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On the natural frequencies and mode shapes of a uniform multi-span beam carrying multiple point masses

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Abstract. Multi-span beams carrying multiple point masses are widely used in engineering applications, but the literature for free vibration analysis of such structural systems is much less than that of single-span beams. The complexity of analytical expressions should be one of the main reasons for the last phenomenon. The purpose of this paper is to utilize the numerical assembly method (NAM) to determine the exact natural frequencies and mode shapes of a multi-span uniform beam carrying multiple point masses. First, the coefficient matrices for an intermediate pinned support, an intermediate point mass, left-end support and right-end support of a uniform beam are derived. Next, the overall coefficient matrix for the whole structural system is obtained using the numerical assembly technique of the finite element method. Finally, the natural frequencies and the associated mode shapes of the vibrating system are determined by equating the determinant of the last overall coefficient matrix to zero and substituting the corresponding values of integration constants into the related eigenfunctions respectively. The effects of in-span pinned supports and point masses on the free vibration characteristics of the beam are also studied.

Key words: single-span beam; multi-span beam; numerical assembly method (NAM); natural frequency; mode shape.

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1. Introduction

The free vibration characteristics of a uniform beam carrying various concentrated elements (such as point masses, rotary inertias, linear springs, rotational springs, spring-mass systems, etc.) is an important problem in engineering, thus, a lot of reports have been published in this area. For example, Liu *et al.* (1988), Gürgöze (1984), Hamdan and Jubran (1991), and Hamdan and Abdel

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(1994) presented various techniques to perform free vibration analysis of beams carrying one or two concentrated elements. Gupta (1970) used a wave approach and the associated propagation constants to determine the natural frequencies and normal modes of a beam over periodic supports. Because the study of Gupta (1970) was restricted to an ideal periodic structure composed of perfectly identical units, Lin and Yang (1974) studied the natural frequencies of a "disordered" periodic beam by using a transfer matrix approach and considering the statistical properties of the beam. However, neither Gupta (1970) nor Lin and Yang (1974) studied the problem regarding free vibration characteristics of a multi-span beam carrying either single or multiple point masses. Bapat and Bapat (1987) developed a transfer matrix method (TMM) to study the natural frequencies of a beam whose individual support was modeled by a linear translational and rotational spring and a point mass, but they did not present the associated mode shapes of the beam. By means of the analytical-and-numerical-combined method (ANCM), Wu and Lin (1990) and Wu and Chou (1998) found the natural frequencies and mode shapes of a uniform beam carrying any number of rigidly-attached point masses and elastically-attached point masses, respectively. Cha (2001) solved the natural frequencies of a linear structure carrying any number of spring-mass systems using the assumed-modes method. Naguleswaran (2003) found the natural frequencies of an Euler-Bernoulli beam with up to five elastic supports (including ends) by using an iterative process. Wu and Chou (1999) obtained the exact solution of a uniform beam carrying any number of spring-mass systems by using the numerical assembly method (NAM). Employing the same technique as Wu and Chou (1999), Chen (2001) studied the free vibration problem concerning uniform and non-uniform beams carrying various concentrated elements.

From the above literature review one sees that exact solutions for the natural frequencies and mode shapes of "single-span" beams carrying either single or multiple point masses were obtained, such as Wu and Chou (1999) and Chen (2001). However, little was found in the literature regarding the exact solutions for the natural frequencies and mode shapes of "multi-span" beams carrying either single or multiple point masses. Thus, this paper aims at studying the last problem. Gürgöze and Erol (2001, 2002) studied the forced vibration responses of a cantilever beam with a single intermediate support, but they did not study the free vibration characteristic of the beam. Wu and Chou (1999) and Chen (2001) determined the exact natural frequencies and mode shapes of a "single-span" uniform beam carrying any number of point masses using the numerical assembly method (NAM). The same NAM is used in this paper, but the problem studied is concerned with the "multi-span" uniform beam with multiple point masses.

Compared with classical explicit analytical methods presented in existing literature, the NAM has the advantage of treating more complicated vibrating systems without much difficulty, as one may see from the numerical examples given in this paper. On the other hand, compared with the TMM, it is found that: (i) The basic concept of the NAM presented by Wu and Chou (1999) is similar to that of the TMM presented by Bapat and Bapat (1987), because both of them are based on the continuum models and the continuity of displacements and slopes, together with the equilibrium of shear forces and bending moments at each support point (or station); (ii) For a "single-span" beam with multiple "elastic" supports, results of the NAM (Wu and Chou 1999) and those of the TMM (Bapat and Bapat 1987) should be identical, because both of them are "exact" solutions. However, this is not true for a beam with other support conditions (such as clamped ends or intermediate simple supports), because the results of the NAM (Wu and Chou 1999) for the last beam, either "single-span" or "multi-span", are still the "exact" solutions and those of the TMM (Bapat and Bapat 1987) are the "approximate" ones only. In the TMM (Bapat and Bapat 1987), as the first "approximation", the properties of each individual support were modeled by a linear translational and rotational spring and a concentrated mass. Thus, the beam studied by Bapat and Bapat (1987) is a single-span beam (rather than a multi-span beam), unless one sets the spring constants at some of the specified support points to approach infinity (or the deflection δ_i of beam at the associated support point i is set to approach zero, i.e., $\delta_i \approx 0$, for i = 1, 2, 3, ...). In theory, the solutions obtained under the assumption that "$\delta_i \approx 0$" (1, 2, 3, ...) are the "approximate" solutions and those based on "$\delta_i \equiv 0$" (such as those in the current paper) are the "exact" ones; (iii) For the free vibration analysis of a beam, one is required to obtain the overall "cumulative" transfer matrix of the entire beam if the TMM is used (Meirovitch 1967). Similarly, one is required to determine the overall property matrices of the entire beam if the conventional finite element method (FEM) is used (Bathe 1982). The efficiency of a numerical method is closely related to the programming technique and therefore will not be studied here. The key point of this paper is placed on the "exact" determination of the natural frequencies together with the associated mode shapes for a multi-span uniform beam carrying multiple point masses. In this paper, the "exact" solution refers to that obtained from the continuum model instead of the discrete model.

