms is determined by the average root mean square error (*RMS*), which is obtained by combining Eqs. (19) and (21) as

$$RMS = \sqrt{RQB^2 + SD^2} = \frac{1}{N_p \|\hat{\mathbf{x}}\|} \sqrt{\frac{1}{N_t} \sum_{t=1}^{N_t} \left\| \mathbf{x}_t^* - \hat{\mathbf{x}} \right\|^2} \tag{22}$$

Notice that variation of the mean square error depends upon both the quadratic bias and the standard deviation. For example, a decrease in the scatter of the parameter estimates around the actual parameters (*RMS*) can be obtained by either decreasing the distance between the expected value of the parameter estimates and the actual parameters (*RQB*) or by reducing the scatter of the parameter estimates with respect to the expected value (*SD*).

The identification errors of the parameter estimation results as defined by Eqs. (19), (21) and (22) for the specified measurement patterns of Case I are shown in Figs. 2-4. In these figures, the values of *RQB*, *SD* and *RMS* are plotted with respect to the reduction factor β of Eq. (17) for different numbers of modes N_m which are used to drive the VRFS and the VMSV algorithms. The identification errors are computed for the population of parameter estimates obtained from Eq. (17) for 100 noisy data sets with a fixed level of noise, $\varepsilon = 20\%$. It should be noted that only the VRFS and the VMSV algorithms are examined in Case I since the key objective of this simulation case is to determine the best reduction factor β_{opt} for these algorithms. It should be noted that the value of β essentially represents the rate of reduction in the value of the α during the search for the optimal regularization factor α_{opt}. Generally, the number of iterations required for this search decreases with the value of β. A decrease in the value of β would increase the computational efficiency of the algorithm. However, a too-low value of β may cause the search to jump over the value of α_{opt}. The situation is more severe when the level of noise in the measurements is high. Therefore, the 20% noisy measurement case is used in the current study as the worst-case scenario.

For the parameter estimation problem associated with the noisy measurements of Case I we use the upper bounds of the estimated parameters of $50\hat{\mathbf{x}}$ and we select the values of the *a priori*

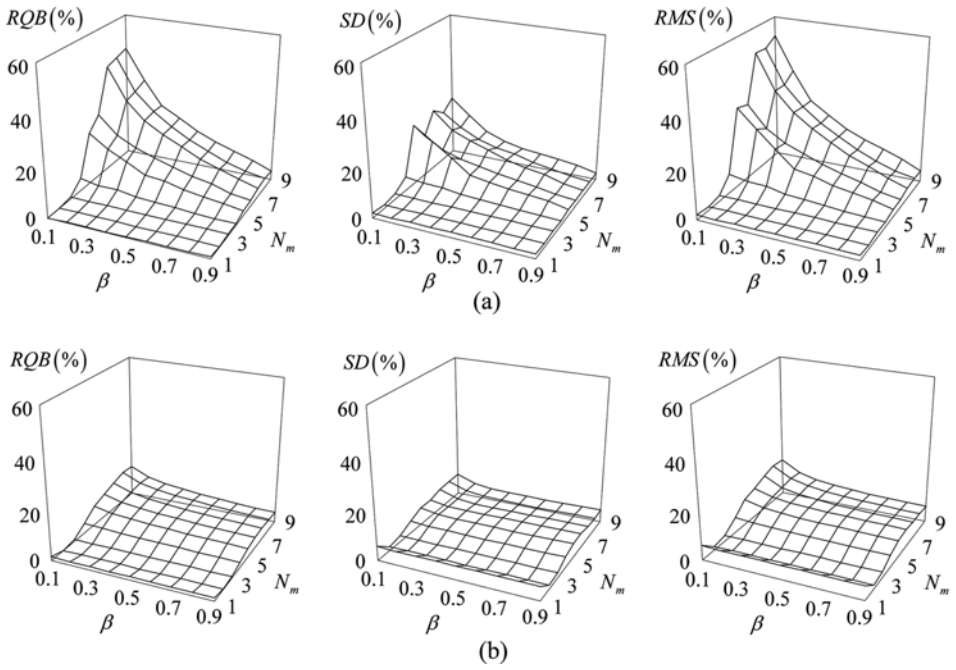

Fig. 2 Variations of *RQB*, *SD* and *RMS* with respect to different values of reduction factor β and number of modes N_m for (a) VMSV and (b) VRFS methods using measurement case (A)

