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Crack effect on the elastic buckling behavior of axially and eccentrically loaded columns

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Abstract. A close form solution of the maximum deflection for cracked columns with rectangular cross-sections was developed and thus the elastic buckling behavior and ultimate bearing capacity were studied analytically. First, taking into account the effect of the crack in the potential energy of elastic systems, a trigonometric series solution for the elastic deflection equation of an arbitrary crack position was derived by use of the Rayleigh-Ritz energy method and an analytical expression of the maximum deflection was obtained. By comparison with the rotational spring model (Okamura *et al.* 1969) and the equivalent stiffness method (Sinha *et al.* 2002), the advantages of the present solution are that there are few assumed conditions and the effect of axial compression on crack closure was considered. Second, based on the above solutions, the equilibrium paths of the elastic buckling were analytically described for cracked columns subjected to both axial and eccentric compressive load. Finally, as examples, the influence of crack depth, load eccentricity and column slenderness on the elastic buckling behavior was investigated in the case of a rectangular column with a single-edge crack. The relationship of the load capacity of the column with respect to crack depth and eccentricity or slenderness was also illustrated. The analytical and numerical results from the examples show that there are three kinds of collapse mechanisms for the various states of cracking, eccentricity and slenderness. These are the bifurcation for axial compression, the limit point instability for the condition of the deeper crack and lighter eccentricity and the fracture for higher eccentricity. As a result, the conception of critical transition eccentricity $(e/h)_c$, from limit-point buckling to fracture failure, was proposed and the critical values of $(e/h)_c$ were numerically determined for various eccentricities, crack depths and slenderness.

Key words: equilibrium path; elastic buckling; bearing capacity; cracked column; eccentricity; slenderness.

1. Introduction

The cracks in structural components are unavoidable defects due to material processing, component manufacturing and a structural poor working environment. The presence of a crack certainly weakens the static and dynamic response and the ultimate bearing capacity of the structure. Therefore, the effects of cracks on the buckling and vibration behavior of the cracked components have received considerable attention. There exist a wealth of analytical, numerical and experimental investigations (Dimarogonas 1996). In order to apply fracture mechanics through the compliance

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concept to the analysis of a structure containing cracked members, a rotational spring model for describing the local flexibility of a crack was established first by Okamura *et al.* (1969). This model has been widely used in the past 20 years for vibration (Ismail *et al.* 1990, Krawczuk *et al.* 1997, Takahashi 1999) or buckling analysis (Takahashi 1999, Anifantis and Dimarogonas 1983, Nikpour 1990, Wang and Chase 2003) of cracked members, because of its simplicity and convenience. Recently, the vibration and buckling behavior of members with an arbitrary number of cracks has been analyzed for both uniform sections (Shifrin and Ruotolo 1999) and non-uniform sections (Li 2003a,b). Some finite elements of prismatic beams were also presented for structural analysis (Gounaris and Dimaragonas 1998, Viola *et al.* 2002). However, the above studies are all based on the use of rotational springs to simulate local flexibility induced by cracks and lead to a system of eigenvalue equations. It is obviously unreasonable in the rotational spring model for compressive column that the effect of axial compression on the crack closure is not considered and sometimes there appears a unreal rotation jump on the deflection curve. Another approximate method for describing the increased flexibility due to a crack is obtained by assuming a variation of the stiffness $EI(x)$ containing some model constants (Sinha *et al.* 2002, Christides and Barr 1984). In this model, the principles of fracture mechanics were not involved and the model constants were empirically or experimentally determined. Taking into account the effect of cracks in the potential energy of elastic system, the Rayleigh-Ritz energy method was recently employed and a trigonometric series solution of the elastic deflection equations was obtained by Zhou (2000, 2002). Especially when the crack is located in the middle section, the solution for the maximum deflection of cracked columns with both ends pinned was analytically given for a rectangular column with a single-edge crack. Furthermore, the two-criteria approach to determinate the stability factor φ has been suggested and its analytical formulae were derived by Zhou and Huang (2005). It should be noted that all studies mentioned above have been confined to eccentric columns and opening cracks. However, the crack closure often appears in the process of buckling and vibration and it changes the response of the structural components. Chondros *et al.* (2001), in their beam vibration analysis, assumed a bilinear-type breathing crack that is in two states, i.e., either fully open or fully closed. However, there is a lack of consideration of crack closure in the buckling analysis for columns. The neglect of crack closure in buckling could lead to foundational deviation.

The objective of the present study is: to develop an analytical solution for maximum deflection of a cracked column under more general conditions; to find a set of analytical equations for the equilibrium path for both the axially and eccentrically loaded column by taking into account the crack closure and fracture condition; and to analytically investigate the effects of both load imperfection (eccentricity) and physical defects (cracks) on the elastic buckling behavior of columns. These objectives are all in order to prepare a theoretical foundation for the further buckling analysis of complex structures containing both geometrical and physical imperfection. As an example, the equilibrium paths and the ultimate bearing capacity of the column were illustrated for a rectangular column with a single-edge crack for various crack depths, load eccentricities and slenderness ratio.

2. Series solution of deflection for eccentric cracked column

2.1 Analytical model

The cracked uniform column with a rectangular cross-section subjected to eccentric compression

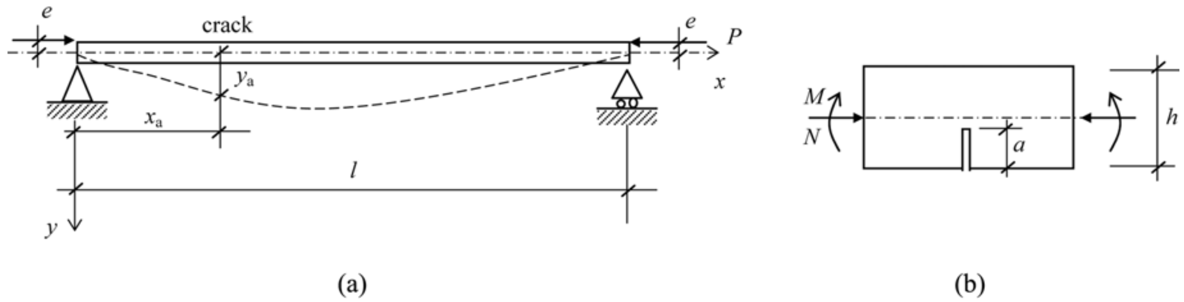


Fig. 1 The eccentric column and the crack model

shown in Fig. 1 will be studied in this paper. The column has a thickness h, a length l and a loading eccentricity e. A model-I crack with the depth a is located at the convex side of the x_a section from the left end of the column. The elastic modulus and Poisson's ratio of the material of the column are E and ν respectively.