2. Equation of motion and displacement function

Fig. 1 shows a sketch of a three-step beam supported by k pins and carrying n lumped masses (●). If each pinned support or lumped mass location is called a "station", then the total number of stations is $N' = k + n$. For convenience, three kinds of coordinates are used as one may see from Fig. 1. Among which, the positions of stations are defined by $x_{v'}(v' = 1 \sim N')$, those of point masses by $x_p^*(p = 1 \sim n)$ and those of pinned supports by $\bar{x}_r(r = 1 \sim k)$. It is obvious that $x_1 = \bar{x}_1 = 0$ and $x_{N'} = \bar{x}_k = \ell$, because the first (left-end) support is at the origin of the coordinates and the final (right-end) support is at the other end of the beam with total length ℓ. In Fig. 1, three kinds of numbering are also used to show the sequences of stations, point masses and pinned supports. The numbers $1'$, $2'$, ..., N' above the x-axis refer to the numbering of the stations, those

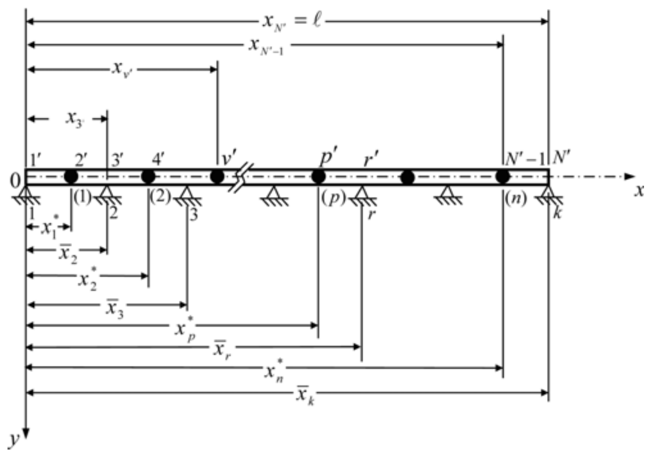

Fig. 1 Sketch for a uniform beam supported by k pins and carrying n point masses and the definitions for coordinates: $x_p^*(p = 1 \sim n)$ for point masses (●), $\bar{x}_r(r = 1 \sim r)$ for pinned supports, and $x_{v'}(v' = 1 \sim N')$ for stations

numbered (1), (2), …, (n) below the x-axis refer to the numbering of the point masses, while those numbered 1, 2, …., k refer to the numbering of the pinned supports. Note that the numbering of the point masses are shown by the parentheses ( ) and those of supports are not.

For free vibration of a uniform Euler-Bernoulli beam, its equation of motion is given by

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \bar{m}\frac{\partial^2 y(x,t)}{\partial^2 t} = 0 \tag{1}$$

where E is the Young's modulus, I is the moment of inertia of the cross-sectional area, $\bar{m}$ is the mass per unit length of the beam, and $y(x, t)$ is the transverse displacement at position x and time t.

For free vibration of the beam, one has

$$y(x,t) = \bar{Y}(x)e^{i\bar{\omega}t} \tag{2}$$

where $\bar{Y}(x)$ is the amplitude of $y(x,t)$, $\bar{\omega}$ is the natural frequency of the beam, and $i = \sqrt{-1}$.

Substitution of Eq. (2) into Eq. (1) gives

$$Y'''' - \beta^4 Y = 0 \tag{3}$$

where

$$\beta^4 = \frac{\bar{\omega}^2\bar{m}}{EI} \tag{4a}$$

or

$$\bar{\omega} = (\beta\ell)^2\left(\frac{EI}{\bar{m}\ell^4}\right)^{1/2} \tag{4b}$$

The general solution of Eq. (3) has the form

$$Y(x) = C_1\sin\beta x + C_2\cos\beta x + C_3\sinh\beta x + C_4\cosh\beta x \tag{5}$$

This equation is the displacement function for each beam segment between any two adjacent stations shown in Fig. 1.

3. Determination of natural frequencies and mode shapes

For an arbitrary point located at $x_{v'}$ (cf. Fig. 1), one obtains from Eq. (5)

$$Y_{v'}(\xi_{v'}) = C_{v',1}\sin\Omega\xi_{v'} + C_{v',2}\cos\Omega\xi_{v'} + C_{v',3}\sinh\Omega\xi_{v'} + C_{v',4}\cosh\Omega\xi_{v'} \tag{6}$$

$$Y'_{v'}(\xi_{v'}) = \Omega C_{v',1}\cos\Omega\xi_{v'} - \Omega C_{v',2}\sin\Omega\xi_{v'} + \Omega C_{v',3}\cosh\Omega\xi_{v'} + \Omega C_{v',4}\sinh\Omega\xi_{v'} \tag{7}$$

$$Y''_{v'}(\xi_{v'}) = -\Omega^2 C_{v',1}\sin\Omega\xi_{v'} - \Omega^2 C_{v',2}\cos\Omega\xi_{v'} + \Omega^2 C_{v',3}\sinh\Omega\xi_{v'} + \Omega^2 C_{v',4}\cosh\Omega\xi_{v'} \tag{8}$$

$$Y'''_{v'}(\xi_{v'}) = -\Omega^3 C_{v',1}\cos\Omega\xi_{v'} + \Omega^3 C_{v',2}\sin\Omega\xi_{v'} + \Omega^3 C_{v',3}\cosh\Omega\xi_{v'} + \Omega^3 C_{v',4}\sinh\Omega\xi_{v'} \tag{9}$$

where

$$\xi_{v'} = \frac{x_{v'}}{\ell} \tag{10}$$

$$\Omega = \beta\ell \tag{11}$$

If the left-end support (i.e., station 1') of the beam is pinned as shown in Fig. 1, then the boundary conditions are:

$$Y_{1'}(0) = Y''_{1'}(0) = 0 \tag{12a,b}$$

In Eqs. (7)-(9) and (12) the primes refer to differentiation with respect to the coordinate $\xi_{v'}$. From Eqs. (6), (8) and (12), one obtains

$$C_{1',2} + C_{1',4} = 0 \tag{13a}$$

$$-C_{1',2} + C_{1',4} = 0 \tag{13b}$$

or in matrix form

$$[B_{1'}]\{C_{1'}\} = 0 \tag{14}$$

where

$$[B_{1'}] = \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \tag{15}$$

$$\{C_{1'}\} = \{C_{1',1} \quad C_{1',2} \quad C_{1',3} \quad C_{1',4}\} \tag{16}$$

In the above expressions, the symbols, [ ] and { }, denote the rectangular matrix and column vector, respectively.

If the station numbering corresponding to the (p)-th intermediate point mass is represented by p', then the continuity of the deformations and the equilibrium of the moments and forces require that

$$Y^L_{p'}(\xi_{p'}) = Y^R_{p'}(\xi_{p'}) \tag{17a}$$

$$Y'^L_{p'}(\xi_{p'}) = Y'^R_{p'}(\xi_{p'}) \tag{17b}$$

$$Y''^L_{p'}(\xi_{p'}) = Y''^R_{p'}(\xi_{p'}) \tag{17c}$$

$$Y'''^L_{p'}(\xi_{p'}) + \Omega^4 m^*_{p'} Y^L_{p'}(\xi_{p'}) = Y'''^R_{p'}(\xi_{p'}) \tag{17d}$$

with

$$\xi_{p'} = \frac{x_{p'}}{\ell}, \quad m^*_{p'} = \frac{m_{p'}}{\bar{m}\ell} \tag{17e,f}$$

where $x_{p'}$ is the coordinate of station p', $m_{p'}$ is the magnitude of point mass, and the right superscripts L and R in Eqs. (17a)-(17d) refer to the "left" and "right" sides of station p'.

