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Use of equivalent spring method for free vibration analyses of a rectangular plate carrying multiple three-degree-of-freedom spring-mass systems

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Abstract. Due to the complexity of mathematical expressions, the literature concerning the free vibration analysis of plates carrying *multiple three*-degree-of-freedom (dof) spring-mass systems is rare. In this paper, the three degrees of freedom (dof's) for a spring-mass system refer to the translational motion of its lumped mass in the vertical ($\bar{z}$) direction and the two pitching motions of its lumped mass about the two horizontal ($\bar{x}$ and $\bar{y}$) axes. The basic concept of this paper is to replace each three-dof spring-mass system by a set of equivalent springs, so that the free vibration characteristics of a rectangular plate carrying any number of three-dof spring-mass systems can be obtained from those of the same plate supported by the same number of sets of equivalent springs. Since the three dof's of the lumped mass for each three-dof spring-mass system are eliminated to yield a set of equivalent springs, the total dof of the entire vibrating system is *not* affected by the total number of the spring-mass systems attached to the rectangular plate. However, this is not true in the conventional finite element method (FEM), where the total dof of the entire vibrating system increases three if one more three-dof spring-mass system is attached to the rectangular plate. Hence, the computer storage memory required by using the presented *equivalent spring method* (ESM) is less than that required by the conventional FEM, and the more the total number of the three-dof spring-mass systems attached to the plate, the more the advantage of the ESM. In addition, since manufacturing a spring with the specified stiffness is much easier than making a three-dof spring-mass system with the specified spring constants and mass magnitude, the presented theory of replacing a three-dof spring-mass system by a set of *equivalent springs* will be also significant from this viewpoint.

Key words: equivalent spring method; finite element method; rectangular plate; three-dof spring-mass system; free vibration.

1. Introduction

Vibration characteristics of a structure mounted with various concentrated elements, such as lumped mass, distributed mass, springs, etc., are important information for machine designers. For this reason, beams and plates carrying various concentrated elements have called for the attentions of several researchers. For example, Rossi *et al.* (1993) have studied the exact solutions for the frequencies and mode shapes of a Timoshenko beam carrying a spring-mass system with three types

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of boundary conditions. Wu and Chou (1998) have calculated the natural frequencies and mode shapes of a Bernoulli-Euler cantilever beam carrying multiple one-dof spring-mass systems using the analytical-and-numerical-combined method and Wu and Chen (2001) have determined those of a Timoshenko beam carrying multiple one-dof spring-mass systems using the numerical assembly technique. Larrondo *et al.* (1992) and Rossit and Laura (2001) have performed the free vibration analysis of a Bernoulli-Euler beam with elastically mounted concentrated masses by means of various analytical approaches. The theoretical results were compared with the experimental ones and satisfactory agreement was achieved. Gúrgóze (1996) has used the Lagrange method to derive the frequency equation of a clamped-free Bernouli-Euler beam mounted with a tip mass and a spring-mass system. Wu and Whittaker (1999), Wu (2002), Chang and Chang (1998) and Jen and Magrab (1993) have investigated those of beams carrying single and multiple two-dof spring-mass systems by means of various approaches. Ingber *et al.* (1992) have performed the free vibration analysis of a clamped plate with concentrated masses and springs by means of mixed boundary-finite element method and confirmed the theoretical results with their experiments. Rossi and Laura (1996) have investigated the influence of Poisson's ratio and position of concentrated mass on the natural frequencies and mode shapes of a cantilever plate using finite element method. Since it may be not always reasonable to simplify the attachments as concentrated masses, Kopmaz and Telli (2002) have studied the free vibrations of a rectangular plate carrying a distributed mass by means of the Galerkin method. Avalos *et al.* (1993, 1994) and Wu and Luo (1997a, 1997b) have respectively investigated the free vibrations of plates with various boundary conditions carrying elastically mounted mass(es) (i.e., one-dof spring-mass system(s)) using the optimized Rayleigh-Ritz method and analytical-and-numerical-combined method, respectively. Recently, Wu (2005) has studied the free vibration characteristics of a rectangular plate carrying multiple three-degree-of-freedom spring-mass systems using the equivalent mass method.

From the review of existing literature, one sees that the literature concerning the *plates* carrying various concentrated elements is much less than that concerning the *beams* carrying various concentrated elements. Besides, due to the complexity of the mathematical expressions, either the concentrated masses or the distributed masses were rigidly attached to the plate in the works of Ingber *et al*. (1992), Rossi and Laura (1996), Kopmaz and Telli (2002) and the literature regarding the elastic attachments is comparatively fewer. The free vibration analysis of a plate carrying a single *one*-dof spring-mass system has been made by Avalos *et al.* (1993, 1994) and that carrying multiple *one*-dof spring-mass systems by Wu and Luo (1997a, 1997b). For the vibration problem of a plate carrying *multiple three*-dof spring-mass systems, the work of Wu (2005) should be one of the most concerned literature. Because the information regarding vibration characteristics of plates carrying multi-dof spring-mass systems is limited, the last problem is further studied. Comparing with the work of Wu (2005), the advantages of the current paper are: (i) This paper uses the *equivalent spring method*, instead of the *equivalent mass method* (Wu 2005), to tackle the problem. This provides a technique for evaluating the overall elastic effect of each three-dof spring-mass system. (ii) Because manufacturing a spring with the specified stiffness is much easier than making a three-dof spring-mass system with the specified spring constants and mass magnitude, the presented theory of replacing a three-dof spring-mass system by a set of *equivalent springs* will be significant in certain practical applications.

This paper starts with the derivation of the stiffness and mass matrices of a three-dof spring-mass system, required by conventional finite element method (FEM), based on the force and moment equilibrium equations and then eliminates the three degrees of freedom (dof's) for the lumped mass of the spring-mass system from the foregoing element stiffness and mass matrices to yield the effective stiffness matrix and a set of equivalent springs for the three-dof spring-mass system, required by the presented equivalent spring method (ESM). Next, the overall property matrices of the entire vibrating system are determined using the standard assembly technique of finite element method (Bathe 1982). The main difference between FEM and ESM is that, in FEM, the total dof of the entire vibrating system increases three if one more three-dof spring-mass system is attached to the plate, while, in ESM, the total dof of the entire vibrating system is not affected by the total number of the three-dof spring-mass systems attached to the plate. This is because all dof's of a three-dof spring-mass system are suppressed by its *equivalent springs*. The last feature of ESM will save much computer storage memory if the total number of spring-mass systems attached to the plate is very large.

For convenience, a rectangular plate is called the *bare plate* if it carries nothing, and is called the *loaded plate* if it carries any number of spring-mass systems in this paper. In FEM, the natural frequencies and the corresponding mode shapes of the loaded plate are determined using Lanczos algorithm (Cullum and Willoughby 2002), however, in ESM, they are determined using half-interval technique and Gauss-Jordan elimination method (Gerald and Wheatley 1998), respectively, because the coefficients of the stiffness matrix for each set of *equivalent springs* are functions of natural frequencies of the loaded plate. For validation, the natural frequencies of a simply supported rectangular plate carrying a three-dof spring-mass system determined by the FEM and ESM are compared with those of the same plate carrying a one-dof spring-mass system. It is found that, if the mass, the resultant spring constant and the attached positions of the three-dof spring-mass system are close to the associated ones of the one-dof spring-mass system, and the mass moments of inertia of the three-dof spring-mass system approach zero, then the natural frequencies of the rectangular plate carrying a three-dof spring-mass system will be close to those of the same plate carrying a one-dof spring-mass system. Finally, the free vibration characteristics of a rectangular plate carrying multiple three-dof spring-mass systems are studied to show the availability of the presented technique.

2. Property matrices for a three-dof spring-mass system

For the rectangular plate carrying a three-dof spring-mass system as shown in Fig. 1, the dynamic equilibrium of the spring-mass system requires that

$$\sum F_z = F_p + F_q + F_r + F_s + F_v - m_z^{(v)} \ddot{u}_v = 0 \tag{1}$$

$$\sum M_x = -(F_p + F_s)a_{y1}^{(v)} + (F_q + F_r)a_{y2}^{(v)} + M_x^{(v)} - J_x^{(v)} \ddot{\theta}_x^{(v)} = 0 \tag{2}$$

$$\sum M_y = (F_p + F_q)a_{x1}^{(v)} - (F_r + F_s)a_{x2}^{(v)} + M_y^{(v)} - J_y^{(v)} \ddot{\theta}_y^{(v)} = 0 \tag{3}$$

where F_v, $M_x^{(v)}$ and $M_y^{(v)}$ represent the external force and moments on the lumped mass ($m_z^{(v)}$) of the v-th spring-mass system in the vertical ($\bar{z}$) direction and about the $\bar{x}$ and $\bar{y}$ axes, respectively, $\ddot{u}_v, \ddot{\theta}_x^{(v)}$ and $\ddot{\theta}_y^{(v)}$ are the associated accelerations, $m_z^{(v)}, J_x^{(v)}$ and $J_y^{(v)}$ are the associated mass and mass moments of inertia for the lumped mass in the vertical ($\bar{z}$) direction and about the $\bar{x}$ and $\bar{y}$ axes, respectively, $a_{x1}^{(v)}$ and $a_{x2}^{(v)}$ are parameters to define the spacing of the four helical springs and

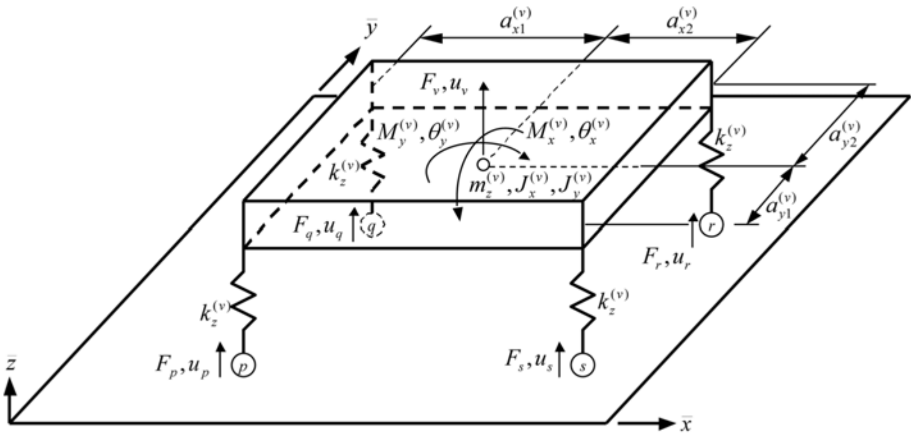

Fig. 1 A rectangular plate carrying an arbitrary three-dof spring-mass system

eccentricity of the center of gravity of the v-th lumped mass ($m_z^{(v)}$) in the $\bar{x}$ direction, while $a_{y1}^{(v)}$ and $a_{y2}^{(v)}$ are those in the $\bar{y}$ direction. Besides, F_i ($i = p, q, r, s$) represent the interactive forces between the three-dof spring-mass system and the plate at the four attaching points, p, q, r and s, given by

$$F_p = k_z^{(v)}(u_p - u_v + a_{y1}^{(v)}\theta_x^{(v)} - a_{x1}^{(v)}\theta_y^{(v)}) \tag{4}$$

$$F_q = k_z^{(v)}(u_q - u_v - a_{y2}^{(v)}\theta_x^{(v)} - a_{x1}^{(v)}\theta_y^{(v)}) \tag{5}$$

$$F_r = k_z^{(v)}(u_r - u_v - a_{y2}^{(v)}\theta_x^{(v)} + a_{x2}^{(v)}\theta_y^{(v)}) \tag{6}$$

$$F_s = k_z^{(v)}(u_s - u_v + a_{y1}^{(v)}\theta_x^{(v)} + a_{x2}^{(v)}\theta_y^{(v)}) \tag{7}$$

where $k_z^{(v)}$ denotes the spring constant for each of the four helical springs, u_v, $\theta_x^{(v)}$ and $\theta_y^{(v)}$ are the displacement and rotational angles for the lumped mass ($m_z^{(v)}$) of the v-th spring-mass system in the vertical ($\bar{z}$) direction and about the $\bar{x}$ and $\bar{y}$ axes, respectively, while u_i denotes the vertical displacement of the rectangular plate at the attaching point i ($i = p, q, r, s$) of the v-th spring-mass system.