According to linear elastic fracture mechanics, the general expression of the stress intensity factor (SIF) at the crack tip of the column, shown in Fig. 1(b), can be written as

$$K_I = \alpha\sigma_E\sqrt{\pi h}\cdot\sqrt{\xi}\left[\omega\cdot\frac{y_a+e}{h}\cdot f_M(\xi)-f_N(\xi)\right] \tag{1}$$

In Eq. (1), $\alpha = P/(A_0\sigma_E)$ is the dimensionless compressive load, in which α_E is the Euler critical stress and A_0 is the uncracked section area; $\xi = a/h$ is the ratio of the crack depth to the height of column; $\omega = A_0h/W$ is the sectional geometric parameter, where W is the section modulus of the cross section; y_a/h denotes the ratio of the cracked section deflection to the column height; $f_M(\xi)$ and $f_N(\xi)$ are the configuration correction factors of the SIF under bending and tension respectively. These SIF factors may be obtained directly from the SIF handbooks or computationally determined.

In the buckling analysis for cracked columns, three possible crack states must be considered. When the stress intensity factor is $K_I \leq 0$, the crack tip is closed and the crack has no effects; when the SIF factor is $K_I > 0$, the crack is fully opened and has some influence on the response of the column; when the SIF factor K_I equals the fracture toughness of the material, K_{IC}, the column fracture will occur.

2.2 Strain energy and external force work

According to the boundary conditions of the column shown in Fig. 1(a), the objective function of the deflection can be assumed to be the trigonometric series form

$$y(x) = h\sum_{m=1}^{\infty} C_m \sin\frac{m\pi x}{l} \tag{2}$$

where C_m may be called the deflection coefficients.

In the final deformation state due to both load and crack, the elastic strain energy of the uncracked column can be obtained from the bending theory of beams, that is

$$U_0 = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{\pi^2 h^2 \sigma_E A_0}{4l}\sum_{m=1}^{\infty} C_m^2 m^4 \tag{3}$$

Assuming the crack is fully opened, it can be obtained from the theories of linear fracture mechanics that the change in elastic strain energy caused by introducing the crack in the column is

$$U_a = -\frac{(1-\nu^2)}{E}\int_0^A K_I^2(A)dA = -\frac{(1-\nu^2)A_0}{E}\int_0^{\xi} K_I^2(\zeta)d\zeta \tag{4}$$

By substituting Eq. (1) into (3) and using Eq. (2), the expression of U_a is derived as

$$U_a = -\frac{\pi^3(1-\nu^2)h\sigma_E A_0 \alpha^2}{\lambda^2}\int_0^{\xi}\zeta\left[\omega\cdot\left(\frac{e}{h}+\sum_{m=1}^{\infty}C_m \sin\frac{m\pi x_a}{l}\right)f_M(\zeta)-f_N(\zeta)\right]^2 d\zeta \tag{5}$$

where λ is the slenderness of the column.

The work performed by external forces is expressed as

$$U_p = \frac{P}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx - 2Pe\left(\frac{dy}{dx}\right)\bigg|_{x=0} = \frac{\pi^2 h^2 \sigma_E A_0 \alpha}{4l}\sum_{m=1}^{\infty} m^2 C_m^2 + \frac{2\pi h^2 \sigma_E A_0 \alpha}{l}\sum_{m=1}^{\infty} C_m m \tag{6}$$

2.3 Series solution of deflection

By means of the deflection function given by Eq. (2), the total potential energy of the elastic system may be written as

$$\Pi(C_1, C_2, \ldots, C_m, \ldots, C_n) = U_0 + U_a - U_p \tag{7}$$

From the principle of minimum potential energy, that is $\delta\Pi = 0$, the energy equilibrium equations can be expressed as

$$\frac{\partial \Pi}{\partial C_m} = \frac{\partial U_0}{\partial C_m} + \frac{\partial U_a}{\partial C_m} - \frac{\partial U_P}{\partial C_m} = 0 \tag{8}$$

The deflection coefficients C_m in the Eq. (2) are determined by Eq. (8) and the deflection curve can be thus obtained. This is the so called the Rayleigh-Ritz energy variational calculation.

Substituting Eqs. (3), (5) and (6) into Eq. (8), the equation of the deflection coefficients is given by

$$C_m = 4\alpha\frac{m\cdot\frac{e}{h}+\pi\eta\alpha\cdot\sin\frac{m\pi x_a}{l}\left[\omega\cdot\left(\frac{e}{h}+\frac{y_a}{h}\right)g_1(\xi)-g_2(\xi)\right]}{\pi m^2(m^2-\alpha)} \quad (m = 1, 2, \ldots, n) \tag{9}$$

where $\eta = \sqrt{3}\,\pi(1-\nu^2)/\lambda$, and the functions $g_1(\xi)$ and $g_2(\xi)$ are respectively defined as

$$\left.\begin{aligned} g_1(\xi) &= \int_0^{\xi}\zeta f_M^2(\zeta)d\zeta \\ g_2(\xi) &= \int_0^{\xi}\zeta f_M(\zeta)\cdot f_N(\zeta)d\zeta \end{aligned}\right\} \tag{10}$$

Finally, substituting Eq. (9) into Eq. (2) leads to the deflection equation of the cracked column

$$y(x)/h = \frac{4\alpha e}{\pi h}\sum_{m=1}^{\infty}\frac{\sin\frac{m\pi x}{l}}{m(m^2-\alpha)} + 4\eta\alpha^2\left[\omega\cdot\left(\frac{e}{h}+\frac{y_a}{h}\right)g_1(\xi) - g_2(\xi)\right]\cdot\sum_{m=1}^{\infty}\frac{\sin\frac{m\pi x_a}{l}\sin\frac{m\pi x}{l}}{m^2(m^2-\alpha)} \tag{11}$$

where the first and second terms correspond to the deflection equation of the uncracked column and the change of deflection owed to the crack, respectively.