Substitution of Eqs.(6)-(9) into Eqs.(17a)-(17d) leads to

$$\begin{aligned} &C_{p'-1,1}\sin\Omega\xi_{p'} + C_{p'-1,2}\cos\Omega\xi_{p'} + C_{p'-1,3}\sinh\Omega\xi_{p'} + C_{p'-1,4}\cosh\Omega\xi_{p'} \\ &\quad - C_{p',1}\sin\Omega\xi_{p'} - C_{p',2}\cos\Omega\xi_{p'} - C_{p',3}\sinh\Omega\xi_{p'} - C_{p',4}\cosh\Omega\xi_{p'} = 0 \end{aligned} \tag{18a}$$

$$\begin{aligned} &C_{p'-1,1}\cos\Omega\xi_{p'} - C_{p'-1,2}\sin\Omega\xi_{p'} + C_{p'-1,3}\cosh\Omega\xi_{p'} + C_{p'-1,4}\sinh\Omega\xi_{p'} \\ &\quad - C_{p',1}\cos\Omega\xi_{p'} + C_{p',2}\sin\Omega\xi_{p'} - C_{p',3}\cosh\Omega\xi_{p'} - C_{p',4}\sinh\Omega\xi_{p'} = 0 \end{aligned} \tag{18b}$$

$$\begin{aligned} &- C_{p'-1,1}\sin\Omega\xi_{p'} - C_{p'-1,2}\cos\Omega\xi_{p'} + C_{p'-1,3}\sinh\Omega\xi_{p'} + C_{p'-1,4}\cosh\Omega\xi_{p'} \\ &\quad + C_{p',1}\sin\Omega\xi_{p'} + C_{p',2}\cos\Omega\xi_{p'} - C_{p',3}\sinh\Omega\xi_{p'} - C_{p',4}\cosh\Omega\xi_{p'} = 0 \end{aligned} \tag{18c}$$

$$\begin{aligned} &- C_{p'-1,1}\cos\Omega\xi_{p'} + C_{p'-1,2}\sin\Omega\xi_{p'} + C_{p'-1,3}\cosh\Omega\xi_{p'} + C_{p'-1,4}\sinh\Omega\xi_{p'} \\ &+ m^*_{p'}\Omega(C_{p'-1,1}\sin\Omega\xi_{p'} + C_{p'-1,2}\cos\Omega\xi_{p'} + C_{p'-1,3}\sinh\Omega\xi_{p'} + C_{p'-1,4}\cosh\Omega\xi_{p'}) \\ &\quad + C_{p',1}\cos\Omega\xi_{p'} - C_{p',2}\sin\Omega\xi_{p'} - C_{p',3}\cosh\Omega\xi_{p'} - C_{p',4}\sinh\Omega\xi_{p'} = 0 \end{aligned} \tag{18d}$$

or

$$[B_{p'}]\{C_{p'}\} = 0 \tag{19}$$

where

$$[B_{p'}] = \begin{array}{c} \begin{array}{cccccccc} 4p-3 & 4p-2 & 4p-1 & 4p & 4p+1 & 4p+2 & 4p+3 & 4p+4 \end{array} \\ \left[\begin{array}{cccccccc} \mathrm{s}\theta_{p'} & \mathrm{c}\theta_{p'} & \mathrm{sh}\theta_{p'} & \mathrm{ch}\theta_{p'} & -\mathrm{s}\theta_{p'} & -\mathrm{c}\theta_{p'} & -\mathrm{sh}\theta_{p'} & -\mathrm{ch}\theta_{p'} \\ \mathrm{c}\theta_{p'} & -\mathrm{s}\theta_{p'} & \mathrm{ch}\theta_{p'} & \mathrm{sh}\theta_{p'} & -\mathrm{s}\theta_{p'} & \mathrm{s}\theta_{p'} & -\mathrm{ch}\theta_{p'} & -\mathrm{sh}\theta_{p'} \\ -\mathrm{s}\theta_{p'} & -\mathrm{c}\theta_{p'} & \mathrm{sh}\theta_{p'} & \mathrm{ch}\theta_{p'} & \mathrm{s}\theta_{p'} & \mathrm{c}\theta_{p'} & -\mathrm{sh}\theta_{p'} & -\mathrm{ch}\theta_{p'} \\ m^*_{p'}\Omega\mathrm{s}\theta_{p'}-\mathrm{c}\theta_{p'} & m^*_{p'}\Omega\mathrm{c}\theta_{p'}+\mathrm{s}\theta_{p'} & m^*_{p'}\Omega\mathrm{sh}\theta_{p'}+\mathrm{ch}\theta_{p'} & m^*_{p'}\Omega\mathrm{ch}\theta_{p'}+\mathrm{sh}\theta_{p'} & \mathrm{c}\theta_{p'} & -\mathrm{s}\theta_{p'} & -\mathrm{ch}\theta_{p'} & -\mathrm{sh}\theta_{p'} \end{array}\right] \end{array} \begin{array}{l} \\ 4p-1 \\ 4p \\ 4p+1 \\ 4p+2 \end{array} \tag{20}$$

$$\{C_{p'}\} = \{C_{p'-1,1} \;\; C_{p'-1,2} \;\; C_{p'-1,3} \;\; C_{p'-1,4} \;\; C_{p',1} \;\; C_{p',2} \;\; C_{p',3} \;\; C_{p',4}\} \tag{21}$$

The symbols appearing in Eq. (20) are defined as

$$\theta_{p'} = \Omega\xi_{p'}, \quad \mathrm{s}\theta_{p'} = \sin\Omega\xi_{p'}, \quad \mathrm{c}\theta_{p'} = \cos\Omega\xi_{p'}, \quad \mathrm{sh}\theta_{p'} = \sinh\Omega\xi_{p'}, \quad \mathrm{ch}\theta_{p'} = \cosh\Omega\xi_{p'} \tag{22}$$