Introducing Eqs. (4)-(7) into Eqs. (1)-(3), one obtains

$$m_z^{(v)}\ddot{u}_v - k_z^{(v)}[u_p + u_q + u_r + u_s - 4u_v + 2(a_{y1}^{(v)} - a_{y2}^{(v)})\theta_x^{(v)} - 2(a_{x1}^{(v)} - a_{x2}^{(v)})\theta_y^{(v)}] = F_v \tag{8}$$

$$\begin{aligned} &J_x^{(v)}\ddot{\theta}_x^{(v)} + k_z^{(v)}[a_{y1}^{(v)}u_p - a_{y2}^{(v)}u_q - a_{y2}^{(v)}u_r + a_{y1}^{(v)}u_s - 2(a_{y1}^{(v)} - a_{y2}^{(v)})u_v \\ &+ 2(a_{y1}^{(v)^2} + a_{y2}^{(v)^2})\theta_x^{(v)} - (a_{x1}^{(v)}a_{y1}^{(v)} - a_{x2}^{(v)}a_{y1}^{(v)} - a_{x1}^{(v)}a_{y2}^{(v)} + a_{x2}^{(v)}a_{y2}^{(v)})\theta_y^{(v)}] = M_x^{(v)} \end{aligned} \tag{9}$$

$$\begin{aligned} &J_y^{(v)}\ddot{\theta}_y^{(v)} - k_z^{(v)}[a_{x1}^{(v)}u_p + a_{x1}^{(v)}u_q - a_{x2}^{(v)}u_r - a_{x2}^{(v)}u_s - 2(a_{x1}^{(v)} - a_{x2}^{(v)})u_v \\ &+ (a_{x1}^{(v)}a_{y1}^{(v)} - a_{x1}^{(v)}a_{y2}^{(v)} + a_{x2}^{(v)}a_{y2}^{(v)} - a_{x2}^{(v)}a_{y1}^{(v)})\theta_x^{(v)} - 2(a_{x1}^{(v)^2} + a_{x2}^{(v)^2})\theta_y^{(v)}] = M_y^{(v)} \end{aligned} \tag{10}$$

Writing Eqs. (4)-(10) in matrix form yields

$$[m]\{\ddot{u}\} + [k]\{u\} = \{F\} \tag{11}$$

where

$$[m] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_z^{(v)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_x^{(v)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & J_y^{(v)} \end{bmatrix} \tag{12a}$$

$$[k] = k_z^{(v)} \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & a_{y1}^{(v)} & -a_{x1}^{(v)} \\ 0 & 1 & 0 & 0 & -1 & -a_{y2}^{(v)} & -a_{x1}^{(v)} \\ 0 & 0 & 1 & 0 & -1 & -a_{y2}^{(v)} & a_{x2}^{(v)} \\ 0 & 0 & 0 & 1 & -1 & a_{y1}^{(v)} & a_{x2}^{(v)} \\ -1 & -1 & -1 & -1 & 4 & X_1 & X_2 \\ a_{y1}^{(v)} & -a_{y2}^{(v)} & -a_{y2}^{(v)} & a_{y1}^{(v)} & X_1 & X_3 & X_4 \\ -a_{x1}^{(v)} & -a_{x1}^{(v)} & a_{x2}^{(v)} & a_{x2}^{(v)} & X_2 & X_4 & X_5 \end{bmatrix} \tag{12b}$$

$$\{u\} = [u_p \;\; u_q \;\; u_r \;\; u_s \;\; u_v \;\; \theta_x^{(v)} \;\; \theta_y^{(v)}]^T \tag{12c}$$

$$\{\ddot{u}\} = [\ddot{u}_p \;\; \ddot{u}_q \;\; \ddot{u}_r \;\; \ddot{u}_s \;\; \ddot{u}_v \;\; \ddot{\theta}_x^{(v)} \;\; \ddot{\theta}_y^{(v)}]^T \tag{12d}$$

$$\{F\} = [F_p \;\; F_q \;\; F_r \;\; F_s \;\; F_v \;\; M_x^{(v)} \;\; M_y^{(v)}]^T \tag{12e}$$

$$X_1 = -2(a_{y1}^{(v)} - a_{y2}^{(v)}) \tag{12f}$$

$$X_2 = 2(a_{x1}^{(v)} - a_{x2}^{(v)}) \tag{12g}$$

$$X_3 = 2(a_{y1}^{(v)^2} + a_{y2}^{(v)^2}) \tag{12h}$$

$$X_4 = -(a_{x1}^{(v)}a_{y1}^{(v)} - a_{x2}^{(v)}a_{y1}^{(v)} - a_{x1}^{(v)}a_{y2}^{(v)} + a_{x2}^{(v)}a_{y2}^{(v)}) \tag{12i}$$

$$X_5 = 2(a_{x1}^{(v)^2} + a_{x2}^{(v)^2}) \tag{12j}$$

In Eqs. (11) and (12), $[m]$ and $[k]$ are respectively the mass matrix and stiffness matrix of the v-th three-dof spring-mass system shown in Fig. 1.

3. Effective stiffness matrix for a three-dof spring-mass system

For free vibration, the external force and moments are zero, i.e., $F_v = M_x^{(v)} = M_y^{(v)} = 0$. Thus, Eqs. (1)-(3) respectively reduce to

$$m_z^{(v)}\ddot{u}_v - F_p - F_q - F_r - F_s = 0 \tag{13}$$

$$J_x^{(v)}\ddot{\theta}_x^{(v)} + (F_p + F_s)a_{y1}^{(v)} - (F_q + F_r)a_{y2}^{(v)} = 0 \tag{14}$$

$$J_y^{(v)}\ddot{\theta}_y^{(v)} - (F_p + F_q)a_{x1}^{(v)} + (F_r + F_s)a_{x2}^{(v)} = 0 \tag{15}$$

Introducing Eqs. (4)-(7) into Eqs. (13)-(15) yields

$$m_z^{(v)}\ddot{u}_v + k_z^{(v)}[-u_p - u_q - u_r - u_s + 4u_v - 2(a_{y1}^{(v)} - a_{y2}^{(v)})\theta_x^{(v)} + 2(a_{x1}^{(v)} - a_{x2}^{(v)})\theta_y^{(v)}] = 0 \tag{16}$$

$$\begin{aligned} J_x^{(v)}\ddot{\theta}_x^{(v)} &+ k_z^{(v)}[u_p a_{y1}^{(v)} - u_q a_{y2}^{(v)} - u_r a_{y2}^{(v)} + u_s a_{y1}^{(v)} - 2(a_{y1}^{(v)} - a_{y2}^{(v)})u_v \\ &+ 2(a_{y1}^{(v)^2} + a_{y2}^{(v)^2})\theta_x^{(v)} + (a_{x1}^{(v)}a_{y2}^{(v)} - a_{x1}^{(v)}a_{y1}^{(v)} + a_{x2}^{(v)}a_{y1}^{(v)} - a_{x2}^{(v)}a_{y2}^{(v)})\theta_y^{(v)}] = 0 \end{aligned} \tag{17}$$

$$\begin{aligned} J_y^{(v)}\ddot{\theta}_y^{(v)} &+ k_z^{(v)}[-u_p a_{x1}^{(v)} - u_q a_{x1}^{(v)} + u_r a_{x2}^{(v)} + u_s a_{x2}^{(v)} + 2(a_{x1}^{(v)} - a_{x2}^{(v)})u_v \\ &+ (a_{x1}^{(v)}a_{y2}^{(v)} - a_{x1}^{(v)}a_{y1}^{(v)} + a_{x2}^{(v)}a_{y1}^{(v)} - a_{x2}^{(v)}a_{y2}^{(v)})\theta_x^{(v)} + 2(a_{x1}^{(v)^2} + a_{x2}^{(v)^2})\theta_y^{(v)}] = 0 \end{aligned} \tag{18}$$

Writing the last three equations in matrix form, one obtains

$$\begin{aligned} &\begin{bmatrix} m_z^{(v)} & 0 & 0 \\ 0 & J_x^{(v)} & 0 \\ 0 & 0 & J_y^{(v)} \end{bmatrix} \begin{Bmatrix} \ddot{u}_v \\ \ddot{\theta}_x^{(v)} \\ \ddot{\theta}_y^{(v)} \end{Bmatrix} + k_z^{(v)} \begin{bmatrix} -1 & -1 & -1 & -1 \\ a_{y1}^{(v)} & -a_{y2}^{(v)} & -a_{y2}^{(v)} & a_{y1}^{(v)} \\ -a_{x1}^{(v)} & -a_{x1}^{(v)} & a_{x2}^{(v)} & a_{x2}^{(v)} \end{bmatrix} \begin{Bmatrix} u_p \\ u_q \\ u_r \\ u_s \end{Bmatrix} + \\ &k_z^{(v)} \begin{bmatrix} 4 & -2(a_{y1}^{(v)} - a_{y2}^{(v)}) & 2(a_{x1}^{(v)} - a_{x2}^{(v)}) \\ -2(a_{y1}^{(v)} - a_{y2}^{(v)}) & 2(a_{y1}^{(v)^2} + a_{y2}^{(v)^2}) & a_{x1}^{(v)}a_{y2}^{(v)} - a_{x1}^{(v)}a_{y1}^{(v)} + a_{x2}^{(v)}a_{y1}^{(v)} - a_{x2}^{(v)}a_{y2}^{(v)} \\ 2(a_{x1}^{(v)} - a_{x2}^{(v)}) & a_{x1}^{(v)}a_{y2}^{(v)} - a_{x1}^{(v)}a_{y1}^{(v)} + a_{x2}^{(v)}a_{y1}^{(v)} - a_{x2}^{(v)}a_{y2}^{(v)} & 2(a_{x1}^{(v)^2} + a_{x2}^{(v)^2}) \end{bmatrix} \begin{Bmatrix} u_v \\ \theta_x^{(v)} \\ \theta_y^{(v)} \end{Bmatrix} \\ &= \{0\} \end{aligned} \tag{19}$$

