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Robustness analysis of vibration control in structures with uncertain parameters using interval method

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Abstract. Variations in system parameters due to uncertainties may result in system performance deterioration. Uncertainties in modeling of structures are often considered to ensure that control system is robust with respect to response errors. Hence, the uncertain concept plays an important role in vibration control of the engineering structures. The paper discusses the robustness of the stability of vibration control systems with uncertain parameters. The vibration control problem of an uncertain system is approximated by a deterministic one. The uncertain parameters are described by interval variables. The uncertain state matrix is constructed directly using system physical parameters and avoided to use bounds in Euclidean norm. The feedback gain matrix is determined based on the deterministic systems, and then it is applied to the actual uncertain systems. A method to calculate the upper and lower bounds of eigenvalues of the close-loop system with uncertain parameters is presented. The lower bounds of eigenvalues can be used to estimate the robustness of the stability the controlled system with uncertain parameters. Two numerical examples are given to illustrate the applications of the present approach.

Key words: uncertain systems; vibration active control; upper and lower bounds of eigenvalues; robustness analysis of the stability; interval analysis.

1. Introduction

In engineering design, it is important to calculate response quantities such as the displacement, stress, vibration frequencies and mode shapes against a given set of design parameters. However, the design parameters may be uncertain because of complexity of structures, manufacture errors and inaccuracy in measurement, etc. In the past, problems with uncertainties have been studied to provide an insight into the statistical response variations. The methods used in these studied were based on probabilistic approaches includes simulation (involving sampling and estimation). Among the most commonly used simulation techniques are direct Monte Carlo simulation, stratified sampling, and Latin hypercube sampling (Larson 1979, Vanmareke 1983). The other probabilistic approaches include numerical integration, second-moment analysis, perturbation analysis, and probabilistic finite element methods (Chen 1992, Chen *et al.* 1992, Contracts 1980).

Despite the success of the above analysis methods, one may recognize that uncertainties in

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parameters can be modeled on the basis of alterative, non-probabilistic conceptual frameworks. One such approach, based on a set-theoretic formulation, is an unknown-but-bounded model (convex model). Such set models of uncertainty have been applied to linear programming and system theory (Dief 1986).

Recently, such set models of uncertainties in parameters have drawn interest both from the system control robustness analysis field and from the structural failure measures field. For example, the convex model was introduced (Ben-Haim and Elishakoff 1990, Lindberg 1991) for the study of dynamic response and failure of structures under pulsed parametric loading; the convex model has been applied in determining the upper and lower bounds of static response for structures (Liu and Chen 1994). The convex model has also been applied to the optimal design of structures with uncertain parameters (Ganzerli and Pantelides 1998, 1999, Pantelides and Booth 2000, Gantelides 2000).

Since the mid-1960s, a new method called the interval analysis has appeared. Moore (1979) and his co-workers, Alefeld and Herzberger (1983) have done the pioneering work. Mathematically, linear interval equations, nonlinear interval equations and interval eigenvalue problems have been resolved partly. Because of the complexity of the algorithm, it is difficult to apply these results to practical engineering problems. Recently, the interval set models have been used in the study of the static response and eigenvalue problems of structures with bounded uncertain parameters (Chen and Qiu 1994). The interval finite element method was presented (Chen *et al.* 2000, 2002, 2003) which makes the method easier to deal with the interval eigenvalues and dynamic response analysis of the complex structures with interval parameters.

The vibration control theory for systems with deterministic parameters has been well developed. For example, the standard methods for vibration control have been developed (Porter and Crossley 1972), and the modal controllability/observability and modal optimal control for defective/near defective systems with repeated/close eigenvalues were discussed (Chen and Liu 2001). As mentioned above, the uncertain concept plays an important role in the control problems of the vibration structures. Many studies have been done from the viewpoint of mathematics about control problems. For example, the sufficient and necessary conditions of the dynamical stability for the uncertain systems were discussed (Mcri and Kokame 1987); the robustness of control systems with uncertain parameters was discussed (Rachid 1989); the stability of an uncertain matrix was discussed (Juang *et al.* 1987).

In recent years, the vibration control problem of structures with uncertain parameters has attracted a great deal of interest. For example, the control problem of uncertain system for helicopter rotor blades was discussed (Krodkiewski 2000); The control problems for a wide class of mechanical system with uncertainties was presented (Ferrara and Giacomini 2000); A systematic approach is proposed for determining the probability of instability for a control structure with real parameter uncertainties which were modeled as random variables with prescribed probability distributions (Spencer and Sain 1992). The robust vibration control of uncertain systems using variable parameter feedback and model-based fuzzy strategies were proposed (Li and Yarn 2001). As mentioned by Li and Yarn (2001), the variable parameter feedback is based on an analytical model with uncertainty bounds in Euclidean norm. However, for a complex structure, the construction of an analytical and determination of bounds in Euclidean norm for uncertainties are difficult.

The robustness of a closed-loop system is one of the most important concerns of control system designers. Variations in system parameters due to uncertainties may result in system performance deterioration. Uncertainties in structural modeling of structures are often considered to ensure that control system is robust with respect to response errors.