2.4 Deflection of cracked section

It should be noticed that the deflection y_a is still undetermined. Letting $x = x_a$ in Eq. (11), the deflection of the cracked section can be shown as

$$\frac{y_a}{h} = \frac{\frac{4\alpha}{\pi}\cdot\frac{e}{h}\cdot\beta_1(\alpha, x_a/l) + 4\eta\alpha^2\beta_2(\alpha, x_a/l)\cdot\left[\omega\cdot\frac{e}{h}g_1(\xi) - g_2(\xi)\right]}{1 - 4\omega\eta\alpha^2\beta_2(\alpha, x_a/l)g_1(\xi)} \tag{12}$$

where $\beta_1(\alpha, x_a/l)$ and $\beta_1(\alpha, x_a/l)$ are defined as

$$\left.\begin{aligned}\beta_1(\alpha, x_a/l) &= \sum_{m=1}^{\infty}\frac{\sin\frac{m\pi x_a}{l}}{m(m^2-\alpha)}\\ \beta_2(\alpha, x_a/l) &= \sum_{m=1}^{\infty}\frac{\sin^2\frac{m\pi x_a}{l}}{m^2(m^2-\alpha)}\end{aligned}\right\} \tag{13}$$

3. Equations describing the equilibrium paths of the elastic buckling

3.1 The maximum deflection

It is easy to understand that when the crack is located at the middle section of the pinned-pinned column shown in Fig. 1 the crack effect is the most harmful. Letting $x_a/l = 1/2$ in Eqs. (12) and (13) yields that the maximum deflection of the column cracked in the middle is

$$\frac{\delta}{h} = \frac{\frac{4\alpha}{\pi}\cdot\frac{e}{h}\cdot\beta_1(\alpha, 0.5) + 4\eta\alpha^2\beta_2(\alpha, 0.5)\cdot\left[\omega\cdot\frac{e}{h}g_1(\xi) - g_2(\xi)\right]}{1 - 4\omega\eta\alpha^2\beta_2(\alpha, 0.5)g_1(\xi)} \tag{14}$$

and it is easily proved that

$$\left.\begin{aligned}\beta_1(\alpha, 0.5) &= \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)[(2m-1)^2-\alpha]} = \frac{\pi}{4\alpha}\left(\sec\frac{\pi\sqrt{\alpha}}{2} - 1\right)\\ \beta_2(\alpha, 0.5) &= \sum_{m=1}^{\infty}\frac{1}{(2m-1)^2[(2m-1)^2-\alpha]} = \frac{\pi}{4\alpha^{3/2}}\left(\tan\frac{\pi\sqrt{\alpha}}{2} - \frac{\pi\sqrt{\alpha}}{2}\right)\end{aligned}\right\} \tag{15}$$

Substituting Eq. (15) into Eq. (14), the closed form solution of the maximum deflection for the eccentrically cracked column with the rectangular cross-section can be obtained by

$$\frac{\delta}{h} = \frac{\frac{e}{h}\left(\sec\frac{\pi\sqrt{\alpha}}{2} - 1\right) + \pi\eta\sqrt{\alpha}\left[\omega\cdot\frac{e}{h}g_1(\xi) - g_2(\xi)\right]\left(\tan\frac{\pi\sqrt{\alpha}}{2} - \frac{\pi\sqrt{\alpha}}{2}\right)}{1 - \pi\omega\eta\sqrt{\alpha}g_1(\xi)\left(\tan\frac{\pi\sqrt{\alpha}}{2} - \frac{\pi\sqrt{\alpha}}{2}\right)} \tag{16}$$

This general δ/h formula is appropriate for rectangular cross-sectional columns with various shapes of model-I cracks. Eq. (16) confirms that, for a certain state of the crack, the crack effect increases with the increase of the parameter η or e/h. In other words, the lower the slenderness λ or the larger the eccentricity e/h, the more significant the crack effect is.

It is most important to emphasize that the formula (16) is only suitable for the column with an opening crack, i.e., $0 < K_I < K_{IC}$. For uncracked columns or closing cracked columns, i.e., $g_1(\xi) = g_2(\xi) = 0$, it can be derived from Eq. (16) that

$$\frac{\delta}{h} = \frac{e}{h}\left(\sec\frac{\pi\sqrt{\alpha}}{2} - 1\right) \tag{17}$$

This is the well-known secant formula of the eccentric column.

3.2 Crack opening and fracture condition

For the rectangular cross-section column with a single-edge through or surface crack, the combined different states of load and crack position causes different performances of cracks in the deformation and the buckling process. If the crack is located at the concave side of the section, the crack is "sleeping" all along and has no effects on the response of the column whether under axial or eccentric compression. However, if the crack is located at the convex side of the column, for the axial compressive column the crack will gradually effect on the post-buckling behavior after a period of time. For the eccentric column the crack is always "awake" and active and has obvious effects on the whole buckling process and the final fracture. Therefore, the establishment of both crack opening and fracture conditions is essential to analyze the equilibrium path of the cracked column.

As stated above, the stress intensity factor $K_I \leq 0$ indicates that the crack tip is closed and the crack has no effects; whereas the crack is fully opened, indicated by $K_I > 0$. Hence, the crack opening condition expressed with the maximum deflection can be obtained by use of Eq. (1), that is

$$\frac{\delta}{h} \geq \left(\frac{\delta}{h}\right)_0 = \frac{1}{\omega}\frac{f_N(\xi)}{f_M(\xi)} - \frac{e}{h} \tag{18}$$

The inequality (18) shows that the crack opening depends on the three ratios of the cracked section deflection, eccentricity and crack depth to column height, i.e., δ/h, e/h and $\xi = a/h$. For example, a curve family of the critical crack opening with eccentricity, i.e., $(\delta/h)_0 \sim e/h$ curves, is shown in Fig. 2, for various crack depths ξ, where a single-edge through crack is located in the middle of the rectangular cross-section column. The points (δ/h, e/h) above the curve correspond to the states of the crack opening, whereas the points below the curve surface correspond to the states

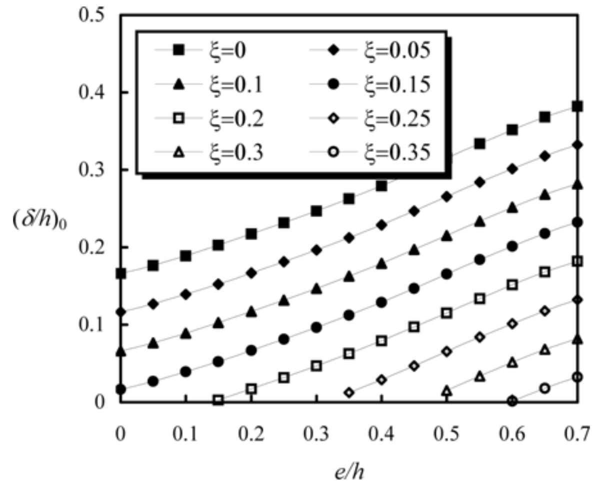


Fig. 2 The curve surface for critical crack opening

of the crack closing. It can be seen from Fig. 2 that the shallower the crack or the higher the eccentricity, the easier the crack opening is. If the eccentricity e/h exceeds 0.4, the crack will be constantly opened from beginning to end.