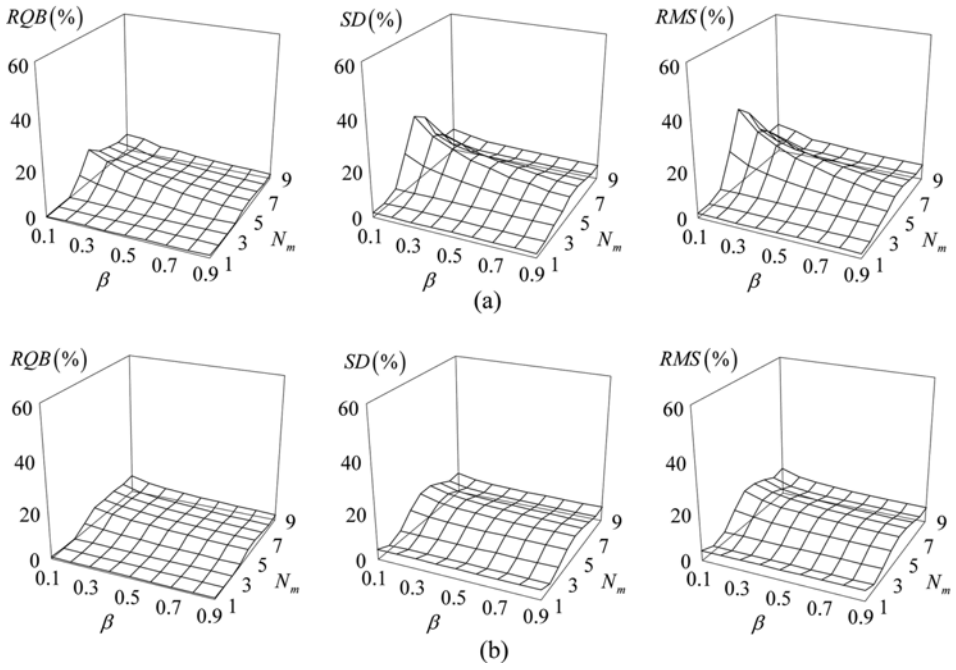

Fig. 3 Variations of *RQB*, *SD* and *RMS* with respect to different values of reduction factor β and number of modes N_m for (a) VMSV and (b) VRFS methods using measurement case (B)