In this paper, the robustness analysis of the stability of the controlled systems with uncertain parameters is discussed. The uncertainties of the structural parameters are described by interval variables. The state matrix is constructed directly using system physical parameters and avoided to use bounds in Euclidean norm. The control problems of the uncertain systems are transformed into ones of the deterministic systems. At first, by using the method of pole allocation, the state feedback gain matrix of the systems with deterministic parameters can be obtained, and then it is applied into the actual uncertain systems. By using interval extension and perturbation method, the expressions can be developed for calculating the upper and lower bounds of eigenvalues of uncertain closed-loop systems. The lower bounds of eigenvalues can be used to estimate the robustness of the stability of the controlled system with uncertain parameters. The method presented in this paper will not require the distribution function of the uncertain parameters of the systems other than their upper and lower bounds. Similarly, the distribution function of eigenvalues of closed-loop systems with uncertain parameters will not be computed other than their upper and lower bounds. So these results are different from those obtained by Spencer and Sain (1992). Two numerical examples are given to illustrate the applications of the approach presented in this study.

2. The definition of the problem

Consider the linear vibration control equation in state space

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1}$$

By using the state feedback, the input vector is

$$\mathbf{u}(t) = \mathbf{G}\mathbf{x}(t) \tag{2}$$

where $\mathbf{x}(t)$ is the $2n \times 1$ state vector, $\mathbf{u}(t)$ is $m \times 1$ input vector, $\mathbf{A}$ is the $2n \times 2n$ asymmetric general state matrix, $\mathbf{B}$ is $2n \times m$ input coefficient matrix, $\mathbf{G}$ is $m \times 2n$ state feedback gain matrix.

The state matrix $\mathbf{A}$ and input coefficient matrix $\mathbf{B}$ of the uncertain system can be expressed as

$$\begin{aligned} \mathbf{A} &= \mathbf{A}_0 + \Delta\mathbf{A} \\ \mathbf{B} &= \mathbf{B}_0 + \Delta\mathbf{B} \end{aligned} \tag{3}$$

where $\mathbf{A}_0$ and $\mathbf{B}_0$ are the deterministic parts of the state matrix and the input coefficient matrix, respectively. $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$ are the uncertain parts of the state matrix and the input coefficient matrix, respectively. Correspondingly, the uncertain state vector $\mathbf{x}$, the uncertain input vector $\mathbf{u}$ and the uncertain gain matrix $\mathbf{G}$ are

$$\begin{aligned} \mathbf{x} &= \mathbf{x}_0 + \Delta\mathbf{x} \\ \mathbf{u} &= \mathbf{u}_0 + \Delta\mathbf{u} \\ \mathbf{G} &= \mathbf{G}_0 + \Delta\mathbf{G} \end{aligned} \tag{4}$$

where $\mathbf{x}_0$, $\mathbf{u}_0$ and $\mathbf{G}_0$ are the deterministic parts of the state vector, the input vector and the gain matrix. $\Delta\mathbf{x}$, $\Delta\mathbf{u}$ and $\Delta\mathbf{G}$ are their uncertain parts, respectively.

Substituting Eqs. (3), (4) to Eqs. (1) and (2) yields

$$\dot{\mathbf{x}}_0 + \Delta\dot{\mathbf{x}} = (\mathbf{A} + \Delta\mathbf{A})(\mathbf{x} + \Delta\mathbf{x}) + (\mathbf{B} + \Delta\mathbf{B})(\mathbf{u} + \Delta\mathbf{u}) \tag{5}$$

and

$$\mathbf{u}_0 + \Delta\mathbf{u} = (\mathbf{G}_0 + \Delta\mathbf{G})(\mathbf{x}_0 + \Delta\mathbf{x}) \tag{6}$$

Expanding Eqs. (5) and (6), we have

$$\dot{\mathbf{x}}_0 + \Delta\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x}_0 + \mathbf{A}_0\Delta\mathbf{x} + \Delta\mathbf{A}\mathbf{x}_0 + \Delta\mathbf{A}\Delta\mathbf{x} + \mathbf{B}_0\mathbf{u}_0 + \mathbf{B}_0\Delta\mathbf{u} + \Delta\mathbf{B}\mathbf{u}_0 + \Delta\mathbf{B}\Delta\mathbf{u} \tag{7}$$

and

$$\mathbf{u}_0 + \Delta\mathbf{u} = \mathbf{G}_0\mathbf{x}_0 + \mathbf{G}_0\Delta\mathbf{x} + \Delta\mathbf{G}\mathbf{x}_0 + \Delta\mathbf{G}\Delta\mathbf{x} \tag{8}$$

Neglecting the higher order terms of the above Eqs. (7) and (8), and equating the coefficients of the same orders of the left and the right sides, we obtain

$$\left.\begin{aligned} \dot{\mathbf{x}}_0 &= \mathbf{A}_0\mathbf{x}_0 + \mathbf{B}_0\mathbf{u}_0 \\ \mathbf{u}_0 &= \mathbf{G}_0\mathbf{x}_0 \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} \Delta\dot{\mathbf{x}} &= \mathbf{A}_0\Delta\mathbf{x} + \Delta\mathbf{A}\mathbf{x}_0 + \mathbf{B}_0\Delta\mathbf{u} + \Delta\mathbf{B}\mathbf{u}_0 \\ \Delta\mathbf{u} &= \mathbf{G}_0\Delta\mathbf{x} + \Delta\mathbf{G}\mathbf{x}_0 \end{aligned}\right\} \tag{10}$$