Assuming that the crack is located in the middle section, it is also readily understood from Eq. (1) that the stress intensity factor K_I increases with the increase of the deflection δ/h. When $K_I = K_{IC}$ occurs, the column will break. Thus the fracture condition expressed with the maximum deflection can be described as

$$\frac{\delta}{h} = \left(\frac{\delta}{h}\right)_F = \frac{\alpha\sqrt{\xi}f_N(\xi) + \theta}{\omega\alpha\sqrt{\xi}f_M(\xi)} - \frac{e}{h} \tag{19}$$

where $\theta = \lambda^2 K_{IC}/(\pi^{2.5}Eh^{0.5})$. The fracture point must be determined by solving simultaneous Eqs. (16) and (19), since the dimensionless load α in Eq. (19) is function of the deflection ratio δ/h.

3.3 Equations of the equilibrium path of the buckling

The equilibrium path, defined as the load-displacement curve in the deformation and buckling process, is most important for analyzing the buckling behavior of the structure. Each point on the path represents an equilibrium configuration of the structure. For engineering structures, which are always imperfect, the determination of the ultimate bearing capacity must be based on the equilibrium path, including the practical axial compressive column with its cracks due to the initial eccentricity or curvature. Otherwise, the results will be unrealistic and sometimes may lead to foundational deviation. So, the equilibrium path has received a considerable amount of attention in the present study. For the uncracked linear elastic column, the equilibrium path is simple and well known. In this paper, therefore, the analysis of the equilibrium path will focus on cracked columns subjected to both axial and eccentric compression.

Based on above analysis and results, the equilibrium path of cracked column under axial or eccentric compression can be computationally determined. On the equilibrium path there are usually

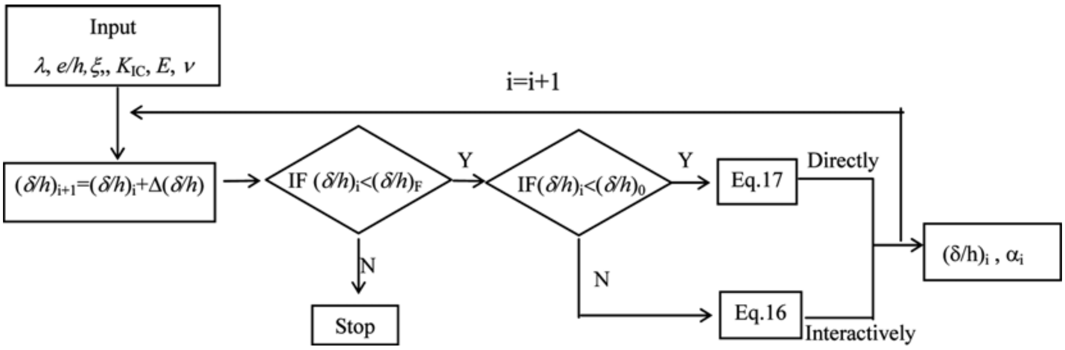


Fig. 3 The schematic diagram of the computational program for determining the equilibrium path

two or more points corresponding to a specific load, while there is only one point corresponding to a definite deflection. Therefore, the deflection-control method should be employed in order to improve the efficiency of calculations. The schematic diagram of the computational program for determining the equilibrium path is shown in Fig. 3.

4. Examples and analysis

The mechanical properties of the steel 16Mn, used for the calculation in this section, are as follows: fracture toughness K_{IC} = 4282 $\text{Nmm}^{1.5}$; elastic modulus E = 206×10^3 N/mm^2, Poisson's ratio $\nu = 0.3$. The sectional parameter $\omega = A_0 h/W$ equals 6 for rectangular columns. A single-edge through crack located at the convex side of the middle section of the column was also considered.

The configuration correction factors of the SIF under bending and tension can be known from Hellan (1984); they are

$$\left.\begin{aligned} f_M(\xi) &= 1.12 - 1.40\xi + 7.33\xi^2 - 13.08\xi^3 + 14.00\xi^4 \\ f_N(\xi) &= 1.12 - 0.23\xi + 10.55\xi^2 - 21.72\xi^3 + 30.39\xi^4 \end{aligned}\right\} \tag{20}$$

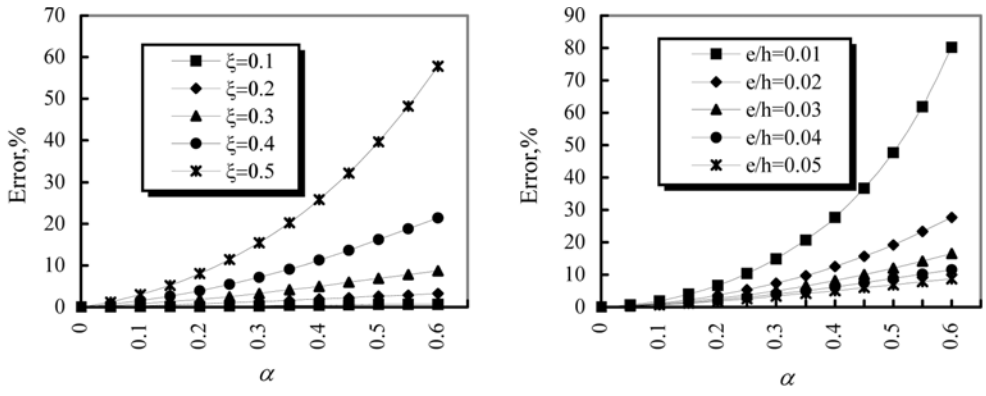
with the accuracy of 0.5% as $\xi \le 0.7$. The functions $g_1(\xi)$ and $g_2(\xi)$ defined by Eq. (10) are thus determined through integration. They are as follows

$$\begin{aligned} g_1(\xi) &= \xi^2(0.63 - 1.05\xi + 4.60\xi^2 - 9.98\xi^3 + 20.30\xi^4 - 32.99\xi^5 + 47.04\xi^6 - 40.69\xi^7 + 19.60\xi^8) \\ g_2(\xi) &= \xi^2(0.63 - 0.61\xi + 5.09\xi^2 - 11.10\xi^3 + 26.33\xi^4 - 47.46\xi^5 + 81.82\xi^6 - 77.95\xi^7 + 42.55\xi^8) \end{aligned} \tag{21}$$