For free vibration of the loaded plate (i.e., the bare plate together with the three-dof spring-mass system), one has

$$u_j = \bar{u}_j e^{i\bar{\omega}t} \quad (j = p, q, r, s, v) \tag{20a}$$

$$\theta_j^{(v)} = \bar{\theta}_j^{(v)} e^{i\bar{\omega}t} \quad (j = x, y) \tag{20b}$$

$$F_j = \bar{F}_j e^{i\bar{\omega}t} \quad (j = p, q, r, s, v) \tag{20c}$$

where $\bar{F}_j$ is the amplitude of F_j ($j = p, q, r, s$), $\bar{\varpi}$ is the natural frequency of the loaded plate; $\bar{u}_j$ ($j = p, q, r, s, v$) and $\bar{\theta}_j^{(v)}$ ($j = x, y$) are respectively the vibration amplitudes of u_j ($j = p, q, r, s, v$) and $\theta_j^{(v)}$ ($j = x, y$); while $i = \sqrt{-1}$.
The substitution of Eqs. (20a) and (20b) into Eq. (19) yields

$$\begin{bmatrix} -m_z^{(v)}\bar{\varpi}^2 & 0 & 0 \\ 0 & -J_x^{(v)}\bar{\varpi}^2 & 0 \\ 0 & 0 & -J_y^{(v)}\bar{\varpi}^2 \end{bmatrix} \begin{Bmatrix} \bar{u}_v \\ \bar{\theta}_x^{(v)} \\ \bar{\theta}_y^{(v)} \end{Bmatrix} + k_z^{(v)} \begin{bmatrix} -1 & -1 & -1 & -1 \\ a_{y1}^{(v)} & -a_{y2}^{(v)} & -a_{y2}^{(v)} & a_{y1}^{(v)} \\ -a_{x1}^{(v)} & -a_{x1}^{(v)} & a_{x2}^{(v)} & a_{x2}^{(v)} \end{bmatrix} \begin{Bmatrix} \bar{u}_p \\ \bar{u}_q \\ \bar{u}_r \\ \bar{u}_s \end{Bmatrix}$$

$$+ k_z^{(v)} \begin{bmatrix} 4 & -2(a_{y1}^{(v)} - a_{y2}^{(v)}) & 2(a_{x1}^{(v)} - a_{x2}^{(v)}) \\ -2(a_{y1}^{(v)} - a_{y2}^{(v)}) & 2(a_{y1}^{(v)^2} + a_{y2}^{(v)^2}) & a_{x1}^{(v)}a_{y2}^{(v)} - a_{x1}^{(v)}a_{y1}^{(v)} + a_{x2}^{(v)}a_{y1}^{(v)} - a_{x2}^{(v)}a_{y2}^{(v)} \\ 2(a_{x1}^{(v)} - a_{x2}^{(v)}) & a_{x1}^{(v)}a_{y2}^{(v)} - a_{x1}^{(v)}a_{y1}^{(v)} + a_{x2}^{(v)}a_{y1}^{(v)} - a_{x2}^{(v)}a_{y2}^{(v)} & 2(a_{x1}^{(v)^2} + a_{x2}^{(v)^2}) \end{bmatrix} \begin{Bmatrix} \bar{u}_v \\ \bar{\theta}_x^{(v)} \\ \bar{\theta}_y^{(v)} \end{Bmatrix}$$
$$= \{0\} \qquad (21)$$

or

$$\begin{Bmatrix} \bar{u}_v \\ \bar{\theta}_x^{(v)} \\ \bar{\theta}_y^{(v)} \end{Bmatrix} = [A]^{-1} \cdot k_z^{(v)} \begin{bmatrix} -1 & -1 & -1 & -1 \\ a_{y1}^{(v)} & -a_{y2}^{(v)} & -a_{y2}^{(v)} & a_{y1}^{(v)} \\ -a_{x1}^{(v)} & -a_{x1}^{(v)} & a_{x2}^{(v)} & a_{x2}^{(v)} \end{bmatrix} \begin{Bmatrix} \bar{u}_p \\ \bar{u}_q \\ \bar{u}_r \\ \bar{u}_s \end{Bmatrix} \qquad (22)$$

where

$$[A] =$$
$$\begin{bmatrix} m_z^{(v)}\bar{\varpi}^2 - 4k_z^{(v)} & 2(a_{y1}^{(v)} - a_{y2}^{(v)})k_z^{(v)} & -2(a_{x1}^{(v)} - a_{x2}^{(v)})k_z^{(v)} \\ 2(a_{y1}^{(v)} - a_{y2}^{(v)})k_z^{(v)} & J_x^{(v)}\bar{\varpi}^2 - 2(a_{y1}^{(v)^2} + a_{y2}^{(v)^2})k_z^{(v)} & (-a_{x1}^{(v)}a_{y2}^{(v)} + a_{x1}^{(v)}a_{y1}^{(v)} - a_{x2}^{(v)}a_{y1}^{(v)} + a_{x2}^{(v)}a_{y2}^{(v)})k_z^{(v)} \\ -2(a_{x1}^{(v)} - a_{x2}^{(v)})k_z^{(v)} & (-a_{x1}^{(v)}a_{y2}^{(v)} + a_{x1}^{(v)}a_{y1}^{(v)} - a_{x2}^{(v)}a_{y1}^{(v)} + a_{x2}^{(v)}a_{y2}^{(v)})k_z^{(v)} & J_y^{(v)}\bar{\varpi}^2 - 2(a_{x1}^{(v)^2} + a_{x2}^{(v)^2})k_z^{(v)} \end{bmatrix} \qquad (23)$$

Substituting Eqs. (20a)-(20c) into Eqs. (4)-(7), introducing Eq. (22) into the resulting expressions and writing the final results in matrix form, one obtains

$$\begin{Bmatrix} \bar{F}_p \\ \bar{F}_q \\ \bar{F}_r \\ \bar{F}_s \end{Bmatrix} = \left(k_z^{(v)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + k_z^{(v)^2} \begin{bmatrix} -1 & a_{y1}^{(v)} & -a_{x1}^{(v)} \\ -1 & -a_{y2}^{(v)} & -a_{x1}^{(v)} \\ -1 & -a_{y2}^{(v)} & a_{x2}^{(v)} \\ -1 & a_{y1}^{(v)} & a_{x2}^{(v)} \end{bmatrix} \cdot [A]^{-1} \cdot \begin{bmatrix} -1 & -1 & -1 & -1 \\ a_{y1}^{(v)} & -a_{y2}^{(v)} & -a_{y2}^{(v)} & a_{y1}^{(v)} \\ -a_{x1}^{(v)} & -a_{x1}^{(v)} & a_{x2}^{(v)} & a_{x2}^{(v)} \end{bmatrix} \right) \begin{Bmatrix} \bar{u}_p \\ \bar{u}_q \\ \bar{u}_r \\ \bar{u}_s \end{Bmatrix} \qquad (24)$$

From Eq. (24), one may obtain the following relation

$$\{\bar{F}\} = [k_{eff}^{(v)}]\{\bar{u}\} \qquad (25)$$

where

$$\{\bar{F}\} = [\bar{F}_p \ \ \bar{F}_q \ \ \bar{F}_r \ \ \bar{F}_s]^T \tag{26a}$$

$$\{\bar{u}\} = [\bar{u}_p \ \ \bar{u}_q \ \ \bar{u}_r \ \ \bar{u}_s]^T \tag{26b}$$

and

$$[k_{eff}^{(v)}] = \begin{bmatrix} k_{eff,11}^{(v)} & k_{eff,12}^{(v)} & k_{eff,13}^{(v)} & k_{eff,14}^{(v)} \\ k_{eff,21}^{(v)} & k_{eff,22}^{(v)} & k_{eff,23}^{(v)} & k_{eff,24}^{(v)} \\ k_{eff,31}^{(v)} & k_{eff,32}^{(v)} & k_{eff,33}^{(v)} & k_{eff,34}^{(v)} \\ k_{eff,41}^{(v)} & k_{eff,42}^{(v)} & k_{eff,43}^{(v)} & k_{eff,44}^{(v)} \end{bmatrix}$$

$$= k_z^{(v)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + k_z^{(v)^2} \begin{bmatrix} -1 & a_{y1}^{(v)} & -a_{x1}^{(v)} \\ -1 & -a_{y2}^{(v)} & -a_{x1}^{(v)} \\ -1 & -a_{y2}^{(v)} & a_{x2}^{(v)} \\ -1 & a_{y1}^{(v)} & a_{x2}^{(v)} \end{bmatrix} \cdot [A]^{-1} \cdot \begin{bmatrix} -1 & -1 & -1 & -1 \\ a_{y1}^{(v)} & -a_{y2}^{(v)} & -a_{y2}^{(v)} & a_{y1}^{(v)} \\ -a_{x1}^{(v)} & -a_{x1}^{(v)} & a_{x2}^{(v)} & a_{x2}^{(v)} \end{bmatrix} \tag{26c}$$

Eq. (25) reveals that the v-th three-dof spring-mass system shown in Fig. 1 can be replaced by an effective stiffness matrix with its coefficients, $k_{eff,ij}^{(v)}$ (i, j = 1, 2, 3, 4), defined by Eqs. (26c). From Eqs. (23)-(26), it can be seen that the last stiffness matrix coefficients are functions of natural frequency $\bar{\omega}$ of the loaded plate.