From the above discussion it can be seen that the uncertain system (1) and (2) has been separated into the deterministic part (9) and the uncertain part (10). The state equation of the closed-loop system corresponding to the deterministic one (9) is

$$\dot{\mathbf{x}}_0(t) = (\mathbf{A}_0 + \mathbf{B}_0\mathbf{G}_0)\mathbf{x}_0(t) \tag{11}$$

and the corresponding eigenvalue problem is

$$S_0\mathbf{u}_0 = (\mathbf{A}_0 + \mathbf{B}_0\mathbf{G}_0)\mathbf{u}_0 \tag{12}$$

3. The feedback gain matrices of the deterministic control systems

In the pole allocation method, to guarantee asymptotic stability, the closed-loop poles can be selected in advance and the gains are determined so as to produce these poles (Porter and Crossley 1972). Thus when the closed-loop eigenvalues of Eq. (11) are assigned to be $S_1^*, S_2^*, \ldots, S_{2n}^*$, by using the pole allocation, the gain matrix $\mathbf{G}_0$ of the deterministic system (9) can be determined.

First, we transform Eq. (9) into the control equation in modal coordinates. It is well known that if $\mathbf{A}_0$ is not a defective matrix, there exists the right and left modal matrices, $\mathbf{U}_0 = [\mathbf{u}_{01}, \mathbf{u}_{02}, \ldots, \mathbf{u}_{02n}]$ and $\mathbf{V}_0 = [\mathbf{v}_{01}, \mathbf{v}_{02}, \ldots, \mathbf{v}_{02n}]$, such that (Porter and Crossley 1972)

$$\mathbf{V}_0^T\mathbf{A}_0\mathbf{U}_0 = \Lambda_0, \qquad \mathbf{V}_0^T\mathbf{U}_0 = \mathbf{I} \tag{13}$$

where $\Lambda_0 = diag(S_{01}, S_{02}, ..., S_{02n})$ is the diagonal matrix of the eigenvalues of the deterministic system. That is the $\mathbf{A}_0$ can be diagonalized.

With the modal transformation

$$\mathbf{x}_0(t) = \mathbf{U}_0\xi(t) \tag{14}$$

the Eq. (9) can be transformated into

$$\dot{\xi}(t) = \Lambda_0\xi(t) + \mathbf{B}_0'\mathbf{u}_0(t) \tag{15}$$

and

$$\mathbf{u}_0(t) = \mathbf{G}_0'\xi(t) \tag{16}$$

$$\mathbf{B}_0' = \mathbf{V}_0^T\mathbf{B}_0 = (b_1', b_2', \ldots, b_{2n}')^T, \qquad \mathbf{G}_0' = \mathbf{G}_0\mathbf{U}_0 = (g_1', g_2', \ldots, g_{2n}') \tag{17}$$

If the single input is used, $\mathbf{B}_0$ is a column vector, $\mathbf{G}_0$ is a row vector.

Substituting Eq. (16) to Eq. (15), one has

$$\dot{\xi}(t) = (\Lambda_0 + \mathbf{B}_0'\mathbf{G}_0')\xi(t) \tag{18}$$

In Eq. (18), suppose the assigned eigenvalues are $S_i^*(i = 1, 2, \ldots, 2n)$ and the corresponding eigenvectors are $W_i(i = 1, 2, \ldots, 2n)$ they satisfy the following eienproblem

$$(\Lambda_0 + \mathbf{B}_0'\mathbf{G}_0')\mathbf{w}_i = S_i^*\mathbf{w}_i \tag{19}$$

That is

$$(\Lambda_0 + \mathbf{B}_0'\mathbf{G}_0' - S_i^*\mathbf{I})\mathbf{w}_i = 0 \tag{20}$$

Because $\mathbf{w}_i \neq 0$, then there exists

$$\det(\Lambda_0 + \mathbf{B}_0'\mathbf{G}_0' - S_i^*\mathbf{I}) = 0 \tag{21}$$

Solving Eq. (21), we obtain

$$g_i' = \frac{\prod_{k=1}^{2n}(s_k^* - s_i)}{b_i'\prod(s_k - s_i)} \tag{22}$$

thus obtaining the matrix $\mathbf{G}_0' = (g_1', g_2', \ldots, g_{2n}')$.

From Eq. (14), we obtain

$$\xi(t) = \mathbf{V}_0^T \mathbf{x}_0(t) \tag{23}$$

Substituting Eq. (23) to Eq. (16) yields

$$\mathbf{u}_0(t) = \mathbf{G}_0' \mathbf{V}_0^T \mathbf{x}_0(t) = \mathbf{G}_0 \mathbf{x}_0(t) \tag{24}$$

where

$$\mathbf{G}_0 = \mathbf{G}_0' \mathbf{V}_0^T \tag{25}$$

Substituting $\mathbf{G}_0$ into Eq. (1) yields

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{G}_0\mathbf{x}(t) \\ &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{26}$$

where $\mathbf{C} = \mathbf{A} + \mathbf{B}\mathbf{G}_0$.