4.1 Comparison between present and Okamura's solution

The existing solution of the maximum deflection δ/h for an eccentric compressive rectangular column with single-edge crack was given by Okamura *et al.* (1969) using the rotational spring model,

$$\frac{\delta}{h} = \frac{e}{h} = \frac{1}{\cos\frac{\pi\sqrt{\alpha}}{2} - 12\eta\sqrt{\alpha}g_1(\xi)\sin\frac{\pi\sqrt{\alpha}}{2}} \tag{22}$$



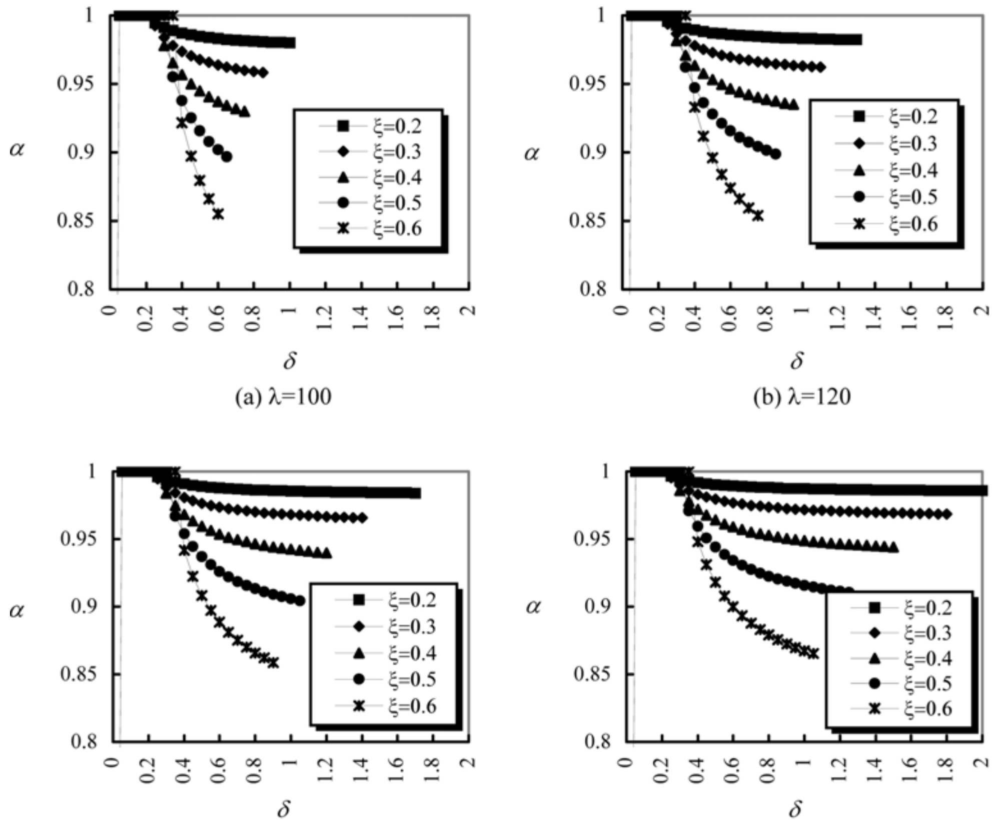
(a) e/h=0.05, λ=100 (b) ξ=0.3, λ=100

Fig. 4 The relative error of Okamura solution

The relative error curves of δ/h vs load level α for different crack depths ξ and eccentricities e/h are shown in Figs. 4(a,b) respectively, where $Error(\%)=100\%[\delta/h(\text{Eq. (22)}) - \delta/h(\text{Eq. (16)})]/\delta/h$ (Eq. (16)). It can be found from Fig. 4 that the heavier the load or the deeper the crack or the lower the eccentricity, the larger the relative error. Fig. 4 also shows that the Okamura's results for δ/h are constantly higher than the closed form resolution obtained in this paper. By comparison with Eq. (16), the function $g_2(\xi)$ does not appear in Eq. (22), that reflects the beneficial effect of axial force N on the crack closure. This is the main reason why the rotational spring model leads to obvious deviation of the deflection under the conditions of a deep crack and low eccentricity.

4.2 Equilibrium paths of cracked columns under axial compression

According to the formulations and programs established in section 3, the equilibrium paths for a cracked column subjected to axial compression were analyzed computationally. A series of equilibrium paths, i.e., dimensionless load-deflection plots, for various crack depths and slenderness of columns are illustrated in Fig. 5. The results show that in the initial stages of the buckling process there is still a bifurcation phenomenon and the critical load remains the same as in the uncracked column. This is because of crack closure. However, in the post-buckling stages, the load begins to decrease with the increase of the deflection after a period of deflection. These facts clarify that for axial compressive columns the crack has effect only on the post-buckling behavior. The conclusion presented in the recent work (Wang and Chase 2003) that the ultimate axial compression capacity of cracked columns decreases due to the crack is obviously inappropriate, because the work (Wang and Chase 2003) has not taken into account the effects of the crack closure. It can also be observed from Fig. 5 that in post-buckling stages the crack has a clear effect on both the speed of the load decrease and the ultimate deflection, which may be defined as the ductility of the column. The deeper the crack, the sharper the load decreases, and the poorer the ductility of the column is. Moreover, for a column with a lower slenderness λ, the effect of the crack on post buckling behavior is more significant.