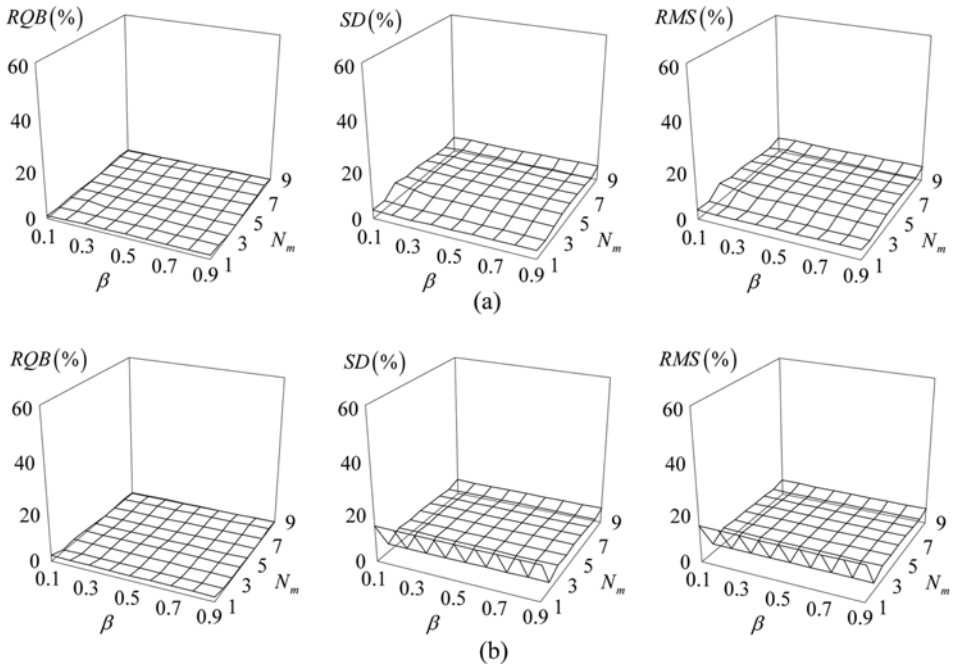

Fig. 4 Variations of *RQB*, *SD* and *RMS* with respect to different values of reduction factor β and number of modes N_m for (a) VMSV and (b) VRFS methods using measurement case (C)

parameter estimates as zero for all stiffness parameters ($\mathbf{x}_0 = \{0, 0, 0, 0, 0, 0, 0, 0, 0\}^T$) since their values are not known in advance.

One can observe that the variation of the identification errors, with different values of β and number of modes, is different for each measurement case. The identification errors for the VRFS and the VMSV algorithms increase as the value of β decreases for all numbers of modes considered. The algorithms perform best when the value of β is equal to 0.9 as indicated by the lowest values of *RQB*, *SD* and *RMS* for all measurement cases. With this "best" value of the reduction factor (β = 0.9), the VMSV algorithm slightly out-performs the VRFS algorithm as evident from the lower identification errors for all measurement cases. It is also observed that when the value of β is 0.9, the performance of the VMSV algorithm is less affected by variation of the number of modes compared to the VRFS algorithm. Furthermore, the VRFS algorithm is observed to lack "consistency" for this same value of β as indicated by variation in the accuracy and precision of the parameter estimates with respect to different amount of information (i.e., the values of *RQB* and *SD*, respectively, vary with the number of modes with response measurements). Nevertheless, the "best" reduction factor, β = 0.9, will be used for the VRFS and the VMSV algorithms for all simulation experiments from this point forward.

The results of the simulation study for different patterns of measurement locations in Case II are illustrated in Figs. 5-7. In these illustrations, the identification errors of the parameter estimation results obtained by using the OEE, VRFS, GMS and VMSV algorithms are shown with respect to different levels of measurement noise and numbers of modes. It should be noted that all the *a priori* parameter estimates in the regularization function proposed in Eq. (7) are (again) set as zero, or assumed unknown, and that the upper bounds of the parameter estimates are set to $50\hat{\mathbf{x}}$.