If the deterministic gain matrix $\mathbf{G}_0$ is applied to the uncertain system, there must exist some errors between the closed-loop eigenvalues and the assigned eigenvalues $S_i^*(i = 1, 2, \ldots, 2n)$. By combining the interval analysis with the perturbation method, the expressions for computing the upper and lower bounds of the closed-loop eigenvalues, $S_i^*(i = 1, 2, \ldots, 2n)$, can be developed.

4. The definitions of the interval and interval operations

In this section, a brief review on the interval operations is given (Moore 1979).

Assume that R is the set of all real numbers, $I(\mathbf{R})$, $I(\mathbf{R}^n)$ and $I(\mathbf{R}^{n\times n})$ denote the sets of all closed real interval numbers, n dimensional real interval vectors and $n \times n$ real interval matrices, respectively. $X^I = [\underline{x}, \overline{x}] \in I(\mathbf{R})$ can be usually written in the following form

$$X^I = [X^c - \Delta X, X^c + \Delta X] \tag{27}$$

in which X^c and ΔX denote the mean (or midpoint) value of x and the uncertainty (or the maximum width) in x, respectively. It follows that

$$X^c = \frac{\underline{x} + \overline{x}}{2} \tag{28}$$

$$\Delta X = \frac{\overline{x} - \underline{x}}{2} \tag{29}$$

In terms of the interval addition, Eq. (27) can be put into the more useful form

$$X^I = X^c + \Delta X^I \tag{30}$$

where $\Delta X^I = [-\Delta X, \Delta X]$

n-dimensional real interval vector $\mathbf{X}^I \in I(\mathbf{R}^n)$ can be expressed as

$$\mathbf{X}^I = (X_1^I, X_2^I, \ldots, X_n^I)^T \tag{31}$$

The mean value and uncertainty of **X** are

$$\mathbf{X}^c = (X_1^c, X_2^c, \ldots, X_n^c)^T \tag{32}$$

$$\Delta\mathbf{X} = (\Delta X_1, \Delta X_2, \ldots, \Delta X_n)^T \tag{33}$$

Similar expression exists for an $n \times n$ interval matrix $\mathbf{A}^I = [\underline{\mathbf{A}}, \overline{\mathbf{A}}] \in I(\mathbf{R}^{n \times n})$

$$\mathbf{A}^I = \mathbf{A}^c + \Delta\mathbf{A}^I \tag{34}$$

where $\Delta\mathbf{A}^I = [-\Delta\mathbf{A}, \Delta\mathbf{A}]$, $\mathbf{A}^c$ and $\Delta\mathbf{A}$ denote the mean matrix of $\mathbf{A}^I$ and the uncertain (or the maximum width) matrix of $\mathbf{A}^I$, respectively. It follows that

$$\mathbf{A}^c = \frac{(\overline{\mathbf{A}} + \underline{\mathbf{A}})}{2} \quad \text{or} \quad a_{ij}^c = \frac{(\bar{a}_{ij} + \underline{a}_{ij})}{2} \tag{35}$$

$$\Delta\mathbf{A} = \frac{(\overline{\mathbf{A}} - \underline{\mathbf{A}})}{2} \quad \text{or} \quad \Delta a_{ij} = \frac{(\bar{a}_{ij} - \underline{a}_{ij})}{2} \tag{36}$$

where $\mathbf{A}^c = (a_{ij}^c)$ and $\Delta\mathbf{A} = (\Delta a_{ij})$

Let $X^I, Y^I \in I(\mathbf{R}), X^I = [\underline{x}, \bar{x}], Y^I = [\underline{y}, \bar{y}]$, then the interval arithmetic operations are

$$X^I + Y^I = [\underline{x}, \bar{x}] + [\underline{y}, \bar{y}] = [\underline{x} + \underline{y}, \bar{x} + \bar{y}] \tag{37}$$

$$X^I - Y^I = [\underline{x}, \bar{x}] - [\underline{y}, \bar{y}][\underline{x} - \bar{y}, \bar{x} - \underline{y}] \tag{38}$$

$$X^I \times Y^I = [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] = [\min(\underline{x} \cdot \underline{y}, \underline{x} \cdot \bar{y}, \bar{x} \cdot \underline{y}, \bar{x} \cdot \bar{y}), \max(\underline{x} \cdot \underline{y}, \underline{x} \cdot \bar{y}, \bar{x} \cdot \underline{y}, \bar{x} \cdot \bar{y})] \tag{39}$$

$$\frac{X^I}{Y^I} = \frac{[\underline{x}, \bar{x}]}{[\underline{y}, \bar{y}]} = [\underline{x}, \bar{x}]\left[\frac{1}{\bar{y}}, \frac{1}{\underline{y}}\right] \tag{40}$$

Let $X_1^I, X_2^I \in \mathbf{I}(R)$. Then the set

$$Z^I = \{a = a_1 + ia_2 | a_1 \in X_1^I, a_2 \in X_2^I\} \tag{41}$$

is called a complex interval.

Let z be an arbitrary complex and r be an arbitrary real with $r \geq 0$, then, the bounded closed set

$$Z = [z, r] = \{z_0 \in C | |z_0 - z| \leq r\} \tag{42}$$

is called a complex circle plate.