(a) λ=100 (b) λ=120

(c) λ=140 (d) λ=160

Fig. 5 A series of equilibrium path diagrams for axial compressive columns

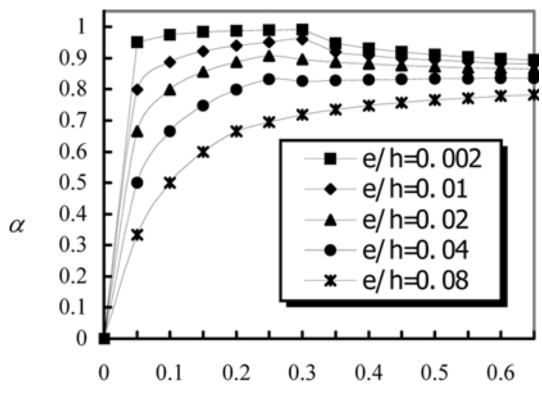


Fig. 6 The equilibrium paths with varied eccentricities

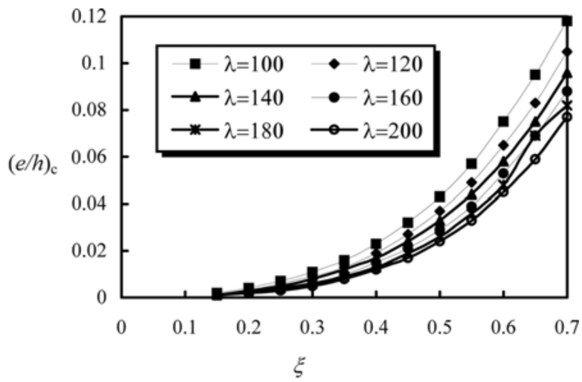


Fig. 7 A curve family of $(e/h)_c$~ξ for varied slenderness λ

4.3 Equilibrium paths of cracked columns under eccentric compression

The equilibrium paths for cracked columns that are subjected to an eccentric compression are depicted systematically and the compound effects of crack, eccentricity and slenderness are analyzed. Fig. 6 shows five equilibrium paths of the cracked columns with the eccentricity *e/h* from 0.002 to 0.08, where $\xi = 0.5$ and $\lambda = 100$.

From Fig. 6 it can be seen that the bifurcation phenomenon disappears in the buckling process of the eccentric column and the shapes of equilibrium paths are varied with respect to the increase of eccentricity. For slightly eccentric columns, there is a limit point on each path, while for highly eccentric columns the path is monotonically increasing until the fracture suddenly occurs. As the eccentricity increases, there is a critical transition point $(e/h)_c$ from limit-point buckling to fracture failure for a certain state of the crack-slenderness ($\xi - \lambda$). The critical eccentricity $(e/h)_c$ is definitely the function of crack depth ξ and column slenderness λ.

A curve family of $(e/h)_c$~ξ is numerically determined as shown in Fig. 7 for varied slenderness λ. If the *e/h* point is below the curve the column failure will be the limit point buckling, while if the *e/h* point is above the curve the column failure will be fracture-dominant. From Fig. 7 it can be

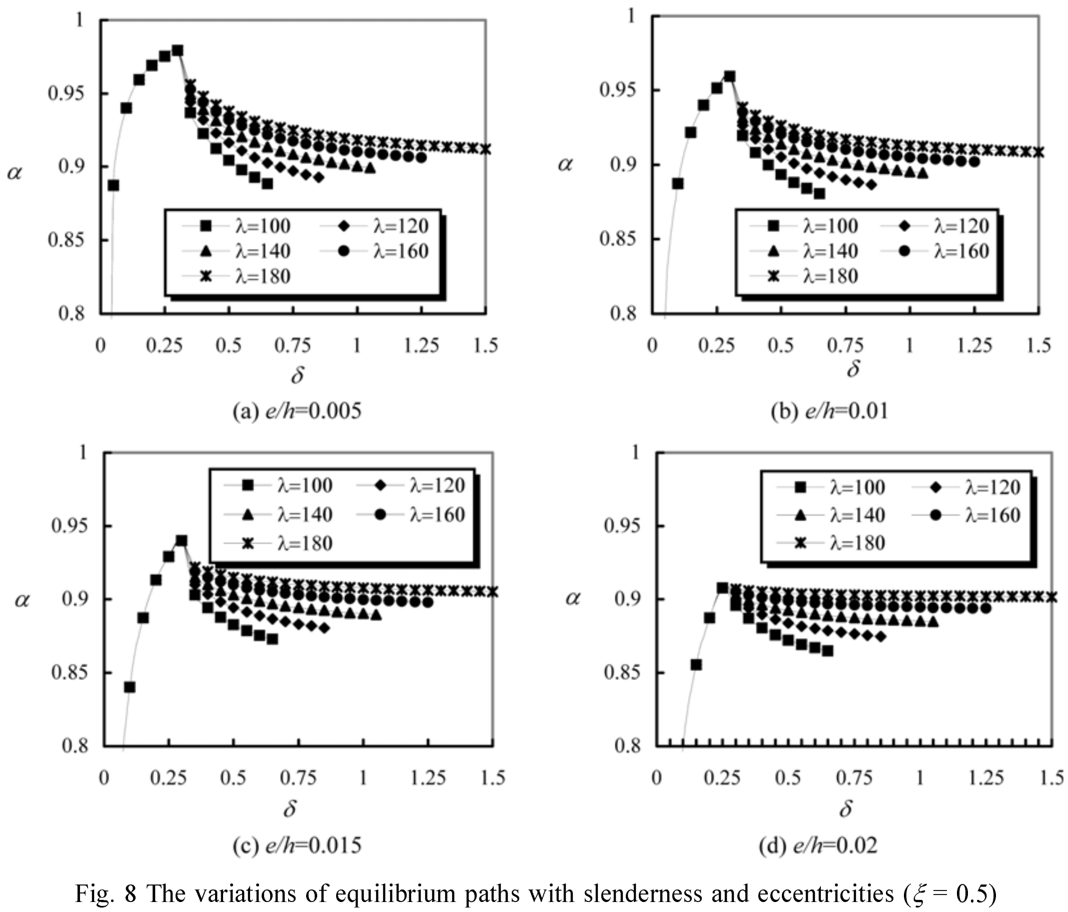


Fig. 8 The variations of equilibrium paths with slenderness and eccentricities ($\xi = 0.5$)