4. Equivalent springs for a three-dof spring-mass system

It is evident that the dynamic characteristics of the rectangular plate carrying a three-dof spring-mass system as shown in Fig. 1 may be obtained from the same bare plate supported by four *equivalent* springs with their spring constants $k_{eq,i}^{(v)}$ ($i = p, q, r, s$) determined by

$$k_{eq,p}^{(v)} = k_{eff,11}^{(v)} + \left(\frac{\bar{u}_q}{\bar{u}_p}\right) k_{eff,12}^{(v)} + \left(\frac{\bar{u}_r}{\bar{u}_p}\right) k_{eff,13}^{(v)} + \left(\frac{\bar{u}_s}{\bar{u}_p}\right) k_{eff,14}^{(v)} \tag{27a}$$

$$k_{eq,q}^{(v)} = \left(\frac{\bar{u}_p}{\bar{u}_q}\right) k_{eff,21}^{(v)} + k_{eff,22}^{(v)} + \left(\frac{\bar{u}_r}{\bar{u}_q}\right) k_{eff,23}^{(v)} + \left(\frac{\bar{u}_s}{\bar{u}_q}\right) k_{eff,24}^{(v)} \tag{27b}$$

$$k_{eq,r}^{(v)} = \left(\frac{\bar{u}_p}{\bar{u}_r}\right) k_{eff,31}^{(v)} + \left(\frac{\bar{u}_q}{\bar{u}_r}\right) k_{eff,32}^{(v)} + k_{eff,33}^{(v)} + \left(\frac{\bar{u}_s}{\bar{u}_r}\right) k_{eff,34}^{(v)} \tag{27c}$$

$$k_{eq,s}^{(v)} = \left(\frac{\bar{u}_p}{\bar{u}_s}\right) k_{eff,41}^{(v)} + \left(\frac{\bar{u}_q}{\bar{u}_s}\right) k_{eff,42}^{(v)} + \left(\frac{\bar{u}_r}{\bar{u}_s}\right) k_{eff,43}^{(v)} + k_{eff,44}^{(v)} \tag{27d}$$

The last expressions are obtained from Eqs. (25) and (26). It is seen that the effective stiffness matrix given by Eq. (26c), $[k_{eff}^{(v)}]$, is equivalent to the set of the four equivalent springs (for the v-th three-dof spring-mass system) with their spring constants defined by Eqs. (27a)-(27d).

5. Solution of the problem

In this paper, two methods, the conventional finite element method (FEM) and the equivalent spring method (ESM), are used to calculate the natural frequencies and mode shapes of the loaded plate. The key points for the last two methods are described in the following.

5.1 By using conventional finite element method (FEM)

First of all, each three-dof spring-mass system is regarded as a finite element with element property matrices given by Eqs. (12a) and (12b). Then, the overall stiffness matrix $[K]$ and overall mass matrix $[M]$ for the loaded plate is determined by means of the standard assembly technique of finite element method (Bathe 1982). Eliminating the rows and columns of $[K]$ and $[M]$, corresponding to the constrained degrees of freedom of the plate, will yield the equations of motion for the entire vibrating system (Clough 1993)

$$[\tilde{M}]\{\ddot{\tilde{u}}\} + [\tilde{K}]\{\tilde{u}\} = \{0\} \tag{28}$$

where $[\tilde{M}]$ and $[\tilde{K}]$ are, respectively, the mass matrix and stiffness matrix obtained from $[M]$ and $[K]$ by imposing the prescribed boundary conditions; while $\{\ddot{\tilde{u}}\}$ and $\{\tilde{u}\}$ are, respectively, the associated acceleration and displacement vectors.
For free vibration of the loaded plate, one has

$$\{\tilde{u}\} = \{\tilde{u}^*\}e^{i\bar{\omega}t} \tag{29a}$$

$$\{\ddot{\tilde{u}}\} = -\bar{\omega}^2\{\tilde{u}^*\}e^{i\bar{\omega}t} \tag{29b}$$

Substitution of the last relations into Eq. (28) leads to

$$([\tilde{K}] - \bar{\omega}^2[\tilde{M}])\{\tilde{u}^*\} = \{0\} \tag{30}$$

Eq. (30) is a typical eigenvalue equation, therefore, many techniques may be used to determine the eigenvalues ($\bar{\omega}_j^2$) and the corresponding eigenvectors $\{\tilde{u}^*\}_j$ (j = 1, 2, ...). In this paper, Eq. (28) was solved by means of the Lanczos method (Cullum and Willoughby 2002).

5.2 By using equivalent spring method (ESM)

Because each three-dof spring-mass system (see Fig. 1) possesses three dof's, the order of the overall property matrices $[\tilde{K}]$ and $[\tilde{M}]$ for the last FEM increases 3 when one more spring-mass system is attached to the plate. However, this is not true for the current ESM because the three dof's of each spring-mass system are eliminated when the spring-mass system is replaced by a set of equivalent springs with the coefficients of its stiffness matrix given by $k^{(v)}_{eff,\,ij}$ $(i, j = 1$ to $4)$ (see Sections 3 and 4). In other words, the order of the overall property matrices for the loaded plate remains unchanged no matter whether or not the total number of the three-dof spring-mass systems attached to the plate is changed.

If the effective stiffness matrix for the vth three-dof spring-mass system shown in Fig. 1 is represented by $[k^{(v)}_{eff}]$ (see Eq. (26c)), then the contribution of the vth three-dof spring-mass system on the overall stiffness matrix of the loaded plate is given by

$$[K_L]_{n\times n} = [K_B]_{n\times n} + [k^{(v)}_{eff}]_{4\times 4} \tag{31}$$

where

$$K_{L,\,ij} = K_{B,\,ij} \qquad (i, j = 1 \text{ to } n) \tag{32}$$

except that

$$K_{L,\,ij} = K_{B,\,ij} + k^{(v)}_{eff,\,IJ} \tag{33}$$

with

$$\begin{array}{llll} i = n_p \ \text{ if } \ I = 1; & i = n_q \ \text{ if } \ I = 2; & i = n_r \ \text{ if } \ I = 3; & i = n_s \ \text{ if } \ I = 4 \\ j = n_p \ \text{ if } \ J = 1; & j = n_q \ \text{ if } \ J = 2; & j = n_r \ \text{ if } \ J = 3; & j = n_s \ \text{ if } \ J = 4 \end{array} \tag{34}$$

In Eq. (31), $[K_L]$ and $[K_B]$ are the overall stiffness matrix of the loaded plate and the bare plate, respectively; while n_p, n_q, n_r and n_s represent the numberings for the degrees of freedom corresponding to the transverse displacements (in $\bar{z}$ direction) of the attaching points p, q, r and s, respectively.

The solution procedures for ESM are:

1. Calculate the overall stiffness matrix $[K]$ and overall mass matrix $[M]$ of the bare plate using the conventional FEM.
2. Give a trial value to the natural frequency $\bar{\omega}$ for the loaded plate and calculate the coefficients of effective stiffness matrix, $k^{(v)}_{eff,\,ij}$ $(i, j = 1, 2, 3, 4)$ with Eqs. (23) and (26c).
3. Add the coefficients of effective stiffness matrix, $k^{(v)}_{eff,\,ij}$ $(i, j = 1, 2, 3, 4)$, for each three-dof spring-mass system to the associated ones of overall stiffness matrix $[K]$ and denote the resulting stiffness matrix by $[K]_s$. Eqs. (32)-(34) show the contribution of the vth three-dof spring-mass system on the overall stiffness matrix $[K]$.
4. Impose the boundary conditions of the plate to determine the effective overall stiffness matrix $[\tilde{K}]_s$ and overall mass matrix $[\tilde{M}]$.
5. Evaluate the value of the determinant

$$\Delta(\bar{\omega}) = \left|[\tilde{K}]_s - \bar{\omega}^2[\tilde{M}]\right| \tag{35}$$

6. If the value of the determinant is equal to zero (i.e., $\Delta(\bar{\omega}) = 0$), then the trial value of $\bar{\omega}$ selected by step 2 is one of the natural frequencies of the loaded plate. Otherwise, steps 2-5 must be repeated with a new trial value of $\bar{\omega}$ until $\Delta(\bar{\omega}) \approx 0$.

Table 1 The four natural frequencies, $\bar{\omega}_j$ (j = 1 to 4), for an undamped uniform SSSS rectangular plate carrying a 1-dof and a three-dof spring-mass systems, as shown in Figs. 2(a) and 2(b), with $m_z^{(1)} = 7.85$ kg, $k_z^{(1)} = 117.4$ N/m and $(\bar{x}^{(1)}, \bar{y}^{(1)}) = (1.5$ m, 0.5 m$)$

Methods	dof of each spring-mass system	Natural frequencies, $\bar{\omega}_j$ (j = 1 to 4) (rad/s)				CPU time (sec)
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	
*ESM	3	94.5099	149.1904	241.5197	717.0250	68.9
*FEM	3	94.5099	149.1904	241.5197	717.0250	68.2
Wu and Luo (1997a)	1	94.5128	149.1958	241.5223	717.0268	----
Avalos *et al.* (1993)	1	95.7515	153.5814	248.0038	706.4264	----

*ESM refers to the equivalent spring method presented in this paper; FEM refers to the conventional finite element method.

7. Determine the corresponding mode shape from the simultaneous equations

$$([\tilde{K}]_s - \bar{\omega}^2[\tilde{M}])\{\tilde{u}^*\} = \{0\} \tag{36}$$

In this paper the iterations for steps 2-6 were performed with the half-interval method and the corresponding mode shape was determined by solving Eq. (36) with the Gauss-Jordan elimination method (Gerald and Wheatley 1998). Clearly, the computer time required by the ESM depends on many factors, such as the scheme of computer program adopted, the lowest (initial) trial value of natural frequency, the total number of natural frequencies (n) determined, the difference between the lowest natural frequency and the highest one among the n natural frequencies, the accuracy of the natural frequencies desired, etc. In general, if only the lowest six (or less than six) natural frequencies of the loaded plate are required then the difference between the computer time required by the ESM and the conventional FEM is insignificant as one may see from the final column of Table 1. In spite of the fact that the computer time required by the ESM may be not necessarily less than that required by the FEM, the ESM has the following advantages: (i) Because the dof's regarding all the 3-dof spring-mass systems are eliminated by the associated effective stiffness matrices, the natural frequencies and mode shapes regarding the local vibrations of all the 3-dof spring-mass systems with respect to the static plate are to disappear in the computer output, therefore, the time saved by the analysis of its output data will be much more than the computer time required by its numerical calculations. (ii) Because each spring-mass system (see Fig. 1) possesses three dof's, the order of the effective overall property matrices $[\tilde{K}]$ and $[\tilde{M}]$ for the FEM increases by 3 when one more spring-mass system is attached to the plate. However, in ESM, the order of the overall property matrices for the loaded plate remains unchanged no matter how many 3-dof spring-mass systems are attached to the plate, because the three dof's of each spring-mass system are eliminated when each 3-dof spring-mass system is replaced by a set of equivalent springs (see Sections 2-4).

6. Numerical results and discussions

For convenience, a four-letter acronym is used to designate the type of support of a rectangular plate starting at the left edge and proceeding in a clockwise direction. Hence, the SSSS plate refers to a rectangular plate with its four edges simply supported, and the SFSF plate refers to a rectangular plate with its two opposite edges normal to the $\bar{x}$ -axis simply supported and the other two edges (normal to the $\bar{y}$ -axis) free. It is evident that the letters, S and F, refer to the *simple* and *free* supports, respectively. In this section, the reliability of the presented theory and the developed computer programs are confirmed first and then the free vibration characteristics of a SFSF square plate carrying four three-dof spring-mass systems are studied.