Let $Z_1^I = X_1^I + iY_1^I, Z_2^I = X_2^I + iY_2^I$, where $X_k^I, Y_k^I \in I(\mathbf{R})$ $(k = 1, 2)$. Then $Z_1^I + Z_2^I$, $Z_1^I - Z_2^I, Z_1^I \times Z_2^I$, and $[Z_1^I/Z_2^I]$ are defined by the following formulas

$$Z_1^I + Z_2^I = (X_1^I + X_2^I) + i(Y_1^I + Y_2^I) \tag{43}$$

$$Z_1^I - Z_2^I = (X_1^I - X_2^I) + i(Y_1^I - Y_2^I) \tag{44}$$

$$Z_1^I \times Z_2^I = (X_1^I X_2^I - Y_1^I Y_2^I) + i(X_1^I Y_2^I + Y_1^I X_2^I) \tag{45}$$

$$\frac{Z_1^I}{Z_2^I} = \frac{X_1^I X_2^I - Y_1^I Y_2^I}{(X_2^I)^2 + (Y_2^I)^2} + i\frac{X_1^I Y_2^I + Y_1^I X_2^I}{(X_2^I)^2 + (Y_2^I)^2} \tag{46}$$

An interval function is an interval-value function of one or more interval arguments. Assume that $\mathbf{F}(\mathbf{X}^I) = \mathbf{F}(X_1^I, X_2^I, \ldots, X_n^I)$ is the interval value function of interval variable $\mathbf{X}^I = (X_1^I, X_2^I, \ldots, X_n^I)$, if $X_i^I \in Y_i^I, i = 1, 2, \ldots, n$, one has

$$\mathbf{F}(X_1^I, X_2^I, \ldots, X_n^I) \in \mathbf{F}(Y_1^I, Y_2^I, \ldots, Y_n^I) \tag{47}$$

We say that the interval value function $\mathbf{F}(\mathbf{X}^I) = \mathbf{F}(X_1^I, X_2^I, \ldots, X_n^I)$ of the interval $\mathbf{X}^I = (X_1^I, X_2^I, \ldots, X_n^I)$ is inclusion monotonic, if $\mathbf{f}$ is the real function of n real variables $x_1, x_2, \ldots, x_n$ and the interval value function $\mathbf{F}$ of n interval variables $X_1^I, X_2^I, \ldots, X_n^I$ satisfy

$$\mathbf{F}(x_1, x_2, \ldots, x_n) = \mathbf{f}(x_1, x_2, \ldots, x_n), \quad x_i \in X_i^I \quad (i = 1, 2, \ldots, n) \tag{48}$$

$\mathbf{F}$ is known as the interval extension of $\mathbf{f}$.

Real rational functions of n real variables may have natural extensions. Given rational expression in real variables, we can replace the real variables by corresponding interval variables and replace the real arithmetic operations by the corresponding interval arithmetic operations to obtain a rational interval function, which is called natural extension of the real rational function. The extensions of the real rational function are inclusion monotonic and they can be calculated through finite-interval arithmetic operations.

Let f be a complex-valued function of n complex variables $z_1, z_2, \ldots, z_n$. A complex circle plate extension of $\mathbf{f}$ means that a complex circle plate value function $\mathbf{F}$ of n complex circle plates $Z_1, Z_2, \ldots, Z_n$ for all $z_i \in Z_i$ $(i = 1, 2, \ldots, n)$ possesses the following property

$$\mathbf{F}([z_1, 0], [z_2, 0], \ldots, [z_n, 0]) = \mathbf{f}(z_1, z_2, \ldots, z_n) \tag{49}$$

It is well known that typical structural response analysis problem resorts to finite element analysis in which the response functions are not analytic. So it is difficult to get the exact interval solutions of the response functions. We can resort to the first-order Taylor expansion to obtain the rational approximation of a complex function and then apply the natural interval extension to the rational approximation to get its interval solution. Thus the rational approximation of a complex function is a linear function of the variables and each variable appears only once, so the interval solution of the rational approximation we obtain is unique (Moore 1979). In order to justify the reasonability of this

Table 1 Comparison for the interval value of $g(x, a)$

Interval variables	δ	Exact solution	Approximate solution	Error of mid-point	Error of interval uncertainty
$x^I = [2.4, 2.6]$ $a^I = [0.4, 0.6]$	0.04 0.2	f^I: [−1.03, −0.65] f^C: −0.8393 $\Delta f = 0.1893$	g^I: [−1.02, −0.64] g^C: −0.8333 $\Delta g = 0.1889$	0.71%	0.21%
$x^I = [2.3, 2.7]$ $a^I = [0.3, 0.7]$	0.08 0.4	f^I: [−1.24, −0.48] f^C: −0.8575 $\Delta f = 0.381$	g^I: [−1.21, −0.46] g^C: −0.8333 $\Delta g = 0.3778$	2.82%	0.84%
$x^I = [2.2, 2.8]$ $a^I = [0.2, 0.8]$	0.12 0.6	f^I: [−1.47, −0.31] f^C: −0.8889 $\Delta f = 0.5778$	g^I: [−1.4, −0.27] g^C: −0.8333 $\Delta g = 0.5667$	6.252%	1.92%