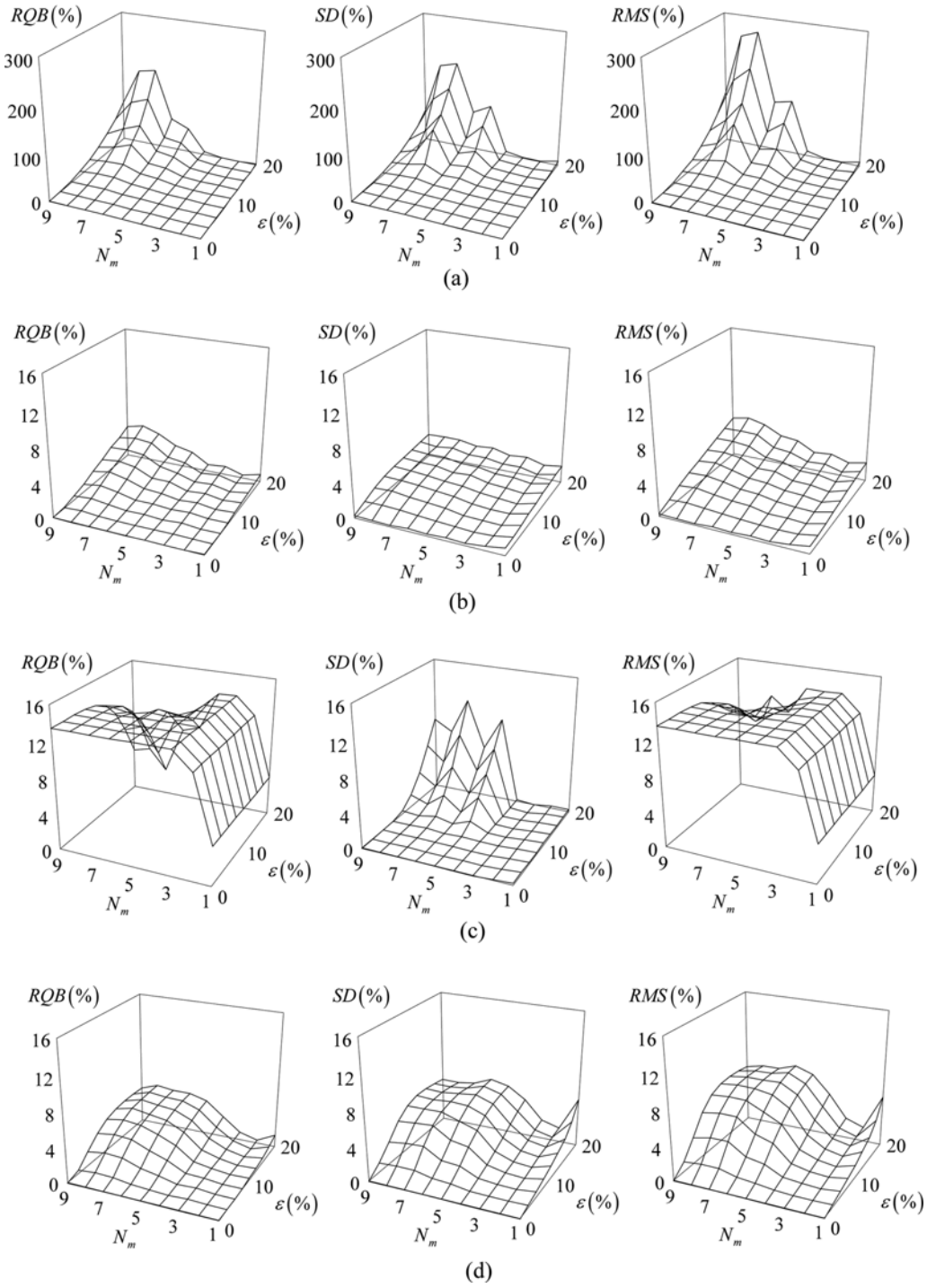

Fig. 5 Variations of *RQB*, *SD* and *RMS* with respect to different levels of noise ε and number of modes N_m for (a) OEE; (b) VMSV; (c) GMS; and (d) VRFS methods using measurement case (A)