Similarly, if the station numbering corresponding to the r-th intermediate support is represented by r', then the continuity of the deformations and the equilibrium of the moments require that

$$Y_{r'}^{L}(\xi_{r'}) = Y_{r'}^{R}(\xi_{r'}) = 0 \tag{23a,b}$$

$$Y_{r'}'^{L}(\xi_{r'}) = Y_{r'}'^{R}(\xi_{r'}) \tag{23c}$$

$$Y_{r'}''^{L}(\xi_{r'}) = Y_{r'}''^{R}(\xi_{r'}) \tag{23d}$$

with

$$\xi_{r'} = \frac{x_{r'}}{\ell} \tag{24}$$

where $x_{r'}$ is the coordinate of station r' at which the r-th intermediate support is located.
Introducing Eqs. (6)-(9) into Eqs. (23), one obtains

$$C_{r'-1,1}\sin\Omega\xi_{r'} + C_{r'-1,2}\cos\Omega\xi_{r'} + C_{r'-1,3}\sinh\Omega\xi_{r'} + C_{r'-1,4}\cosh\Omega\xi_{r'} = 0 \tag{25a}$$

$$C_{r',1}\sin\Omega\xi_{r'} + C_{r',2}\cos\Omega\xi_{r'} + C_{r',3}\sinh\Omega\xi_{r'} + C_{r',4}\cosh\Omega\xi_{r'} = 0 \tag{25b}$$

$$\begin{aligned} C_{r'-1,1}\cos\Omega\xi_{r'} - C_{r'-1,2}\sin\Omega\xi_{r'} + C_{r'-1,3}\cosh\Omega\xi_{r'} + C_{r'-1,4}\sinh\Omega\xi_{r'} - C_{r',1}\cos\Omega\xi_{r'} \\ + C_{r',2}\sin\Omega\xi_{r'} - C_{r',3}\cosh\Omega\xi_{r'} - C_{r',4}\sinh\Omega\xi_{r'} = 0 \end{aligned} \tag{25c}$$

$$\begin{aligned} - C_{r'-1,1}\sin\Omega\xi_{r'} - C_{r'-1,2}\cos\Omega\xi_{r'} + C_{r'-1,3}\sinh\Omega\xi_{r'} + C_{r'-1,4}\cosh\Omega\xi_{r'} + C_{r',1}\sin\Omega\xi_{r'} \\ + C_{r',2}\cos\Omega\xi_{r'} - C_{r',3}\sinh\Omega\xi_{r'} - C_{r',4}\cosh\Omega\xi_{r'} = 0 \end{aligned} \tag{25d}$$

or

$$[B_{r'}]\{C_{r'}\} = 0 \tag{26}$$

where

$$[B_{r'}] = \begin{array}{c} \begin{array}{cccccccc} 4r-3 & 4r-2 & 4r-1 & 4r & 4r+1 & 4r+2 & 4r+3 & 4r+4 \end{array} \\ \left[\begin{array}{cccccccc} \mathrm{s}\Omega\xi_{r'} & \mathrm{c}\Omega\xi_{r'} & \mathrm{sh}\Omega\xi_{r'} & \mathrm{ch}\Omega\xi_{r'} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{s}\Omega\xi_{r'} & \mathrm{c}\Omega\xi_{r'} & \mathrm{sh}\Omega\xi_{r'} & \mathrm{ch}\Omega\xi_{r'} \\ \mathrm{c}\Omega\xi_{r'} & -\mathrm{s}\Omega\xi_{r'} & \mathrm{ch}\Omega\xi_{r'} & \mathrm{sh}\Omega\xi_{r'} & -\mathrm{c}\Omega\xi_{r'} & \mathrm{s}\Omega\xi_{r'} & -\mathrm{ch}\Omega\xi_{r'} & -\mathrm{sh}\Omega\xi_{r'} \\ -\mathrm{s}\Omega\xi_{r'} & -\mathrm{c}\Omega\xi_{r'} & \mathrm{sh}\Omega\xi_{r'} & \mathrm{ch}\Omega\xi_{r'} & \mathrm{s}\Omega\xi_{r'} & \mathrm{c}\Omega\xi_{r'} & -\mathrm{sh}\Omega\xi_{r'} & -\mathrm{ch}\Omega\xi_{r'} \end{array}\right] \end{array} \begin{array}{l} \\ 4r-1 \\ 4r \\ 4r+1 \\ 4r+2 \end{array} \tag{27}$$

$$\{C_{r'}\} = \{C_{r'-1,1} \;\; C_{r'-1,2} \;\; C_{r'-1,3} \;\; C_{r'-1,4} \;\; C_{r',1} \;\; C_{r',2} \;\; C_{r',3} \;\; C_{r',4}\} \tag{28}$$

where

$$\mathrm{s}\Omega\xi_{r'} = \sin\Omega\xi_{r'}, \quad \mathrm{c}\Omega\xi_{r'} = \cos\Omega\xi_{r'}, \quad \mathrm{sh}\Omega\xi_{r'} = \sinh\Omega\xi_{r'}, \quad \mathrm{ch}\Omega\xi_{r'} = \cosh\Omega\xi_{r'} \tag{29}$$

From Fig. 1 one sees that the right-end support (i.e., station N') of the beam is pinned, thus its boundary conditions are

$$Y_{N'}(\ell) = Y''_{N'}(\ell) = 0 \tag{30a,b}$$

Substituting Eqs. (6) and (8) into Eq. (30) gives

$$C_{N',1}\sin\Omega + C_{N',2}\cos\Omega + C_{N',3}\sinh\Omega + C_{N',4}\cosh\Omega = 0 \tag{31a}$$

$$-C_{N',1}\sin\Omega - C_{N',2}\cos\Omega + C_{N',3}\sinh\Omega + C_{N',4}\cosh\Omega = 0 \tag{31b}$$

or

$$[B_{N'}]\{C_{N'}\} = 0 \tag{32}$$

$$[B_{N'}] = \begin{array}{cccccc} & 4N_i+1 & 4N_i+2 & 4N_i+3 & 4N_i+4 & \\ & \left[\begin{matrix} \sin\Omega \\ -\sin\Omega \end{matrix}\right. & \begin{matrix} \cos\Omega \\ -\cos\Omega \end{matrix} & \begin{matrix} \sinh\Omega \\ \sinh\Omega \end{matrix} & \left.\begin{matrix} \cosh\Omega \\ \cosh\Omega \end{matrix}\right] & \begin{matrix} q-1 \\ q \end{matrix} \end{array} \tag{33}$$

$$\{C_{N'}\} = \{C_{N',1} \quad C_{N',2} \quad C_{N',3} \quad C_{N',4}\} \tag{34}$$

In Eq. (33), N_i denotes the total number of intermediate stations given by (cf. Fig. 1)

$$N_i = N' - 2 \tag{35}$$

and q denotes the total number of equations for the integration constants given by

$$q = 4N_i + 4 \tag{36}$$

From the above derivations one obtains four equations from each intermediate station (at which either a point mass or a pinned support is located). In addition, one obtains two equations from either the left-end station or the right-end station of the beam. Therefore, the total number of equations for the integration constants is $q = 4N_i + 4$. Note that in Eq. (35), N' is the total number of stations as shown in Fig. 1.