approach, we take a function given by $g(x, a) = \frac{ax}{1-x} x \neq 1, a \neq 0$. The exact solutions of the interval value for different interval variables are easy to calculation. Now we use Taylor expansion to expand the function about the mid-points of the interval variables to get the approximation of the interval value. In Table 1, we give the comparison for the interval value of the exact solution and the approximate solution for different interval variables, where δ is the relative uncertainty of a interval which is defined by $\delta = \frac{\Delta X}{|x^c|}$. Suppose the mid-point and the uncertainty of the exact solution are denoted as f^C and Δf, respectively. Similarly, we denote the mid-point and the uncertainty of the approximate solution as g^C and Δg, respectively. The error of the mid-point is the value of $|(g^c - f^c)/f^c|$, and the error of the interval uncertainty is the value of $|(\Delta g - \Delta f)/\Delta f|$.

From Table 1, we can see that the errors of the mid-point and the interval uncertainty go up as the relative uncertainties of the interval variables increase. In fact, the relative uncertainties of the interval variables are small in practical engineering problems, so the approximate approach is acceptable for practical applications.

5. The state matrix with interval parameters

It has been pointed out that the classical formulation of the system matrices does not take into account the way the matrices are built for the system physical parameters (Li and Yarn 2001). In this section we will present a way to build the interval matrices with the system physical parameters.

Assume that the interval structural parameters of the structures are denoted by $\mathbf{b}^I$

$$\mathbf{b}^I = (b_1^I, b_2^I, \ldots, b_m^I)^T = \mathbf{b}^c + \Delta\mathbf{b}^I$$

$$\mathbf{b}^c = (b_1^c, b_2^c, \ldots, b_m^c)^T \qquad \Delta\mathbf{b}^I = (\Delta b_1^I, \Delta b_2^I, \ldots, \Delta b_m^I)^T \tag{50}$$

where m is the number of interval parameters. For any component

$$b_i^I = [\underline{b}_i, \overline{b}_i] = b_i^c + \Delta b_i e_i \tag{51}$$

where $\Delta b_i = (\overline{b}_i - \underline{b}_i)/2$ and $e_i = [-1, 1]$

From Eq. (26), the control equation of the close-loop system with the uncertain parameters can be expressed as

$$\dot{\mathbf{x}}(t) = (\mathbf{A} + \mathbf{BG}_0)\mathbf{x}(t) = \mathbf{Cx}(t) \tag{52}$$

where $\mathbf{C} = \mathbf{A} + \mathbf{BG}_0$

For any $b \in b^I$, using Taylor series and expanding the state matrix, $\mathbf{C(b)}$, around the mean values $\mathbf{b}^c$, one has

$$\mathbf{C(b)} = \mathbf{C(b}^c) + \sum_{j=1}^{m}\left(\frac{\partial\mathbf{C(b}^c)}{\partial b_j}\right)(b_j - b_j^c) \tag{53}$$

Using the natural interval extension, the interval state matrix can be obtained

$$\mathbf{C(b}^I) = \mathbf{C(b}^c) + \sum_{j=1}^{m}\frac{\partial\mathbf{C(b}^c)}{\partial b_j}(b_j^I - b_j^c) = \mathbf{C(b}^c) + \sum_{j=1}^{m}\frac{\partial\mathbf{C(b}^c)}{\partial b_j}\Delta b_j e_j = \mathbf{C(b^c)} + \Delta\mathbf{C(b^I)} \tag{54}$$

$$\Delta\mathbf{C(b}^I) = \sum_{j=1}^{m}\left(\frac{\partial\mathbf{C}^e\mathbf{(b)}}{\partial b_j}\right)\Delta b_j e_j \tag{55}$$

6. Matrix perturbation analysis for eigenvalues

Consider the eigenproblem

$$\mathbf{C}_0\mathbf{u}_{k0} = S_{k0}\mathbf{u}_{k0} \qquad \mathbf{C}_0^T\mathbf{v}_{k0} = S_0\mathbf{v}_{k0} \tag{56}$$

where $\mathbf{C}_0$ is the state matrix of the deterministic system, S_{k0} is the kth eigenvalue and $\mathbf{u}_{k0}$ is the kth eigenvector, $\mathbf{v}_{k0}^T$ is the corresponding left eigenvector. When the small changes of the parameters are introduced into the state matrix $\mathbf{C}_0$, the eigenvalue problem becomes

$$(\mathbf{C}_0 + \Delta\mathbf{C})(\mathbf{u}_{k0} + \Delta\mathbf{u}_k) = (S_{k0} + \Delta S_k)(\mathbf{u}_{k0} + \Delta\mathbf{u}_k) \tag{57}$$

where $\Delta\mathbf{C}$ are the increment of $\mathbf{C}_0$. The eigensolutions of the perturbed system are

$$S_k = S_{k0} + S_k \qquad \mathbf{u}_k = \mathbf{u}_{k0} + \Delta\mathbf{u}_k \tag{58}$$

According to the perturbation theory (Chen 1999), we have

$$S_k = S_{k0} + S_k \qquad S_{k1} = \mathbf{v}_{k0}^T\Delta\mathbf{C}\mathbf{u}_{k0} \tag{59}$$

where $k = 1, 2, 3, ..., 2n$, $\mathbf{u}_{k0}$ and S_{k0} are the kth original eigenvector and eigenvalue; $\mathbf{u}_{k1}$ and S_{k1} are the first-order perturbation of the kth eigensolution.