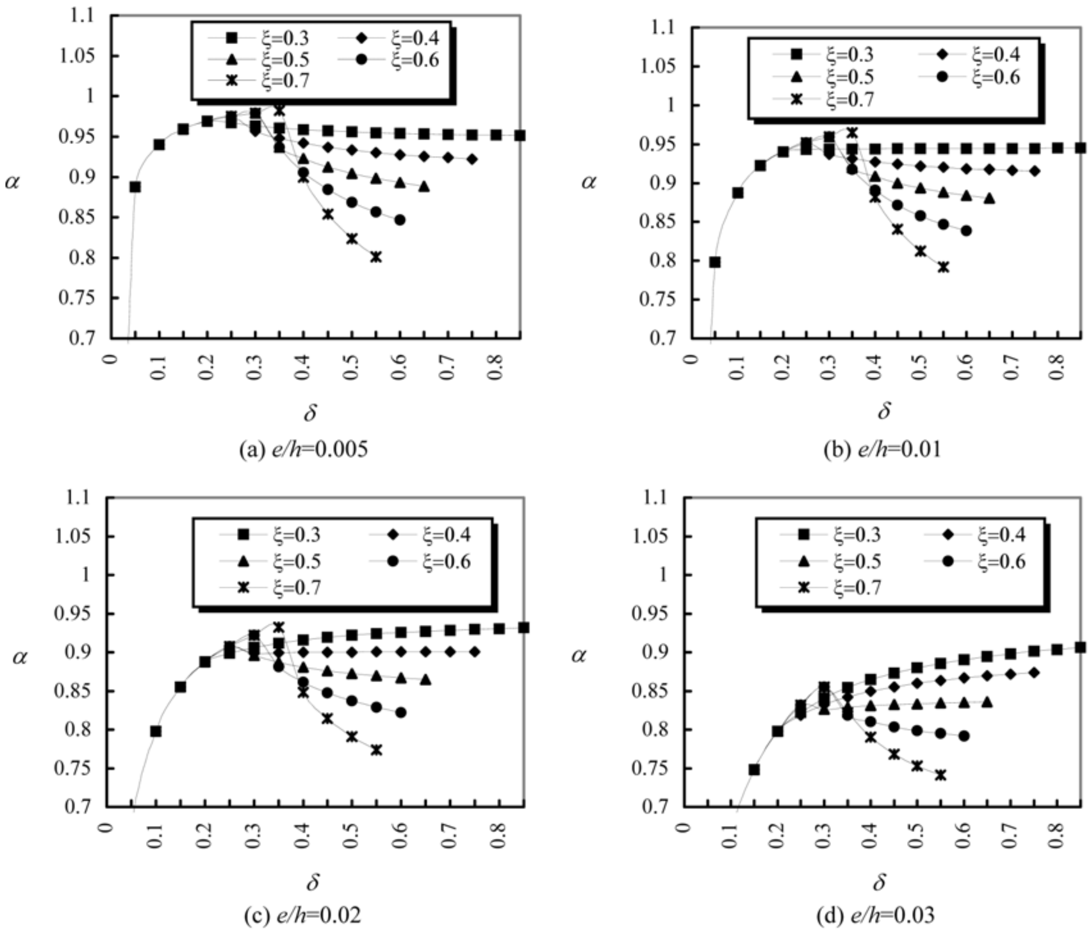


Fig. 9 The variations of the equilibrium path with crack depth and eccentricity (λ = 100)

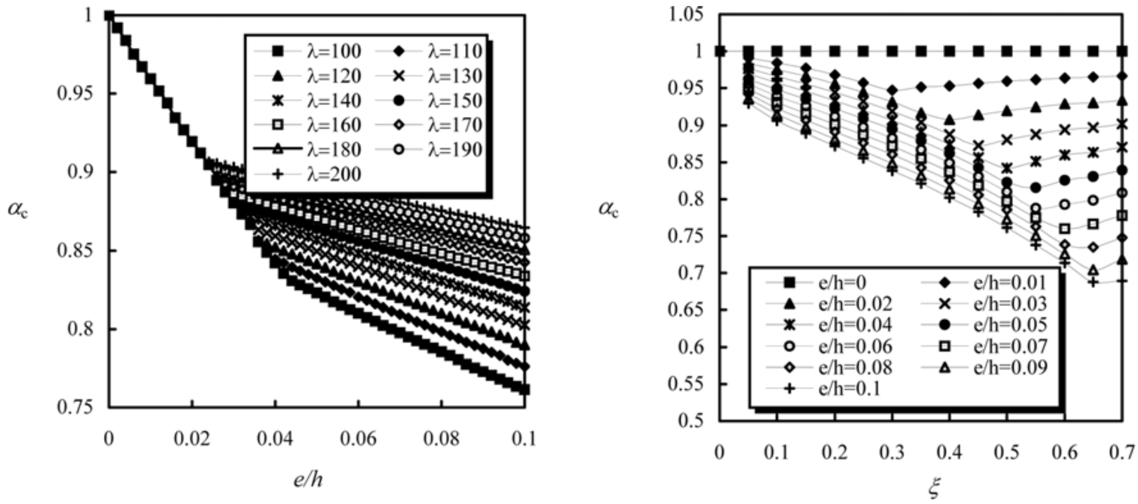
observed that the slighter the slenderness or the deeper the crack is, the higher the value $(e/h)_c$, i.e., the range of the eccentricity e/h is more extensive for the happening of the limit-point buckling.

A group of the equilibrium path curves with limit points are illustrated in Fig. 8. It shows that as the eccentricity increases, the flexibility of pre-buckling deformation is increasing and the limit load is decreasing. Another important discovery is that the post-buckling behavior and the fracture point are related to the slenderness, while the pre-buckling behavior is insensitive to the slenderness.

The effects of cracks on the equilibrium paths of eccentric columns are depicted in Fig. 9. It shows that the crack performance influencing the buckling behavior of the eccentric column is similar to the uncracked columns. Both the post-buckling process and the fracture point are influenced by the appearance of the crack. The deeper the crack, the sharper the load decreases and the poorer the ductility of the column. However, the pre-buckling behavior and limit load are insensitive to the crack. Moreover, as eccentricity e/h increases, the limit load decreases and the flexibility of the pre-buckling stages is increases gradually.