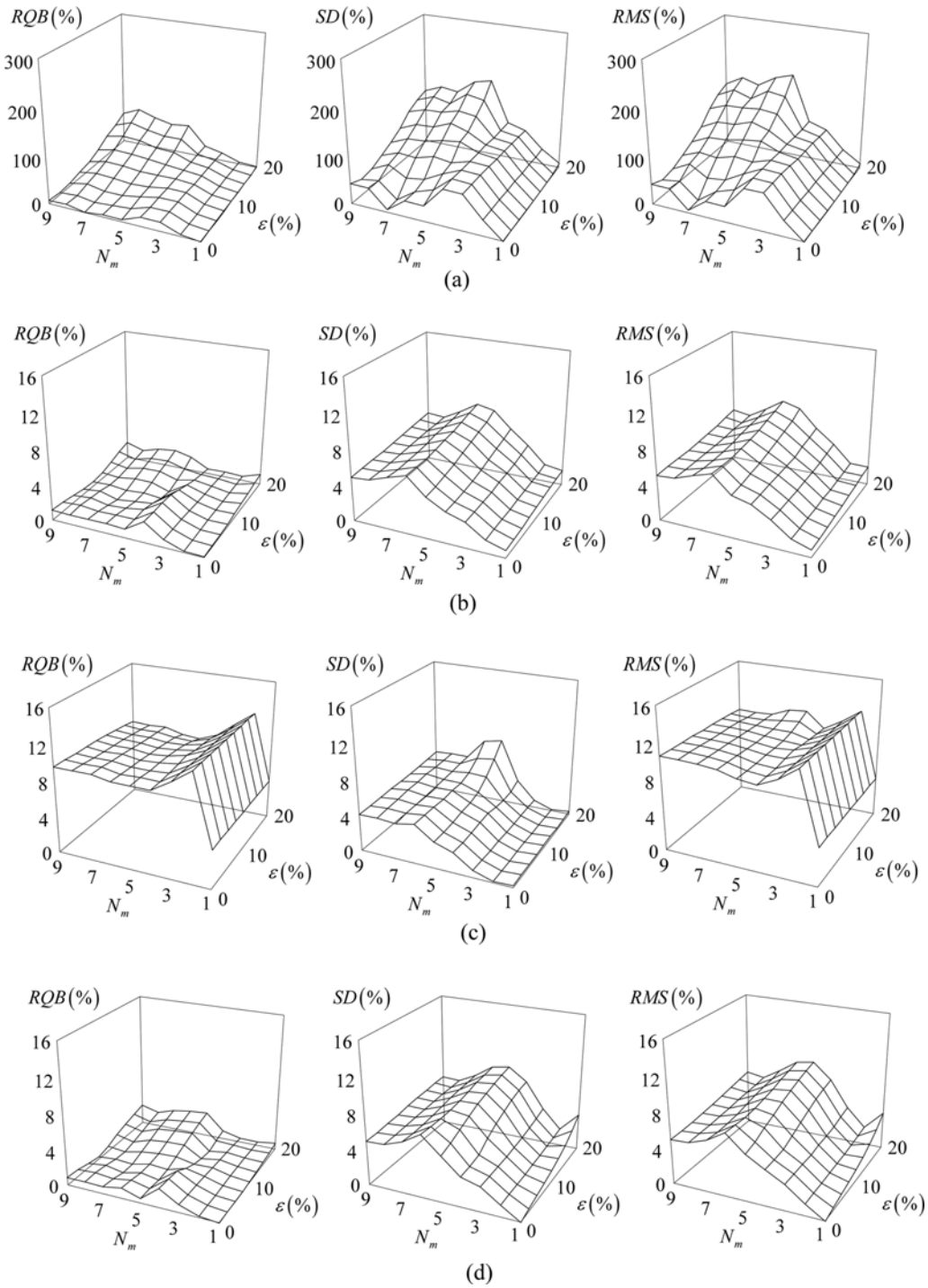

Fig. 6 Variations of *RQB*, *SD* and *RMS* with respect to different levels of noise ε and number of modes N_m for (a) OEE; (b) VMSV; (c) GMS; and (d) VRFS methods using measurement case (B)