7. Robustness analysis of the stability of closed-loop systems with interval parameters

Applying the feedback gain matrix $\mathbf{G}_0$ to the uncertain system, the control equation of the closed-loop system becomes

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{G}_0\mathbf{x}(t) \tag{60}$$

Letting $\mathbf{C} = \mathbf{A} + \mathbf{B}\mathbf{G}_0$ Eq. (60) can be expressed as

$$\dot{\mathbf{x}}(t) = \mathbf{C}\mathbf{x}(t) \tag{61}$$

For any $\mathbf{b} \in \mathbf{b}^I$, the eigenproblem is

$$\mathbf{C}(\mathbf{b})\mathbf{u} = S\mathbf{u} \tag{62}$$

With the interval matrix expression and the interval extension, Eq. (62) can be expressed as

$$\mathbf{C}(\mathbf{b}^I)\mathbf{u} = S^I\mathbf{u} \tag{63}$$

Eq. (63) is called interval eigenvalue problem. It is the basic problem for given interval state matrix $\mathbf{C}(\mathbf{b}^I)$, to find the interval eigenvalue S^I which is not only the smallest interval but enclose all possible eigenvalues S, satisfying $\mathbf{Cu} = S\mathbf{u}$. In other words, we seek a hull

$$\Gamma = \{S{:}[\mathbf{C}(\mathbf{b}) + S\mathbf{I}]\mathbf{u} = 0, \quad \mathbf{C}(\mathbf{b}) \in \mathbf{C}(\mathbf{b}^I)\}$$

$$\Gamma = \{S{:}[\mathbf{C}(\mathbf{b}) + S\mathbf{I}]^T\mathbf{v} = 0, \quad \mathbf{C}(\mathbf{b}) \in \mathbf{C}(\mathbf{b}^I)\} \tag{64}$$

to the set

$$\underline{S} = \min S(\mathbf{C}(\mathbf{b}))$$

$$\overline{S} = \max S(\mathbf{C}(\mathbf{b})) \tag{65}$$

In Eqs. (63) and (64), $\mathbf{C}(\mathbf{b}) \in \mathbf{C}(\mathbf{b}^I)$. It should be noted that the number of the eigensolutions satisfying Eq. (63) may be infinite and thus it is difficult to solve using the standard methods.

In terms of the interval expression, the interval matrix $\mathbf{C}(\mathbf{b}^I)$ can be expressed as

$$\mathbf{C}(\mathbf{b}^I) = \mathbf{C}(\mathbf{b}^c) + \Delta\mathbf{C}(\mathbf{b}^I) \tag{66}$$

Thus, the interval eigenproblem (63) can be written as

$$[\mathbf{C}(\mathbf{b}^c) + \Delta\mathbf{C}(\mathbf{b}^I)]\mathbf{u} = S^I\mathbf{u} \tag{67}$$

where $\Delta\mathbf{C}(\mathbf{b}^I)$ are given by Eq. (55). For any $\mathbf{b} \in \mathbf{b}^I$, there is a group o $\delta\mathbf{C}(\mathbf{b})$ which satisfies

$$\delta\mathbf{C}(\mathbf{b}) \in \Delta\mathbf{C}(\mathbf{b}^I)$$

The corresponding eigenproblem is

$$[\mathbf{C}(\mathbf{b}^c) + \delta\mathbf{C}(\mathbf{b})]\mathbf{u} = S\mathbf{u} \tag{68}$$

According to the matrix perturbation theory, one can obtain the eigenvalue of Eq. (68)

$$S_k = S_k^c + S_{k1} \qquad S_{k1} = \mathbf{v}_{k0}^T \delta\mathbf{C}(\mathbf{b})\mathbf{u}_{k0} \tag{69}$$

Applying natural interval extension to Eq. (69), one can obtain the interval eigenvalues, i.e.,

$$S_k^I = S_k^c + S_{k1}^I \tag{70}$$

where

$$S_{k1}^I = \mathbf{v}_{k0}^T \Delta\mathbf{C}(b^I)\mathbf{u}_{k0} \tag{71}$$

Substituting Eq. (55) into Eq. (71) yields

$$\begin{aligned} S_{k1}^I &= (\mathbf{v}_{k0})^T \sum_{j=1}^{m} \frac{\partial\mathbf{C}(\mathbf{b}^c)}{\partial b_j}\Delta b_j e_j \mathbf{u}_{k0} \\ &= \sum_{j=1}^{m} (|S_R^j| + i|S_I^j|)e_j \end{aligned} \tag{72}$$

Letting

$$\Delta S_R^{(k)} = \sum_{j=1}^{m} |S_R^j| \qquad \Delta S_I^{(k)} = \sum_{j=1}^{m} |S_I^j| \tag{73}$$

we have

$$S_k^I = S_k^c + S_k^I = S_k^c + \Delta S_R^{(k)} e_j + i\Delta S_J^{(k)} e_j \tag{74}$$