The integration constants relating to the left-end support (i.e., station 1') and those relating to the right-end support (i.e., station N') of the beam are determined by Eqs. (14) and (32), while those relating to the intermediate stations (i.e., 2 to $N' - 1$) are determined by Eq. (19) or (26) depending upon point mass or pinned support is located there. The associated coefficient matrices are given by $[B_{1'}]$, $[B_{N'}]$, $[B_{p'}]$ and $[B_{r'}]$ as one may see from Eqs. (15), (33), (20) and (27), respectively. From the last four equations one may see that the identification number for each element of the last four coefficient matrices is shown on the top side and right side of each matrix. Therefore, using the numerical assembly technique as done by the conventional finite element method one may obtain an equation for all integration constants of the entire beam

$$[\bar{B}]\{\bar{C}\} = 0 \tag{37}$$

The non-trivial solution of Eq. (37) requires that

$$|\bar{B}| = 0 \tag{38}$$

which is the frequency equation for the present problem.

In this paper, the half-interval method is used to find natural frequencies $\bar{\omega}_s$ (s = 1, 2, ...) of the multi-span beam carrying multiple point masses. For each natural frequency $\bar{\omega}_s$, from Eq. (37), one may obtain a set of simultaneous equations of the form: $\sum_{j=1}^{n-1} a_{ij}C_j = -a_{in}C_n$ (i = 1 to n). Solving the first n − 1 equations one obtains the values of C_1 to C_{n-1} in terms of C_n. For convenience, one sets $C_n = 1$ to define all the integration constants C_j (j = 1 to n). It is noted that the last values of C_j satisfy the final equation $\sum_{j=1}^{n-1} a_{nj}C_j = -a_{nn}C_n$. Substitution of the last integration constants into the displacement functions of the associated beam segments allows one to determine the corresponding mode shape of the beam, $Y^{(i)}(\xi)$. For reference, the overall coefficient matrix $[\bar{B}]$ for a two-span uniform beam with one intermediate point mass (or a pinned-pinned uniform beam with one intermediate pinned support and one intermediate point mass) is shown in the Appendix.

4. Coefficient matrices for the boundary stations of a cantilever beam

For a cantilever beam with its left end at station 1' and its right end at station N', the boundary conditions are

$$Y_{1'}(0) = Y'_{1'}(0) = 0 \tag{39a, b}$$

$$Y''_{N'}(\ell) = Y'''_{N'}(\ell) = 0 \tag{40a, b}$$

From Eqs. (17), (39) and (40) one obtains the following boundary coefficient matrices

$$[B_{1'}] = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ 1 \\ 2 \end{array} \tag{41}$$

$$[B_{N'}] = \begin{array}{c} \begin{array}{cccc} 4N_i+1 & 4N_i+2 & 4N_i+3 & 4N_i+4 \end{array} \\ \begin{bmatrix} -\sin\Omega & -\cos\Omega & \sinh\Omega & \cosh\Omega \\ -\cos\Omega & \sin\Omega & \cosh\Omega & \sinh\Omega \end{bmatrix} \end{array} \begin{array}{l} \\ q-1 \\ q \end{array} \tag{42}$$

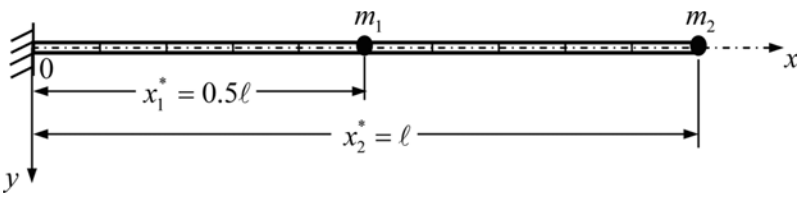

Fig. 2 A uniform cantilever beam carrying two point masses m_i located at x_i^*, $i = 1, 2$ (example 1)

Table 1 The lowest five dimensionless frequency parameters, $\Omega_i = (\bar{\omega}_i\sqrt{\bar{m}\ell^4/(EI)})^{1/2}$ (i = 1 to 5), for the uniform cantilever beam carrying two concentrated masses

Methods	Frequency parameters, $\Omega_i = (\bar{\omega}_i\sqrt{\bar{m}\ell^4/(EI)})^{1/2}$				
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
Present	1.338179	2.984562	7.365617	9.163801	13.497616
Hamdan and Abdel (1994)	1.338179	2.984562	7.365617	9.163802	13.497617

5. Numerical results and discussions

Before the free vibration analysis of a multi-span uniform beam carrying multiple point masses is performed, the reliability of the theory and computer program developed in this paper are confirmed by comparing the present results with those obtained from the existing literature or the conventional finite element method (FEM). Unless otherwise mentioned, all numerical results of this paper are obtained based on a uniform Euler-Bernoulli beam with the following given data: Young's modulus $E = 2.069 \times 10^{11}$ N/m^2, moment of inertia of cross-sectional area $I = 3.06796 \times 10^{-7}$ m^4, mass per unit length $\bar{m}$ = 15.3875 kg/m, and total length $\ell = 1$ m. When applying FEM, the two-node beam elements are used and each continuous beam is subdivided into 40 beam elements. Since each node has two degrees of freedom (DOF's), the total DOF for the entire beam is 2(40 + 1) = 82.

5.1 Reliability of the developed computer program

The first example (see Fig. 2) studied is a uniform cantilever beam carrying two concentrated masses (m_1 and m_2) located at $x_1^* = 0.5\ell = 0.5$ m and $x_2^* = \ell = 1.0$ m, respectively. The first point mass is $m_1 = 5(\bar{m}\ell) = 76.9375$ kg and the second point mass is $m_2 = 0.1(\bar{m}\ell) = 1.53875$ kg. The non-dimensional parameters for the current example are identical with those of Hamdan and Abdel (1994), i.e., $m_1^* = m_1/(\bar{m}\ell) = 5.0$, $m_2^* = m_2/(\bar{m}\ell) = 0.1$, $\xi_1^* = x_1^*/\ell = 0.5$ and $\xi_2^* = x_2^*/\ell$ = 1.0. From the lowest five frequency parameters, $\Omega_i = (\bar{\omega}_i\sqrt{\bar{m}\ell^4/(EI)})^{1/2}$ (i = 1 to 5), shown in Table 1, it is found that the numerical results of this paper and those of Hamdan and Abdel (1994) are in excellent agreement.