6.1 Reliability of the theory and the computer programs

To confirm the reliability of the presented technique and the developed computer programs, a uniform undamped SSSS rectangular plate carrying a one-dof spring-mass system (see Fig. 2(a)) and the same plate carrying a three-dof spring-mass system (see Fig. 2(b)) are investigated in this subsection. The material properties and dimensions of the SSSS plate are: length a = 2.0 m, width b = 1.0 m, thickness h = 0.005 m, mass density ρ = 7850 kg/m^3, Young's modulus E = 2.051 × 10^{11} N/m^2, Poisson's ratio ν = 0.3 and bending rigidity $D = Eh^3/[12(1-\nu^2)] = 2.348 \times 10^3$ Nm. Besides, the physical properties of the spring-mass systems shown in Fig. 2 are: spring stiffness

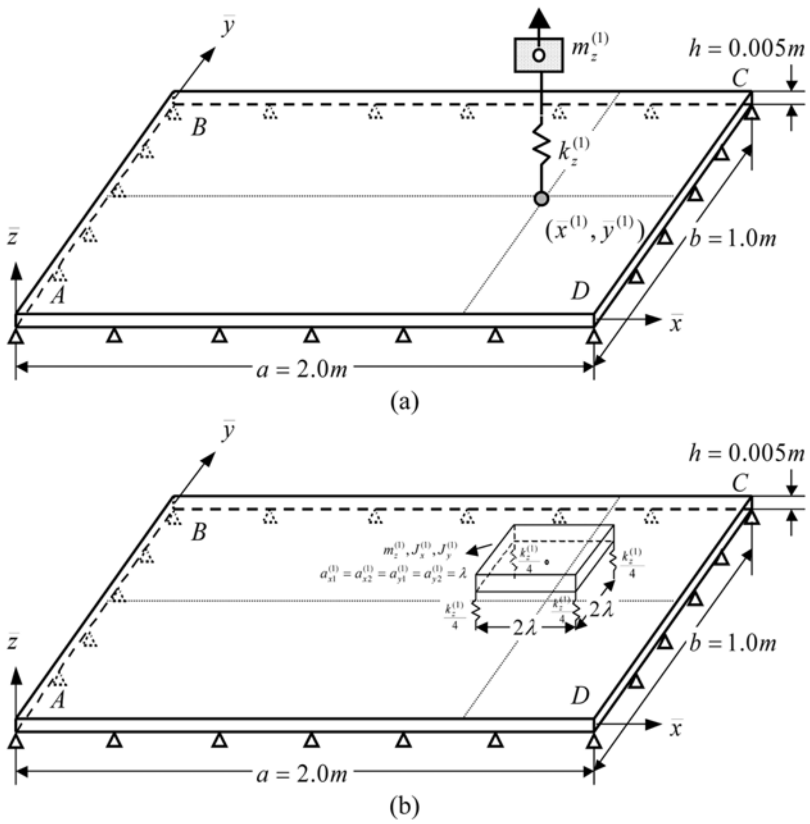

Fig. 2 A rectangular SSSS plate carrying (a) a one-dof spring-mass system and (b) a three-dof spring-mass system with $m_z^{(1)}$ = 7.85 kg, $k_z^{(1)}$ = 117.4 N/m and $(\bar{x}^{(1)}, \bar{y}^{(1)})$ = (1.5 m, 0.5 m)

$k_z^{(1)} = 117.4$ N/m and lumped mass $m_z^{(1)} = 7.85$ kg. It is noted that the last information concerning the plate and the spring-mass system is exactly the same as that of Wu and Luo (1997a).

From Figs. 2(a) and 2(b), one may infer that the natural frequencies of the plate carrying a three-dof spring-mass system shown in Fig. 2(b) will approach to the corresponding ones of the same plate carrying a one-dof spring-mass system shown in Fig. 2(a), if the following conditions are satisfied: (i) The mass moments of inertia, $J_x^{(1)}$ and $J_y^{(1)}$, and the spacing λ for the three-dof spring-mass system approach zero; (ii) The resultant spring constant and the $\bar{x}$ and $\bar{y}$ co-ordinates for the centre of gravity of the three-dof spring-mass system are identical to the corresponding ones of the one-dof spring-mass system. In this paper, $J_x^{(1)} = J_y^{(1)} = 10^{-6}\text{kg}\cdot\text{m}^2 \approx 0$, $\lambda = 0.01\ \text{m} \approx 0$, $k_p^{(1)} = k_q^{(1)} = k_r^{(1)} = k_s^{(1)} = k_z^{(1)}/4 = 117.4/4 = 29.35$ N/m and $(\bar{x}^{(1)}, \bar{y}^{(1)}) = (1.5\ \text{m}, 0.5\ \text{m})$ are used.

Table 1 shows the four natural frequencies, $\bar{\omega}_j$ (j = 1 to 4), of the last loaded plate. The natural frequencies listed in 5th and 6th rows of the table are for the SSSS plate carrying a one-dof spring-mass system given by Wu and Luo (1997a) and Avalos *et al.* (1993), while those listed in 3rd and 4th rows are for the same plate carrying a three-dof spring-mass system obtained from this paper. Among which the data listed in 5th and 6th rows are, respectively, calculated by using the finite element method (Wu and Luo 1997a) and the analytical method (Avalos *et al.* 1993), respectively, while those listed in 3rd and 4th rows are determined by the presented effective spring method (ESM) and the conventional finite element method (FEM) with property matrices of each three-dof spring-mass system given by Eqs. (12a) and (12b). From the table, one finds that the natural frequencies of the SSSS plate carrying a three-dof spring-mass system, obtained from this paper using either ESM or FEM, are very close to the corresponding ones of the same plate carrying a one-dof spring-mass system given by the existing literature. For this reason, one believes that the theory and the computer programs presented in this paper should be viable for investigating the dynamic characteristics of a rectangular plate carrying multiple three-dof spring-mass systems in this research.

6.2 Free vibration analysis of a SFSF plate carrying a three-dof spring-mass system

In this subsection, a uniform undamped SFSF square plate carrying a three-dof spring-mass system is studied (see Fig. 3). The material properties and dimensions of the SFSF square plate are: length $a = 1.0$ m, width $b = 1.0$ m, thickness $h = 0.003$ m, mass density $\rho = 7820$ kg/m^3, Young's modulus E = 206.8 GN/m^2 and Poisson's ratio ν = 0.29. The physical properties of the three-dof spring-mass system are: lumped mass $m_z^{(1)} = 23.46$ kg, mass moments of inertia $J_x^{(1)} = J_y^{(1)} = 10^{-6}$ kg $\cdot$ m^2, spacings $a_{x1}^{(1)} = a_{x2}^{(1)} = 0.125$ m and $a_{y1}^{(1)} = a_{y2}^{(1)} = 0.375$ m. The $\bar{x}$ and $\bar{y}$ co-ordinates of the attaching points, p, q, r and s, are: $(\bar{x}_p^{(1)}, \bar{y}_p^{(1)}) = (0.375\ \text{m}, 0.125\ \text{m})$, $(\bar{x}_q^{(1)}, \bar{y}_q^{(1)}) = (0.375\ \text{m}, 0.875\ \text{m})$, $(\bar{x}_r^{(1)}, \bar{y}_r^{(1)}) = (0.625\ \text{m}, 0.875\ \text{m})$ and $(\bar{x}_s^{(1)}, \bar{y}_s^{(1)}) = (0.625\ \text{m}, 0.125\ \text{m})$. Three cases are studied and the associated spring constants for each of the four helical springs of the three-dof spring-mass system are: $k_z^{(1)} = 3750.0$, 5250.0 and 6750.0 N/m, respectively. From the last given data for the three-dof spring-mass system, one sees that the $\bar{x}$ and $\bar{y}$ co-ordinates for the centre of gravity of the spring-mass system are identical to those of the centre of the SFSF square plate.

The lowest six natural frequencies of the bare SFSF plate are shown in the final row of Table 2 and the corresponding mode shapes are plotted by meshes of Fig. 4 and dashed curves in contour plots of Figs. 4-7 and 9. Besides, the lowest six mode shapes of the loaded SFSF plate with $k_z^{(1)} =$ 3750.0 N/m obtained by using the FEM and ESM are, respectively, plotted by solid curves in contour plots of Figs. 5 and 6. The 3rd to 8th rows of Table 2 list the lowest six natural frequencies, $\bar{\omega}_j$ (j = 1 to 6), of the SFSF square plate carrying a three-dof spring-mass system with different

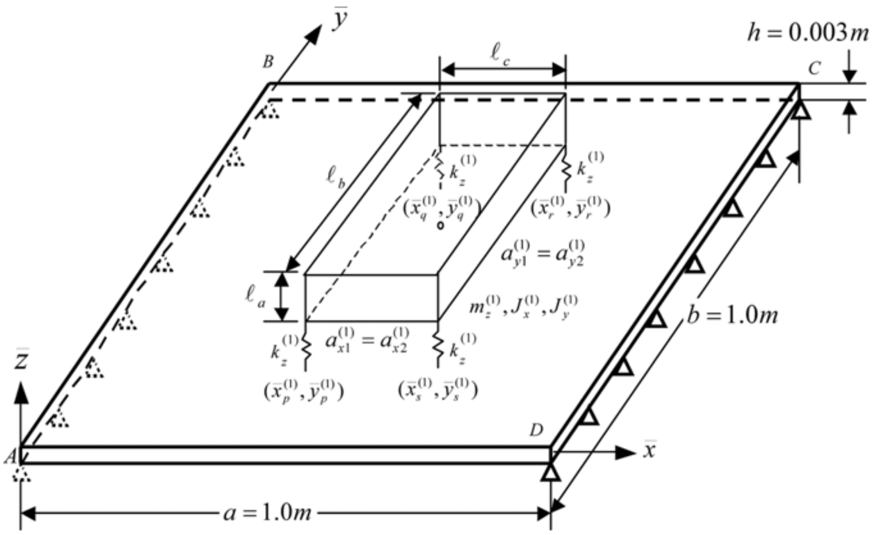

Fig. 3 A SFSF square plate carrying a three-dof spring-mass system with $m_z^{(1)} = 23.46$ kg, $a_{x1}^{(1)} = a_{x2}^{(1)} = 0.125$ m, $a_{y1}^{(1)} = a_{y2}^{(1)} = 0.375$ m, $J_x^{(1)} = J_y^{(1)} = 10^{-6}$ kg · m^2 and $(\bar{x}_p^{(1)}, \bar{y}_p^{(1)}) = (0.375$ m, 0.125 m$)$, $(\bar{x}_q^{(1)}, \bar{y}_q^{(1)}) = (0.375$ m, 0.875 m$)$, $(\bar{x}_r^{(1)}, \bar{y}_r^{(1)}) = (0.625$ m, 0.875 m$)$ and $(\bar{x}_s^{(1)}, \bar{y}_s^{(1)}) = (0.625$ m, 0.125 m$)$. The stiffness for each of the four helical springs of the three-dof spring-mass system are: $k_z^{(1)} = 3750.0$, 5250.0 or 6750.0 N/m