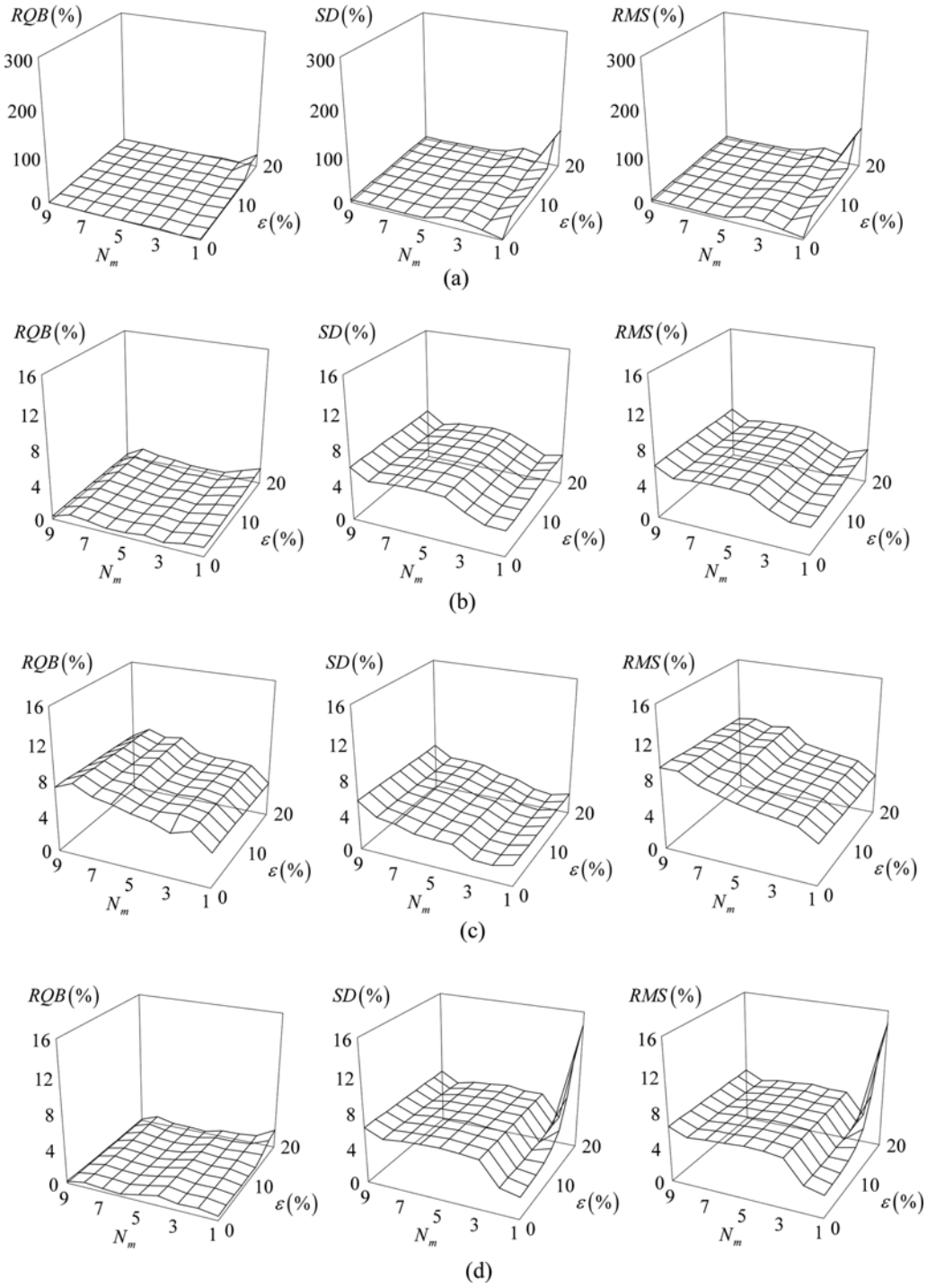

Fig. 7 Variations of *RQB*, *SD* and *RMS* with respect to different levels of noise ε and number of modes N_m for (a) OEE; (b) VMSV; (c) GMS; and (d) VRFS methods using measurement case (C)

The case of complete measurements (A) is illustrated in Fig. 5. It is observed that the VMSV algorithm performs best for the investigated measurement cases with the maximum values of *RQB*, *SD* and *RMS* equal to 4.3%, 2.97% and 5.22%, respectively, which are the lowest among the investigated algorithms. The worst overall performance belongs to the OEE algorithm in which the highest values of *RQB*, *SD* and *RMS* are obtained, respectively, as 197.83%, 197.71% and 267.24% for high levels of noise. The GMS algorithm shows bias in the parameter estimates, as indicated by non-zero values of *RQB*, even when the noise-free data are used. This phenomenon is likely an artifact of the assumption that the values of the *a priori* estimates $\mathbf{x}_0$ are not known rather than a characteristic of the algorithm. As such, the accuracy of the GMS algorithm can be jeopardized if