The second example studied is also a uniform cantilever beam (see Fig. 3), but m_1 in the last example (Fig. 2) is replaced by an intermediate pinned support, and $m_2 = 0$. The data E, I, $\bar{m}$ and ℓ were given previously. Table 2 shows the influence of supporting location on the lowest three natural frequencies of beam, $\bar{\omega}_i$ (i = 1 to 3). From Table 2 one sees that the lowest two natural frequencies of the cantilever beam increase with increasing distance between the pinned support and

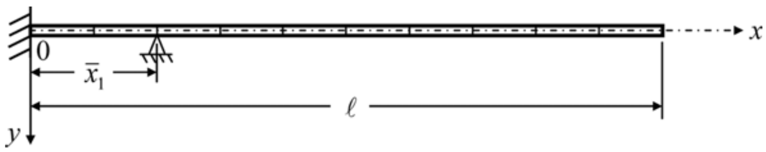

Fig. 3 A cantilever beam with single intermediate pinned support located at $\bar{x}_1$ (example 2)

Table 2 Influence of location of the intermediate support on the lowest three natural frequencies of the cantilever beam shown in Fig. 3 (example 2)

Location of intermediate support $\bar{\xi}_1 = \bar{x}_1/\ell$	Methods	Natural frequencies (rad/sec)		
		$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$
0.2	Present	315.4023	2013.4007	5703.1626
	Karnovsky and Lebed (2001)	315.4023	2013.4007	5703.1626
0.4	Present	484.3618	3245.3331	7227.5760
	Karnovsky and Lebed (2001)	484.3618	3245.3331	7227.5760
0.6	Present	871.2308	3350.7914	7167.1205
	Karnovsky and Lebed (2001)	871.2308	3350.7914	7167.1205
0.8	Present	1408.2916	3362.8779	6088.5710
	Karnovsky and Lebed (2001)	1408.2916	3362.8779	6088.5710

the fixed end of beam, $\bar{\xi}_1 = \bar{x}_1/\ell$ because the effective stiffness of beam regarding to the lowest two mode shapes increases with increasing the value of $\bar{\xi}_1 = \bar{x}_1/\ell$. The numerical results of this example are compared with the exact solutions obtained from Karnovsky and Lebed (2001). It is obvious that the agreement between them is very good as one may see from Table 2. The last exact solutions (Karnovsky and Lebed 2001) are obtained from the following frequency equation for a clamped-pinned beam with overhang:

$$S[\beta(\ell-\bar{x}_1)]\cdot\{S(\beta\ell)V(\beta\bar{x}_1)-T(\beta\ell)U(\beta\bar{x}_1)\}+T[\beta(\ell-\bar{x}_1)]$$
$$\cdot\{S(\beta\ell)U(\beta\bar{x}_1)-V(\beta\ell)V(\beta\bar{x}_1)\} = 0 \tag{43}$$

where S, T, U and V are Krylov-Duncan functions given by

$$S(\beta x) = \frac{1}{2}(\cosh\beta x+\cos\beta x) \tag{44a}$$

$$T(\beta x) = \frac{1}{2}(\sinh\beta x+\sin\beta x) \tag{44b}$$

$$U(\beta x) = \frac{1}{2}(\cosh\beta x-\cos\beta x) \tag{44c}$$

$$V(\beta x) = \frac{1}{2}(\sinh\beta x-\sin\beta x) \tag{44d}$$

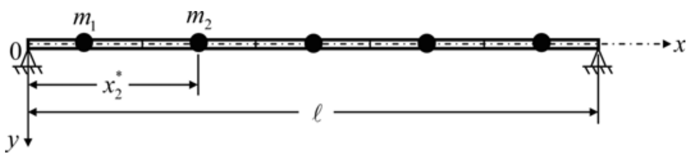

Fig. 4 A pinned-pinned beam with multiple intermediate point masses m_i located at x_i^* , $i = 1,2,\ldots$ (example 3)

Table 3 The lowest five natural frequencies of the pinned-pinned beam carrying multiple intermediate point masses shown in Fig. 4 (example 3)

No. of point masses, n	Methods	Natural frequencies (rad/sec)				
		$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$
$3^{(1)}$	Present	423.9717	1793.4811	3264.8800	7052.5025	10365.4514
	Chen (2001)	423.9718	1793.4807	3264.8793	7052.5020	10365.4516
$5^{(2)}$	Present	339.4906	1371.5926	2979.7831	4793.1061	7569.8126
	Chen (2001)	339.4903	1371.5926	2979.7824	4793.1055	7569.8121

Note: (1) $m_1^* = 0.2$, $m_2^* = 0.5$, $m_3^* = 1.0$ located at $\xi_1^* = 0.1$, $\xi_2^* = 0.5$, $\xi_3^* = 0.9$, respectively.
(2) $m_1^* = 0.2$, $m_2^* = 0.3$, $m_3^* = 0.5$, $m_4^* = 0.65$, $m_5^* = 1.0$ located at $\xi_1^* = 0.1$, $\xi_2^* = 0.3$, $\xi_3^* = 0.5$, $\xi_4^* = 0.7$, $\xi_5^* = 0.9$, respectively.

The third example (cf. Fig. 4) is the uniform pinned-pinned beam carrying three to five intermediate point masses studied by Chen (2001). Table 3 shows the influence of the magnitudes and distributions of point masses on the lowest five natural frequencies of the beam, $\overline{\omega}_i$ ($i = 1$ to 5). For the case with three point masses, the magnitudes and locations of point masses are: $m_1^* = 0.2$, $m_2^* = 0.5$, and $m_3^* = 1.0$ located at $\xi_1^* = 0.1$, $\xi_2^* = 0.5$, and $\xi_3^* = 0.9$, respectively, where $m_i^* = m_i/(\overline{m}\ell)$ and $\xi_i^* = x_i^*/\ell$ for $i = 1, 2, 3$. While for the case with five point masses, the associated data are: $m_1^* = 0.2$, $m_2^* = 0.3$, $m_3^* = 0.5$, $m_4^* = 0.65$ and $m_5^* = 1.0$ located at $\xi_1^* = 0.1$, $\xi_2^* = 0.3$, $\xi_3^* = 0.5$, $\xi_4^* = 0.7$, and $\xi_5^* = 0.9$, respectively. From Table 3 one sees that the numerical results of this paper are in good agreement with the corresponding ones obtained by Chen (2001).

5.2 Free vibration analysis of a two-span beam with single point mass

In the previous examples, the beam is either carrying an intermediate point mass (or masses) or is supported by an intermediate pin. However, the beam studied in this subsection is both carrying an intermediate point mass and supported by an intermediate pin as shown in Fig. 5. The beam is pinned at its two ends and has the same physical quantities as that in the last (third) example. The location of point mass is at $\xi_1^* = x_1^*/\ell = 0.5$ and that of pinned support is at $\overline{\xi}_1 = \overline{x}_1/\ell = 0.4$. The magnitude of point mass is $m_1 = m_1^*(\overline{m}\ell) = 0.5(15.3875 \times 1) = 7.69375$ kg. For reference, the explicit form for the overall coefficient matrix $[B]$ for the present example is given in the Appendix at the end of this paper. The lowest five natural frequencies of the beam are shown in Table 4 and the associated mode shapes are plotted in Fig. 6. In addition to the results of the NAM, those of the FEM are also given in Table 4. It is seen that the agreement between the corresponding results is excellent.