Table 2 Influence of stiffness $k_z^{(1)}$ for each of the four helical springs of the three-dof spring-mass system on the lowest six natural frequencies, $\bar{\omega}_j$ (j = 1 to 6), of the loaded SFSF square plate shown in Fig. 3, with $m_z^{(1)} = 23.46$ kg, $a_{x1}^{(1)} = a_{x2}^{(1)} = 0.125$ m, $a_{y1}^{(1)} = a_{y2}^{(1)} = 0.375$ m, $J_x^{(1)} = J_y^{(1)} = 10^{-6}$ kg · m^2 and $(\bar{x}_p^{(1)}, \bar{y}_p^{(1)}) = (0.375$ m, 0.125 m$)$, $(\bar{x}_q^{(1)}, \bar{y}_q^{(1)}) = (0.375$ m, 0.875 m$)$, $(\bar{x}_r^{(1)}, \bar{y}_r^{(1)}) = (0.625$ m, 0.875 m$)$ and $(\bar{x}_s^{(1)}, \bar{y}_s^{(1)}) = (0.625$ m, 0.125 m$)$

Spring constant, $k_z^{(1)}$ (N/m)	Methods	Natural frequencies, $\bar{\omega}_j$ (j = 1 to 6) (rad/s)					
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$
3750.0	ESM	58.2010	72.9789	161.0306	183.4530	215.0264	300.9359
	FEM	58.2010	72.9789	161.0306	183.4530	215.0264	300.9359
5250.0	ESM	63.1064	72.9789	161.8907	183.4530	215.9455	300.9359
	FEM	63.1064	72.9789	161.8907	183.4530	215.9456	300.9359
6750.0	ESM	67.6585	72.9789	162.7923	183.4529	216.8556	300.9359
	FEM	67.6585	72.9789	162.7923	183.4529	216.8557	300.9359
Bare plate	FEM	45.0048	72.9793	159.0543	183.4556	212.6873	300.9381

spring constant $k_z^{(1)}$. From the last figures and Table 2, one finds that the installation of a three-dof spring-mass system to the SFSF square plate will influence the 1st and 3rd natural frequencies and mode shapes of the bare SFSF plate to some degree. This is a reasonable result because the center of gravity for the three-dof spring-mass system is very close to the crest of the 1st and 3rd mode shapes of the bare SFSF plate, as one may see from Fig. 4. In addition to 1st and 3rd modes, the three-dof spring-mass also slightly influences the 5th natural frequencies of the bare SFSF plate. This is because the locations of the four helical springs are very close to the crest of the 5th mode

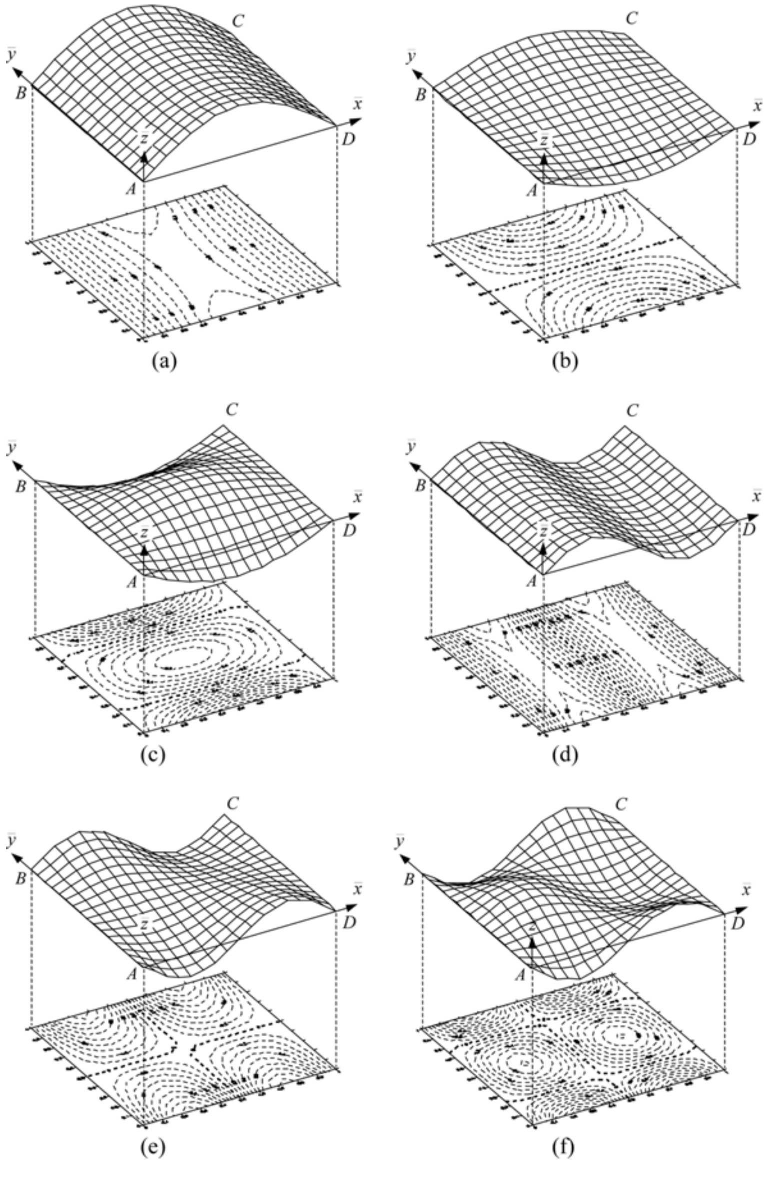

Fig. 4 Meshes and contour plots for the lowest six mode shapes of the bare SFSF plate corresponding to (a) $\bar{\omega}_1 = 45.0048$ rad/s, (b) $\bar{\omega}_2 = 72.9793$ rad/s, (c) $\bar{\omega}_3 = 159.0543$ rad/s, (d) $\bar{\omega}_4 = 183.4556$ rad/s, (e) $\bar{\omega}_5 = 212.6873$ rad/s, (f) $\bar{\omega}_6 = 300.9381$ rad/s

shape shown in Fig. 4(e).

From the 3rd to 8th rows of Table 2, one also sees that the larger the spring constant $k_z^{(1)}$ for each of the four helical springs of the three-dof spring-mass system, the higher the natural frequencies, $\bar{\omega}_j$ (j = 1 to 6), of the loaded SFSF plate. Moreover, the mode shapes obtained from FEM and ESM are in close agreement (see Figs. 5 and 6) and the natural frequencies of the loaded plate determined by using FEM and ESM are in good agreement (see Table 2). These results further confirm the reliability of the presented equivalent spring method.

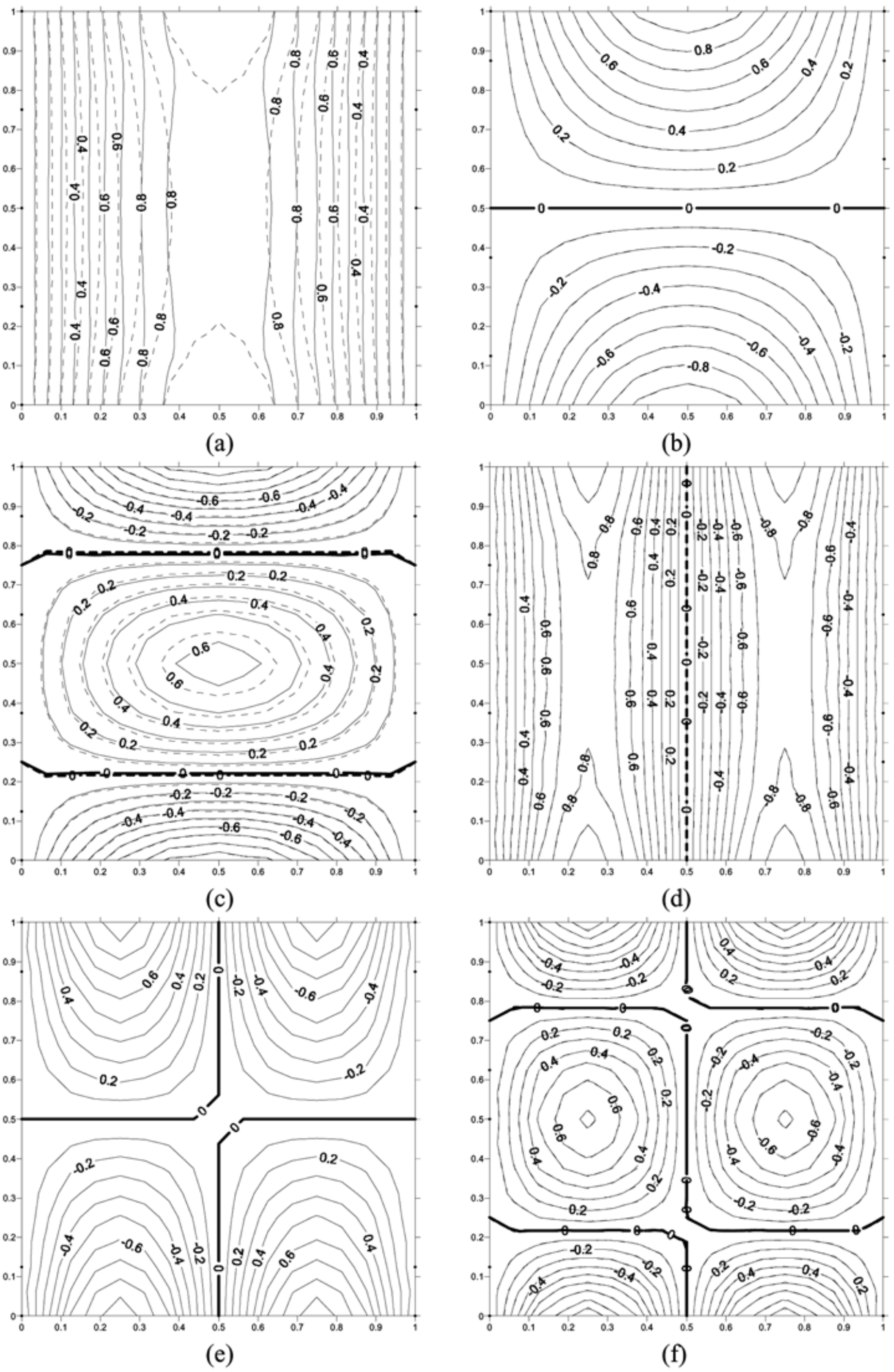

Fig. 5 Contour plots for the (a) 1st, (b) 2nd, (c) 3rd, (d) 4th, (e) 5th and (f) 6th mode shapes of the SFSF square plate carrying a three-dof spring-mass system, with $k_z^{(1)}$ = 3750.0 N/m, obtained from FEM (—) and the corresponding ones of the bare SFSF plate (----)