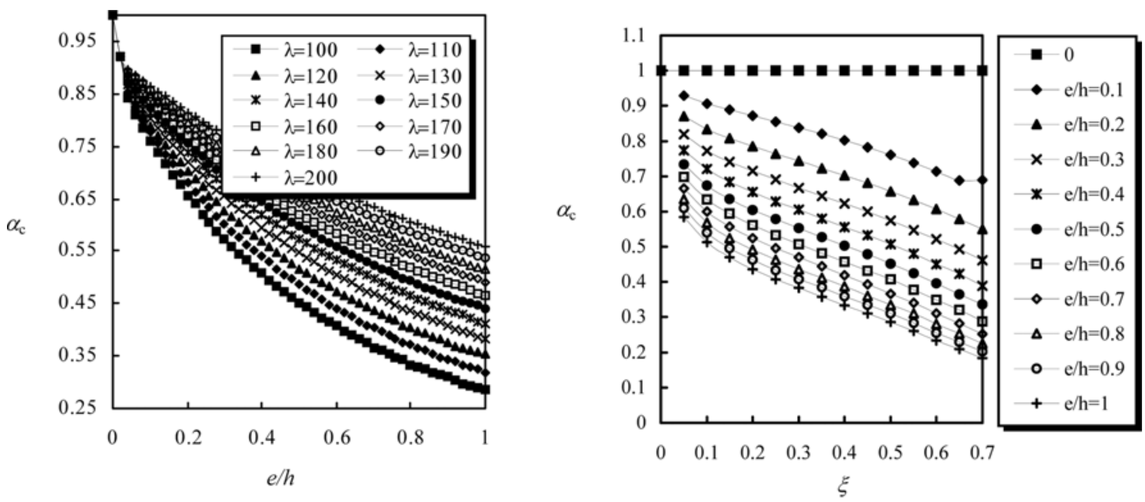
4.4 Effects of a crack on ultimate bearing capacity

For the cracked eccentric columns, the ultimate bearing capacity is a function of slenderness λ, eccentricity e/h and crack depth ξ. Moreover, there are three kinds of possible failure mechanisms: bifurcation buckling, limit point instability and fracture. Hence the determination of the load capacity should be based on the equilibrium paths. For a small range of eccentricities, the two curve families of the critical load parameter α_c with respect to the eccentricity e/h and the relative crack depth ξ are shown in Figs. 10(a) and (b) respectively, where λ = 100. The two curve families of the



(a) A curve family of load capacity α_c - e/h (b) A curve family of load capacity α_c $-\xi$

Fig. 10 Two curve families of load capacity for a small range of eccentricity, ξ = 0.5



(a) A curve family of load capacity α_c - e/h (b) A curve family of load capacity α_c -ξ

Fig. 11 The curve families of load capacity for a large range of eccentricity, λ = 100 critical load parameter α_c are shown in Figs. 11(a) and (b) for a large range of eccentricities, where $\xi = 0.5$.

From Fig. 10, it can be observed that for the cases of small eccentricity $e/h < 0.1$, the capacity parameter α_c monotonically and bilinearly decreases with the increase of the eccentricity e/h but non-monotonically varies with the relative crack depth ξ. It is a universal phenomenon that on each curve there is a critical transition point from the limit point buckling to the fracture-dominate failure. For the minute eccentricities of $e/h < 0.02$, which approximately fall in the range of the limit point buckling, the parameter α_c linearly decreases with the increase of the eccentricity e/h but is insensitive to the variations of the slenderness λ.

Fig. 11 indicates that for large eccentricities $e/h > 0.1$, that fall in the range of the fracture type of failure, the bearing capacity parameter α_c monotonically decreases with the increase of both eccentricity e/h and crack depth ξ, while it monotonically increases with the increase of the slenderness λ.

In summary, under the failure conditions of the fracture-dominate type, the higher the eccentricity e/h or the deeper the crack ξ, or lower the slenderness λ is, the lower the ultimate bearing capacity α_c. Under the failure conditions of limit point buckling, the capacity α_c is insensitive to the variation of the slenderness λ. It is also most important to emphasize that the variation laws of the capacity parameter α_c with the slenderness λ discussed above are different from the variation laws of the ultimate load P_c or the ultimate stress σ_c with the slenderness λ, because $P_c = A_0\sigma_c = A_0\pi E/\lambda^2$.

5. Conclusions

In the present work, the elastic buckling behavior and ultimate bearing capacity were systematically studied for cracked uniform columns with rectangular cross-sections subjected to both axial and eccentric compression. The crack closure and the fracture conditions were addressed in the equilibrium path analysis. The main contributions are the establishment of the close-form solution for maximum deflection by use of the energy method to describe analytically the equilibrium paths of the elastic buckling. And the combined influences of crack depth, load eccentricity and slenderness on the elastic buckling behaviors and the ultimate bearing capacity were also investigated. This study provides the theoretical foundation for further analyzing the combined influence of geometrical imperfections and physical defects on the buckling behavior of complex structures. The numerical results from the examples show that there are three kinds of collapse mechanisms for various states of cracks, eccentricity and slenderness, which are the bifurcation for axial compression; the limit point instability for deeper cracks and light eccentricity conditions; and the fracture for higher eccentricity. Moreover, the parameter $(e/h)_c$ of the critical transition eccentricity from limit-point buckling to fracture failure was proposed and determined numerically for various eccentricities, crack depths and slenderness.

Based on the analytical and numerical results from the cracked columns, the following brief conclusions can be drawn:

(1) Okamura's solution of the maximum deflection from rotational springs for eccentric columns is only suitable for the conditions of a lower load, shallower crack and lighter eccentricity. The main reason for this is that the beneficial effect of the axial force N on a crack closure has not been reflected;

(2) With the decrease of the slenderness λ or the increase of the eccentricity e/h, the crack effect

on the elastic response is more significant;

(3) The shallower the crack or the higher the eccentricity, the easier the crack opens. If the eccentricity e/h exceeds 0.4, the crack will be constantly open;

(4) For axial compressive columns, the crack only has effect on the post-buckling behavior owed to the crack opening. The conclusion drawn in the recent work (Wang and Chase 2003) that the ultimate bearing capacity of the cracked column decreases because of the crack is incorrect;

(5) For eccentric columns, as the eccentricity increases, the critical eccentricity $(e/h)_c$ is certainly a function of crack depth ξ and column slenderness λ. The slighter the slenderness or the deeper the crack, the higher is the value $(e/h)_c$;

(6) With the increase of the eccentricity, the flexibility of the pre-buckling deformation is increasing and the limit load is decreasing. Additionally, the post-buckling behavior and the fracture point are related to the changes in slenderness, but pre-buckling behavior is insensitive to it;

(7) Under the failure conditions of the fracture-dominate type, the higher the eccentricity e/h or the deeper the crack ξ or lower the slenderness λ is, the lower the load capacity parameter α_c becomes.

It should be emphasized that although the analysis and conclusions in this paper are all on columns with both ends pinned, the method is easily utilized for columns with other types of constraint conditions. In addition, the present work is based on the conditions of elastic deformation and small rotation. However, some of the ideas, techniques and conclusions proposed may be helpful for analyzing the nonlinear and elastic-plastic buckling problems found in complex structures with geometrical imperfections and physical defects.

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