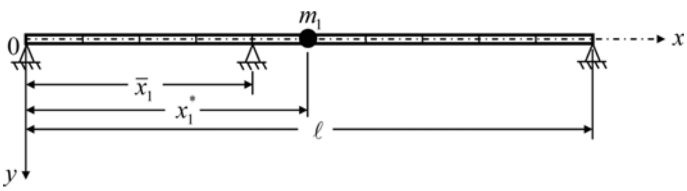

Fig. 5 A uniform pinned-pinned beam with an intermediate support located at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.4$ and carrying one intermediate point mass $m_1^* = m_1/(\overline{m}\ell) = 0.5$ located at $\xi_1^* = x_1^*/\ell = 0.5$ (example 4)

Table 4 The lowest five natural frequencies of the uniform pinned-pinned beam with an intermediate support located at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.4$ and carrying one intermediate point mass $m_1^* = m_1/(\overline{m}\ell) = 0.5$ at $\xi_1^* = x_1^*/\ell = 0.5$ (example 4)

Methods	Natural frequencies (rad/sec)				
	$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$
Present	1884.0997	4603.2739	6417.4170	12798.6756	18372.0114
FEM	1884.0996	4603.2790	6417.4299	12798.7945	18372.3955

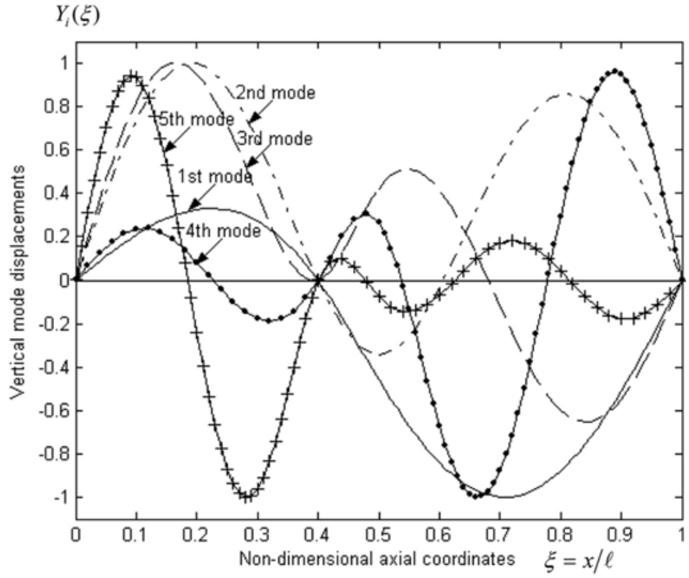

Fig. 6 The lowest five mode shapes for a uniform pinned-pinned beam with an intermediate support located at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.4$ and carrying one intermediate point mass $m_1^* = m_1/(\overline{m}\ell) = 0.5$ located at $\xi_1^* = x_1^*/\ell = 0.5$

From the lowest five mode shapes shown in Fig. 6 one sees that all the five curves pass through the intermediate pinned support located at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.4$, besides, the total number of intersecting points of each curve with the abscissa (i.e., the "node" for each mode shape) increases with increasing the mode number, as expected.

5.3 Free vibration analysis of a multi-span beam with multiple point masses

In example 3 (see Fig. 4), a uniform pinned-pinned beam carrying three to five intermediate point masses is studied and in the present example, the same pinned-pinned beam carrying five intermediate point masses and with one to four intermediate pinned supports is studied (cf. Fig. 7). The magnitudes and distributions of the five point masses are as shown in Note (2) below Table 3. Four cases are studied: (i) one support at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.2$; (ii) one support at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.4$; (iii) two supports at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.4$ and $\bar{\xi}_2 = \bar{x}_2/\ell = 0.6$, respectively; (iv) four supports at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.2$, $\bar{\xi}_2 = \bar{x}_2/\ell = 0.4$, $\bar{\xi}_3 = \bar{x}_3/\ell = 0.6$ and $\bar{\xi}_4 = \bar{x}_4/\ell = 0.8$, respectively. The lowest five natural frequencies of the multi-span beam carrying multiple point masses are shown in Table 5, and the associated mode shapes for the case of two supports located at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.4$ and $\bar{\xi}_2 = \bar{x}_2/\ell = 0.6$ are plotted in Fig. 8. From Table 5 one sees that the lowest five natural frequencies of the beam increase with increasing number of intermediate supports as expected and the present results are very close to those obtained from the FEM. It is noted that the total number of intermediate stations for the present problem is $N_i = 9$, thus, according to Eq. (36), the total number of equations for the integration constants is $q = 4N_i + 4 = 40$. In other words, the order of the overall coefficient matrix $[B]$ is 40×40. It is obvious that the classical explicit analytical methods will suffer much difficulty in such a case.

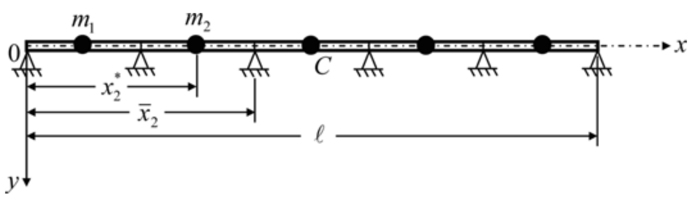

Fig. 7 A uniform pinned-pinned beam carrying five intermediate point masses m_i at $\xi_i^* = x_i^*/\ell$ ($i = 1 - 5$) and with multiple intermediate supports located at $\bar{\xi}_j = \bar{x}_j/\ell$ (example 5)

Table 5 Influence of total number and location of intermediate supports on the lowest five natural frequencies of a uniform pinned-pinned beam carrying five intermediate point masses (example 5)

No. of supports k	Locations of supports $\bar{\xi}_i = \bar{x}_i/\ell$	Methods	Natural frequencies (rad/sec) $\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
1	0.2	Present	675.1635	2234.4879	4386.4858	7109.2055	12197.0443
		FEM	675.1632	2234.4878	4386.4857	7109.2109	12197.0845
1	0.4	Present	1022.7077	2952.4270	4003.1320	6516.1612	9998.6141
		FEM	1022.7072	2952.4272	4003.1331	6516.1648	9998.6317
2	0.4, 0.6	Present	2205.0012	3490.7278	5832.2267	8642.4383	11290.6774
		FEM	2205.0010	3490.7286	5832.2276	8642.4485	11290.6967
4	0.2, 0.4, 0.6, 0.8	Present	5328.3373	7611.3321	9445.7897	11205.5248	14530.7043
		FEM	5328.3380	7611.3361	9445.8000	11205.5426	14530.7504