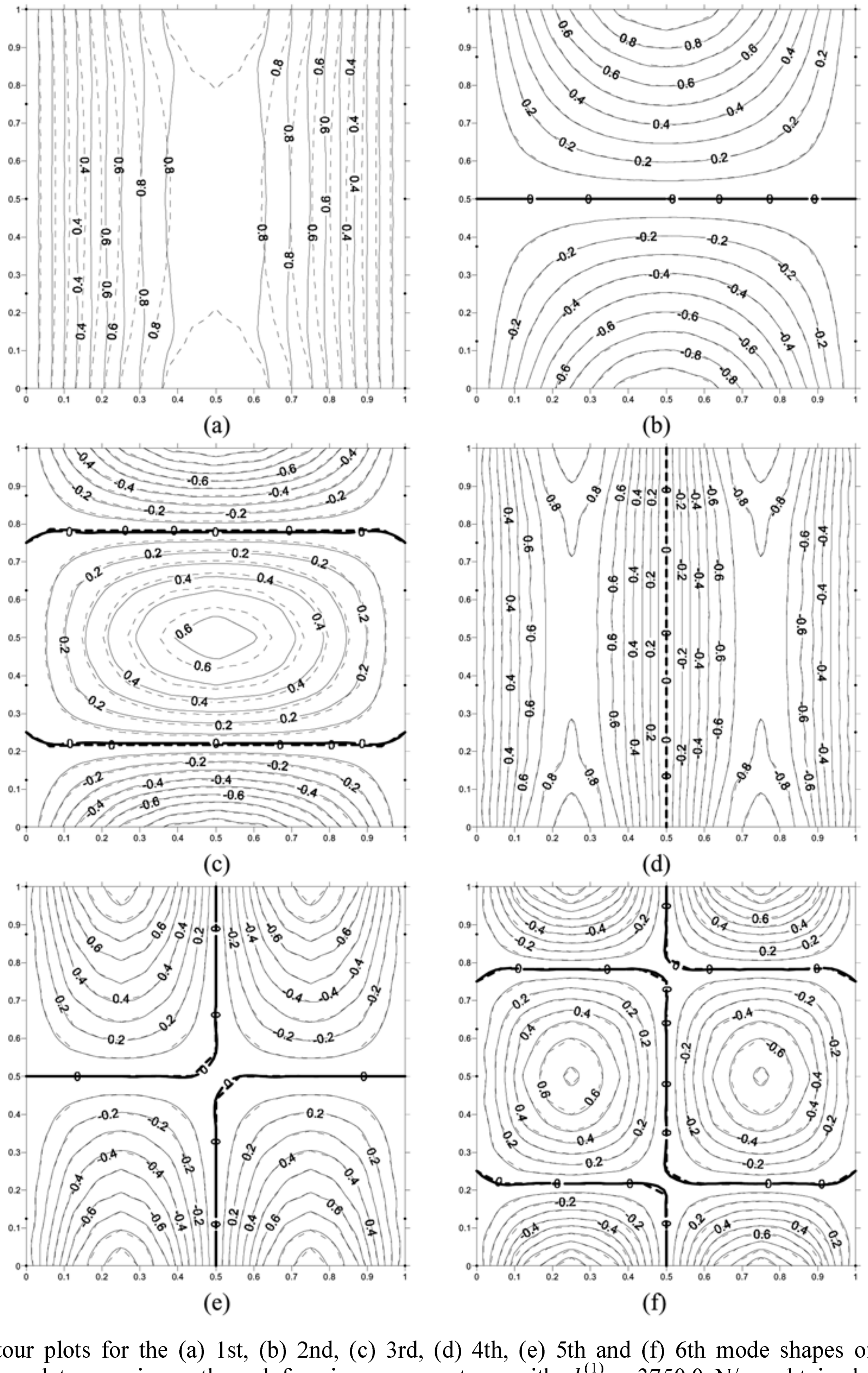

Fig. 6 Contour plots for the (a) 1st, (b) 2nd, (c) 3rd, (d) 4th, (e) 5th and (f) 6th mode shapes of the SFSF square plate carrying a three-dof spring-mass system, with $k_z^{(1)}$ = 3750.0 N/m, obtained from ESM (—) and the corresponding ones of the bare SFSF plate (----)

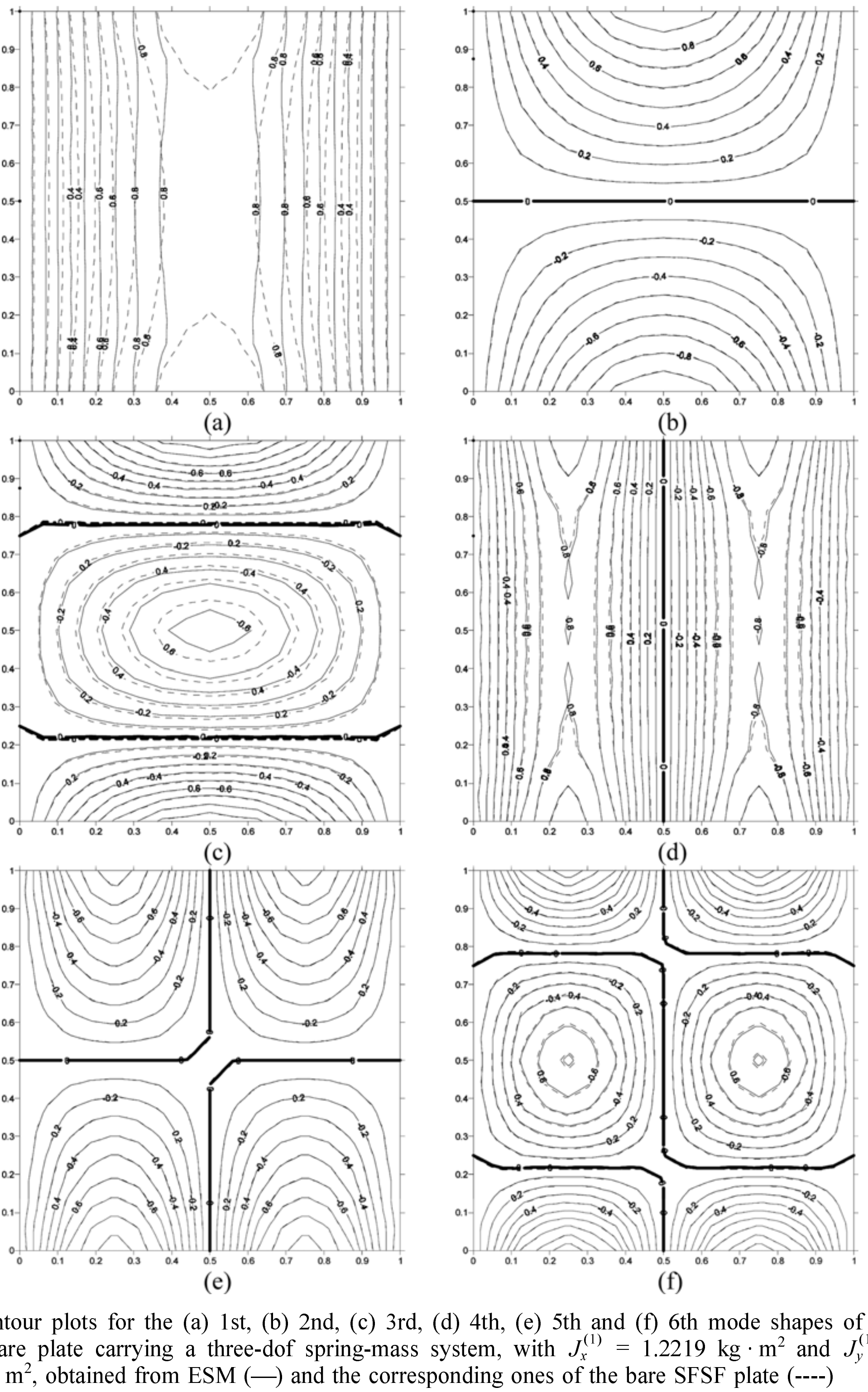

Fig. 7 Contour plots for the (a) 1st, (b) 2nd, (c) 3rd, (d) 4th, (e) 5th and (f) 6th mode shapes of the SFSF square plate carrying a three-dof spring-mass system, with $J_x^{(1)}$ = 1.2219 kg · m^2 and $J_y^{(1)}$= 0.2444 kg · m^2, obtained from ESM (—) and the corresponding ones of the bare SFSF plate (----)

Table 3 Influence of mass ($m_z^{(1)}$) and its mass moments of inertia ($J_x^{(1)}$ and $J_y^{(1)}$) for the lumped mass of the three-dof spring-mass system on the lowest six natural frequencies, $\bar{\omega}_j$ (j = 1 to 6), of the loaded SFSF square plate (carrying a three-dof spring-mass system) (cf. Fig. 3) with $m_z^{(1)}$=23.46 kg, $k_z^{(1)}$ = 3750.0 N/m, $a_{x1}^{(1)} = a_{x2}^{(1)} = 0.125$ m, $a_{y1}^{(1)} = a_{y2}^{(1)} = 0.375$ m, and $(\bar{x}_p^{(1)}, \bar{y}_p^{(1)})$ = (0.375 m, 0.125 m), $(\bar{x}_q^{(1)}, \bar{y}_q^{(1)})$ = (0.375 m, 0.875 m), $(\bar{x}_r^{(1)}, \bar{y}_r^{(1)})$ = (0.625 m, 0.875 m) and $(\bar{x}_s^{(1)}, \bar{y}_s^{(1)})$ = (0.625 m, 0.125 m)

Mass moments of inertia, $J_x^{(1)}$ and $J_y^{(1)}$ (kg · m^2)	Methods	Natural frequencies, $\bar{\omega}_j$ (j = 1 to 6) (rad/s)					
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$
10^{-6}	ESM	58.2010	72.9789	161.0306	183.4530	215.0264	300.9359
	FEM	58.2010	72.9789	161.0306	183.4530	215.0264	300.9359
$J_x^{(1)}$ = 1.2219	ESM	58.2010	87.1636	161.0306	185.3038	215.0264	301.4907
$J_y^{(1)}$ = 0.2444	FEM	58.2010	87.1636	161.0306	185.3038	215.0264	301.4907
Bare plate	FEM	45.0048	72.9793	159.0543	183.4556	212.6873	300.9381

6.3 Influence of mass and its mass moments of inertia of the three-dof spring-mass system

This subsection studies the effects of mass and its mass moments of inertia, $J_x^{(1)}$ and $J_y^{(1)}$, for the lumped mass of the three-dof spring-mass system on the free vibration characteristics of the last SFSF square plate. All the physical properties of the plate and the three-dof spring-mass system are exactly the same as those of the last subsection except that the spring constant for each of the four helical springs of the spring-mass system is $k_z^{(1)}$= 3750.0 N/m and mass moments of inertia are: $J_x^{(1)} = J_y^{(1)} = 10^{-6}$ or $J_x^{(1)}$ = 1.2219 kg · m^2 and $J_y^{(1)}$= 0.2444 kg · m^2, respectively.