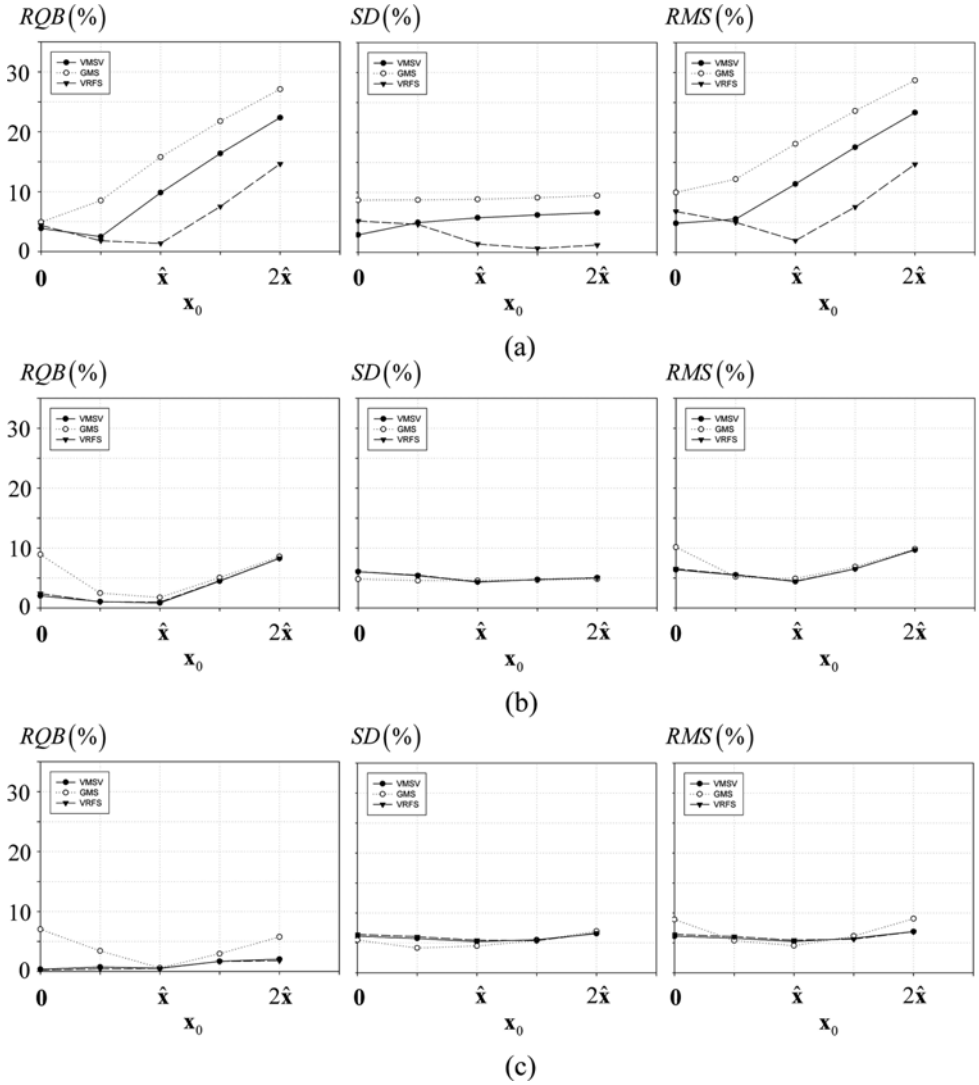

Fig. 8 Variations of *RQB*, *SD* and *RMS* with respect to different values of *a priori* estimates $\mathbf{x}_0$ for the VMSV, GMS and VRFS methods using different measurement cases: (a) Case (A); (b) Case (B); and (c) Case (C)

the *a priori* estimates are unknowns (as in the present case). For noise-free measurements one can observe that there is a scatter of the parameter estimates for the VMSV algorithm as evident from the non-zero *SD* values. This phenomenon is due mainly to the effect of the regularization function which is added to the initial objective function in Eq. (7). Nonetheless, the VMSV algorithm shows the least variation of identification errors for the parameter estimates with respect to different levels of measurement noise and numbers of modes used in the estimation.

The same trends are observed in Figs. 6 and 7 for the incomplete measurement cases (B) and (C), respectively. In these measurement cases, the scatter of the parameter estimates for the OEE algorithm is somewhat reduced as indicated by the lower *SD* values at high levels of noise. For lower levels of noise in the measurements all of the investigated algorithms show lower bias for the parameter estimation results on the whole except for the noise-free measurements in which non-zero values of *RQB* are observed for the OEE, VRFS and VMSV algorithms. As for the complete measurement case, the VMSV algorithm shows the best performance with the lowest values of the maximum identification errors among all of the algorithms under consideration. Moreover, the VMSV algorithm shows the highest level of the overall regularization effect with the least variation of the identification errors.