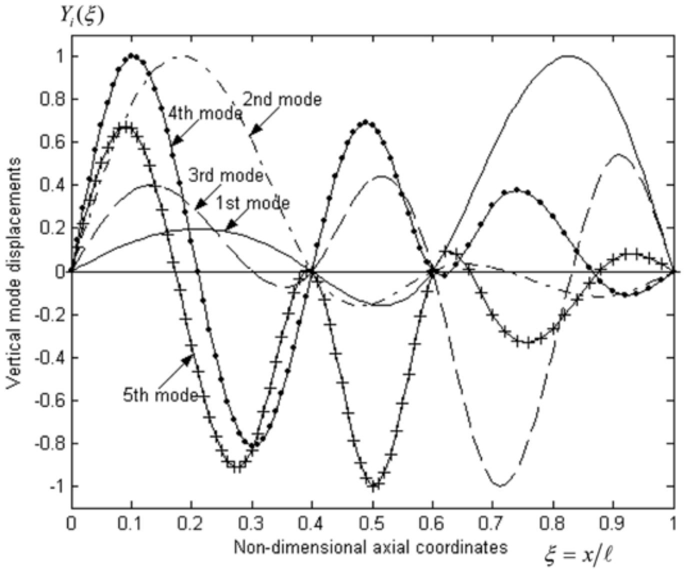

Fig. 8 The lowest five mode shapes for a uniform pinned-pinned beam carrying five intermediate point masses and with two intermediate supports located at $\bar{\xi}_1 = 0.4$ and $\bar{\xi}_2 = 0.6$, respectively. The magnitudes and locations of the five point masses are: $m_1^* = 0.2, m_2^* = 0.3, m_3^* = 0.5, m_4^* = 0.65$ and $m_5^* = 1.0$ located at $\xi_1^* = 0.1$, $\xi_2^* = 0.3$, $\xi_3^* = 0.5$, $\xi_4^* = 0.7$ and $\xi_5^* = 0.9$, respectively

In the case of pinned-pinned beam with two intermediate supports located at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.4$ and $\bar{\xi}_2 = \bar{x}_2/\ell = 0.6$ (cf. Fig. 7), in spite of the fact that the configuration of the beam itself is symmetrical with respect to its central point C and the spacings for the five intermediate point masses are also equal to each other, the lowest five mode shapes shown in Fig. 8 are not symmetrical with respect to the central point C of beam. This is because the magnitudes of the five intermediate point masses ($m_1^* = 0.2, m_2^* = 0.3, m_3^* = 0.5, m_4^* = 0.65, m_5^* = 1.0$) are not equal to each other, where $m_i^* = m_i/(\bar{m}\ell)$, ($i$ = 1 to 5). Of course, all the five curves pass through the two pinned supports located at $\bar{\xi}_1 = \bar{x}_1/\ell = 0.4$ and $\bar{\xi}_2 = \bar{x}_2/\ell = 0.6$ as one may see from Fig. 8.

6. Conclusions

In general, the accuracy of a numerical method is evaluated by comparing its numerical result with the associated "exact" solution. For this reason, many researchers devote themselves to obtaining exact solutions of various problems. The free vibration analysis of a multi-span beam carrying multiple intermediate point masses is an important engineering problem, but its exact solution for natural frequencies and mode shapes is rare. In this paper the numerical assembly method (NAM) has been successfully applied to determine the exact solution for the natural frequencies and mode shapes of a multi-span beam carrying multiple intermediate point masses.

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Appendix

The overall coefficient matrix $[B]$ for a uniform pinned-pinned beam with one intermediate point mass and one intermediate pinned support is found to be

$$\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
s\Omega\bar{\xi}_1 & c\Omega\bar{\xi}_1 & sh\Omega\bar{\xi}_1 & ch\Omega\bar{\xi}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s\Omega\bar{\xi}_1 & c\Omega\bar{\xi}_1 & sh\Omega\bar{\xi}_1 & ch\Omega\bar{\xi}_1 & 0 & 0 & 0 & 0 \\
c\Omega\bar{\xi}_1 & -s\Omega\bar{\xi}_1 & ch\Omega\bar{\xi}_1 & sh\Omega\bar{\xi}_1 & -c\Omega\bar{\xi}_1 & s\Omega\bar{\xi}_1 & -ch\Omega\bar{\xi}_1 & -sh\Omega\bar{\xi}_1 & 0 & 0 & 0 & 0 \\
-s\Omega\bar{\xi}_1 & -c\Omega\bar{\xi}_1 & sh\Omega\bar{\xi}_1 & ch\Omega\bar{\xi}_1 & s\Omega\bar{\xi}_1 & c\Omega\bar{\xi}_1 & -sh\Omega\bar{\xi}_1 & -ch\Omega\bar{\xi}_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & s\Omega\xi_1^* & c\Omega\xi_1^* & sh\Omega\xi_1^* & ch\Omega\xi_1^* & -s\Omega\xi_1^* & -c\Omega\xi_1^* & -sh\Omega\xi_1^* & -ch\Omega\xi_1^* \\
0 & 0 & 0 & 0 & c\Omega\xi_1^* & -s\Omega\xi_1^* & ch\Omega\xi_1^* & sh\Omega\xi_1^* & -c\Omega\xi_1^* & s\Omega\xi_1^* & -ch\Omega\xi_1^* & -sh\Omega\xi_1^* \\
0 & 0 & 0 & 0 & -s\Omega\xi_1^* & -c\Omega\xi_1^* & sh\Omega\xi_1^* & ch\Omega\xi_1^* & s\Omega\xi_1^* & c\Omega\xi_1^* & -sh\Omega\xi_1^* & -ch\Omega\xi_1^* \\
0 & 0 & 0 & 0 & m_1^*\Omega s\Omega\xi_1^* - c\Omega\xi_1^* & m_1^*\Omega c\Omega\xi_1^* + s\Omega\xi_1^* & m_1^*\Omega sh\Omega\xi_1^* + ch\Omega\xi_1^* & m_1^*\Omega ch\Omega\xi_1^* + sh\Omega\xi_1^* & c\Omega\xi_1^* & -s\Omega\xi_1^* & -ch\Omega\xi_1^* & -sh\Omega\xi_1^* \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s\Omega & c\Omega & sh\Omega & ch\Omega \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -s\Omega & -c\Omega & sh\Omega & ch\Omega
\end{bmatrix}$$

where

$$s\Omega\bar{\xi}_1 = \sin\Omega\bar{\xi}_1, \quad c\Omega\bar{\xi}_1 = \cos\Omega\bar{\xi}_1, \quad sh\Omega\bar{\xi}_1 = \sinh\Omega\bar{\xi}_1, \quad ch\Omega\bar{\xi}_1 = \cosh\Omega\bar{\xi}_1$$

$$s\Omega\xi_1^* = \sin\Omega\xi_1^*, \quad c\Omega\xi_1^* = \cos\Omega\xi_1^*, \quad sh\Omega\xi_1^* = \sinh\Omega\xi_1^*, \quad ch\Omega\xi_1^* = \cosh\Omega\xi_1^*$$