The influences of mass ($m_z^{(1)}$) and its mass moments of inertia ($J_x^{(1)}$ and $J_y^{(1)}$) for the lumped mass of the three-dof spring-mass system on the lowest six natural frequencies and mode shapes of the loaded SFSF plate are shown in 3rd to 6th rows of Table 3 and Fig. 7, respectively. For comparison, the lowest six natural frequencies of the bare SFSF plate are listed in the final row of Table 3. From the table, one sees that the lumped mass ($m_z^{(1)}$) and its mass moments of inertia ($J_x^{(1)}$ and $J_y^{(1)}$) have significant influence on the lowest six natural frequencies of the bare SFSF plate. For the case of neglecting the mass moments of inertia ($J_x^{(1)}$ and $J_y^{(1)}$), the lumped mass ($m_z^{(1)}$) has the most significant influence on the first natural frequency of the bare SFSF plate (see 3rd, 4th and 7th rows of Table 3). This is a reasonable result because the center of gravity for lumped mass $m_z^{(1)}$ is very close to the crest of the first mode shape of the plate (cf. Figs. 4(a) and 7(a)). In addition, one also sees that the change of $J_x^{(1)}$ and $J_y^{(1)}$ will influence the 2nd, 4th and 6th natural frequencies, $\bar{\omega}_2$, $\bar{\omega}_4$ and $\bar{\omega}_6$, of the SFSF square plate to some degree (see 3-6 rows of Table 3). This is to be expected because the mode shapes corresponding to the 2nd, 4th and 6th natural frequencies are the torsional (vibration) modes of the bare SFSF plate, as one may see from Figs. 4 and 7.

6.4 Natural frequencies and mode shapes of a SFSF plate with four identical 3-dof spring-mass systems

To show the applicability of the presented technique, the lowest six natural frequencies of a SFSF square plate carrying four identical three-dof spring-mass systems, SM1, SM2, SM3 and SM4, as shown in Fig. 8, are investigated here. The locations and physical properties of the four identical three-dof spring-mass systems are shown in Table 4. Table 5 shows the lowest six natural

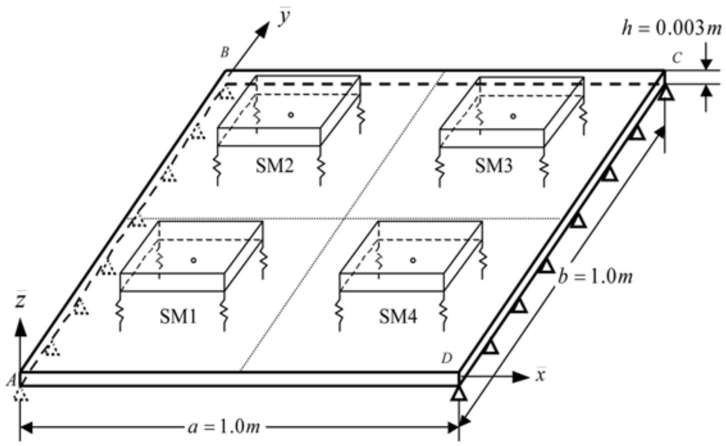

Fig. 8 A SFSF square plate carrying four identical three-dof spring-mass systems with their locations and physical properties shown in Table 4

Table 4 The locations and physical properties for the four three-dof spring-mass systems, SM1, SM2, SM3 and SM4, attached to the SFSF square plate shown in Fig. 8

Locations and physical properties	Spring-mass systems attached to the SFSF square plate			
	SM1	SM2	SM3	SM4
*$(\bar{x}_p, \bar{y}_p)$	(0.125,0.125)	(0.125,0.625)	(0.625,0.625)	(0.625,0.125)
$(\bar{x}_q, \bar{y}_q)$	(0.125,0.375)	(0.125,0.875)	(0.625,0.875)	(0.625,0.375)
$(\bar{x}_r, \bar{y}_r)$	(0.375,0.375)	(0.375,0.875)	(0.875,0.875)	(0.875,0.375)
$(\bar{x}_s, \bar{y}_s)$	(0.375,0.125)	(0.375,0.625)	(0.875,0.625)	(0.875,0.125)
Lumped mass m_z (kg)	23.46/4 = 5.865			
Spring constant k_z (N/m)	3750/4 = 937.5			
Mass moments of inertia $J_x = J_y$ (kg · m^2)	$J_x = J_y = \frac{1}{12} \times 5.865 \times (0.25^2 + 0.25^2) = 0.06104$			
Spacings $a_{x1} = a_{x2} = a_{y1} = a_{y2}$ (m)	0.125			

*The unit of coordinates is meters.

Table 5 The lowest six natural frequencies, $\bar{\omega}_j$ (j = 1 to 6) , for the SFSF square plate carrying four identical three-dof spring-mass systems (cf. Fig. 8) with their locations and physical properties shown in Table 4

Vibrating systems	Methods	Natural frequencies, $\bar{\omega}_j$ (j = 1 to 6) (rad/s)					
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$	$\bar{\omega}_6$
Loaded plate	ESM	53.4559	77.3556	160.5900	185.1642	214.0428	301.7541
	FEM	53.4559	77.3556	160.5900	185.1642	214.0428	301.7541
Bare plate	FEM	45.0048	72.9793	159.0543	183.4556	212.6873	300.9381

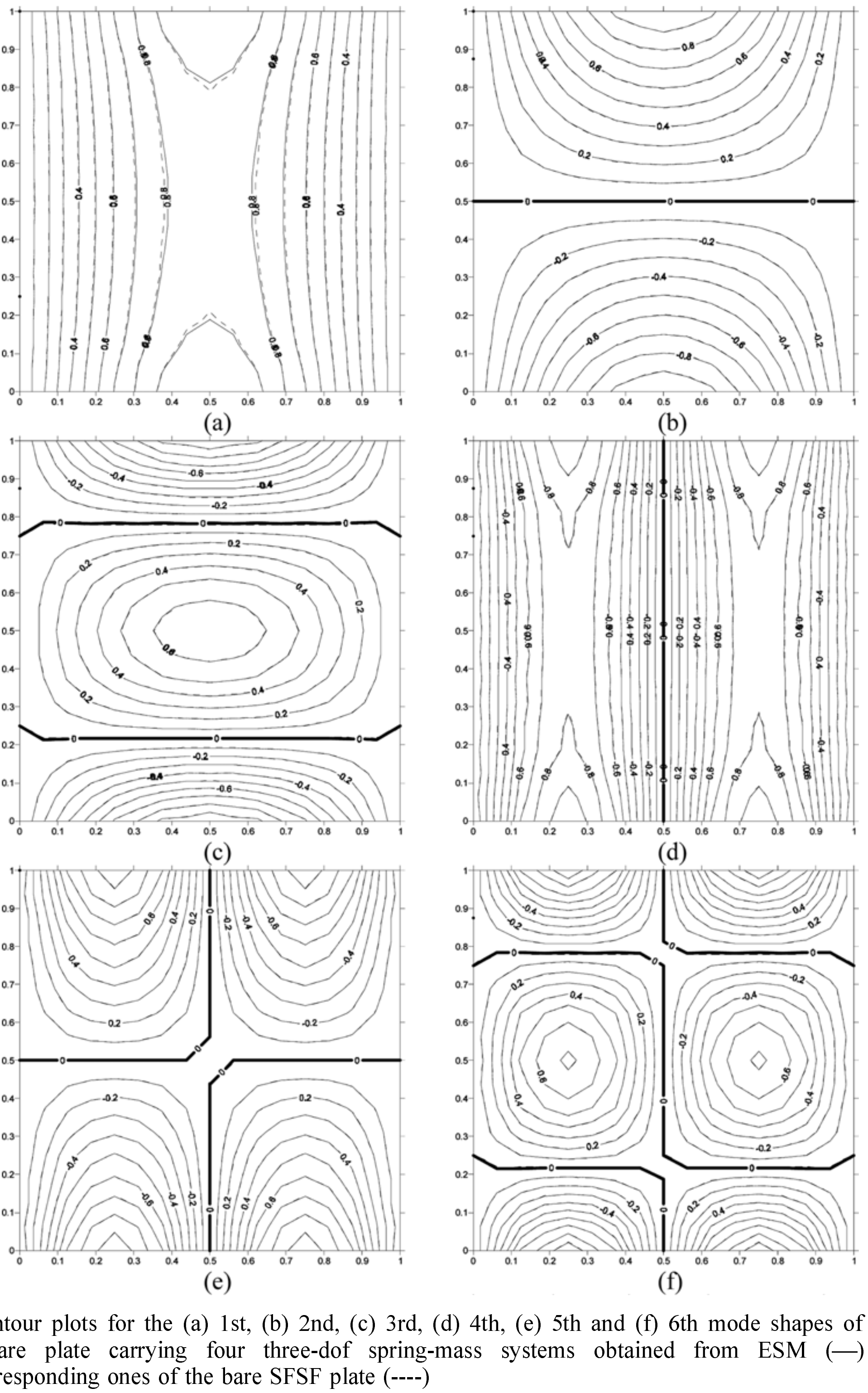

Fig. 9 Contour plots for the (a) 1st, (b) 2nd, (c) 3rd, (d) 4th, (e) 5th and (f) 6th mode shapes of the SFSF square plate carrying four three-dof spring-mass systems obtained from ESM (—) and the corresponding ones of the bare SFSF plate (----)

frequencies, $\bar{\omega}_j$ (j = 1 to 6), of the loaded plate. The mode shapes of the loaded plate are shown in Fig. 9. From Figs. 6 and 9, one finds that the influence of multiple spring-mass systems on the mode shapes of the bare SFSF plate is less than that of single spring-mass system. In general, the influence of a concentrated load on the dynamic characteristics of a plate is more than that of a distributed load if the magnitude of the concentrated load is equal to that of the distributed load. Therefore, the distribution of the spring-mass system on the plate is also an important factor affecting the vibration characteristics of the plate.

7. Conclusions

1. A technique of replacing a three-degree-of-freedom (three-dof) spring-mass system by a set of equivalent springs has been presented. By means of this equivalent spring method (ESM), one may obtain the dynamic characteristics of a rectangular plate carrying any number of three-dof spring-mass systems from the same plate supported by the same sets of equivalent springs. If n denotes the total number of three-dof spring-mass systems attached to the plate, then the total dof of the entire vibrating system using ESM is $3n$ less than those using the conventional FEM. In the conventional FEM, all natural frequencies and the associated mode shapes, either related to the plate or to the attached three-dof spring-mass systems, will appear in the computer output. However, in the presented ESM, only those related to the plate are output, this will significantly reduce the trivial output data when n is very large.
2. All parameters concerning a three-dof spring-mass system, such as lumped mass, spring stiffness, mass moments of inertia, the attaching position on the plate and the spacing between the helical springs, will affect the natural frequencies of the plate to some degree. It is hoped that the technique presented in this paper will provide a useful tool for solving the relevant problems.

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