Variation of the performance of the VRFS, GMS and VMSV algorithms with respect to different *a priori* parameter estimates $\mathbf{x}_0$ is illustrated in Fig. 8 for all of the measurement cases shown in Fig. 1. In this figure one can observe similar trends for each of the algorithms considered. One can observe that the best performance for the VMSV algorithm is achieved when the values of $\mathbf{x}_0$ are set to zero (that is, with the absence of $\mathbf{x}_0$), and that performance of the GMS and VRFS algorithms strictly depends upon the choice of $\mathbf{x}_0$.

For the complete measurement case (A) the VRFS algorithm performs comparatively better when the values of $\mathbf{x}_0$ are used exactly as the actual parameters. The same trend is seen for the GMS algorithm for the incomplete measurement cases (B) and (C). This phenomenon is somehow foreseen since, unlike the VMSV algorithm, the solutions to the parameter estimation problem as defined in Eq. (13) obtained by using the GMS and VRFS algorithms are clearly affected by the values of $\mathbf{x}_0$. On the other hand, the better performance of the VMSV algorithm in the absence of $\mathbf{x}_0$ (i.e., $\mathbf{x}_0 = \mathbf{0}$) exhibits the minimal effect of $\mathbf{x}_0$ on the outcome of parameter estimation. It is interesting to see that for the incomplete measurement case (C) the value of *RQB* is slightly lower when $\mathbf{x}_0 = \mathbf{0}$ compared to when $\mathbf{x}_0 = \hat{\mathbf{x}}$. This counter-intuitive observation somewhat illustrates an intrinsic bias of solutions to the parameter estimation problem when the measurements are incomplete. In any cases, this observation calls upon question the reliability of using the *a priori* parameter estimates in the estimation of the system parameters.

5. Conclusions

The key culprit for algorithms in which structural parameters are estimated based upon certain response measurements is the existence of the measurement error. The effect of measurement errors is generally manifested as the discontinuity of solutions to the governed mathematical parameter-estimation problems, and is often observed through errors of the parameter estimates. A regularization scheme has been presented to reduce the sensitivity of the parameter estimates to the measurement error. The method introduces a regularization function as a Euclidean norm of the difference between the values of the current and *a priori* parameter estimates to the parameter

estimation objective function. The purpose of this function is to penalize any divergence of the parameter estimates from some *a priori* parameter values. The overall regularization effect is adjusted by using a specific weight factor typically known as a regularization factor.

The selection of an optimal regularization factor is essential for obtaining the maximal improvements on the parameter estimation outcome. During the course of the iterative optimization process, the regularization factor must be optimally adjusted to maintain a well-balanced regularization effect on the parameter estimation results. Several algorithms are used in the current study to perform this task. The variable regularization factor scheme (VRFS) proposed by Lee *et al.* (1999) adjusts the regularization factor by using a specified reduction factor. The geometric mean scheme (GMS) proposed by Park *et al.* (2001) selects the optimal regularization factor to simultaneously balance the effects of the *a priori* parameter estimates and the *a posteriori* solution. The variable maximum singular value (VMSV) algorithm proposed in the present paper specifies an optimal regularization factor that corresponds with a typical situation in which the *a priori* parameter values are taken as unknowns. The concept behind the algorithm seems attractive since, for most parameter estimation problems, the *a priori* estimates of the model parameters are usually not known in advance of the estimation.

It has been illustrated through a simple numerical example that the VMSV algorithm can be used to select an optimal regularization factor with success in improving the outcome of parameter estimation in the absence of the *a priori* parameter estimates. The algorithm performs well in comparison with the other algorithms under consideration, even with the lack of completeness of the measured data and noise in the measurements. Although the example model chosen for the current study is quite simple, a thorough procedure for the investigation on the performance of the proposed algorithm has been established and presented. In further investigations the more complex structural models could be used.

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