Letting $S_k^I = [\underline{S}_{kR} + i\underline{S}_{kI}, \bar{S}_{kR} + i\bar{S}_{kI}]$, the lower and upper bounds of the real parts and imaginary parts of the complex eigenvalues can be obtained

$$\begin{aligned} \underline{S}_{kR} &= S_{kR}^c - \Delta S_R^{(k)} \\ \bar{S}_{kR} &= S_{kR}^c + \Delta S_R^{(k)} \\ \underline{S}_{kI} &= S_{kI}^c - \Delta S_I^{(k)} \\ \bar{S}_{kI} &= S_{kI}^c + \Delta S_I^{(k)} \end{aligned} \tag{75}$$

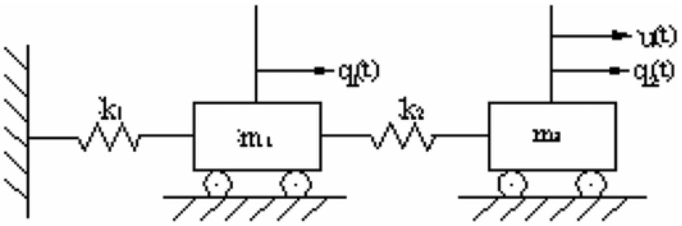

Fig. 1 A vibration control system

The following condition can be used to estimate the stability robustness

$$0 < \left|S_{kR}^{c}\right| - \left|\Delta S_{R}^{(k)}\right| \le \left|\alpha_k\right| \le \left|S_{kR}^{c}\right| + \left|\Delta S_{R}^{(k)}\right| \qquad (k = 1, 2, \ldots, 2n) \tag{76}$$

where α_k is the real part of the kth eigenvalue of uncertain close-loop system. It is obvious that if S_{kR}^{c} $(k = 1, 2, \ldots, 2n)$ are large enough in designing the feedback control law, the stability of the uncertain closed-loop system will be remained.

8. Numerical example

In order to illustrate the application of the present method, two numerical examples are given as follows.

8.1 Example 1

Consider a vibration control system shown in Fig. 1. A control force is imposed on m_2. Assume that the mass coefficients m_1 and m_2 are deterministic, and the stiffness coefficients of springs, k_1 and k_2, have some errors in the manufacturing process. The k_1 and k_2 can be expressed as $k_1^I = (1 + \alpha_1^I)k_0$, $k_2^I = (1 + \alpha_2^I)k_0$, where k_0 is a constant, α_1 and α_2 are interval uncertain parameter. Assume $m_1 = 1$, $m_2 = 2$, and $k_0 = 1$, $\alpha_1^c = \alpha_2^c = 0$, $\alpha_1^I = [-0.03, 0.03]$, $\alpha_2^I = [-0.04, 0.04]$.

The mass matrix is deterministic

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The stiffness matrix of the system with uncertain parameters is $\mathbf{K}^I = \mathbf{K}_0 + \alpha_1^I \mathbf{K}_1 + \alpha_2^I \mathbf{K}_2$
where

$$\mathbf{K}_0 = \begin{bmatrix} 2k_0 & -k_0 \\ -k_0 & k_0 \end{bmatrix}, \quad \mathbf{K}_1 = \begin{bmatrix} k_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} k_0 & -k_0 \\ -k_0 & k_0 \end{bmatrix}$$

Suppose the state vector is

$$\mathbf{x}(t) = [\mathbf{q}_1(t) \;\; \mathbf{q}_2(t) \;\; \dot{\mathbf{q}}_1(t) \;\; \dot{\mathbf{q}}_2(t)]^T$$

Then the state matrix of the system is

$$\mathbf{A}^I = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}^I & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}_0 & \mathbf{0} \end{bmatrix} + \alpha_1^I \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{K}_1 & \mathbf{0} \end{bmatrix} + \alpha_2^I \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{K}_2 & \mathbf{0} \end{bmatrix}$$

$$= \mathbf{A}_0 + \alpha_1^I \mathbf{A}_1 + \alpha_2^I \mathbf{A}_2$$

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}_0 & \mathbf{0} \end{bmatrix}, \quad \mathbf{A}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{K}_1 & \mathbf{0} \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{K}_2 & \mathbf{0} \end{bmatrix}$$

where $\mathbf{A}_0$ is the state matrix with deterministic parameters, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the state sub-matrices corresponding to the uncertain parameters α_1 and α_2, respectively. The eigenvalues of $\mathbf{A}_0$ are $S_{01} = 1.51022i$, $S_{02} = -1.51022i$, $S_{03} = 0.46821i$, $S_{04} = -0.46821i$.

If the frequencies of the system are unchanged, and only the damping of the system is assigned, that is the real parts of the eigenvalues of the system can be assigned as –0.50000. Using Eq. (22), the state feedback gain matrix for the system with deterministic parameters can be obtained

$$\mathbf{G}_0 = [4.6250 \quad -3.0000 \quad 2.0000 \quad -4.000]$$

If $\mathbf{G}_0$ is applied to the actual uncertain system with interval parameters given by $\alpha_i^I = \alpha_i^c + \Delta\alpha e_i$ $(i = 1, 2)$, the lower and upper bounds and the mean values of the eigenvalues of the closed-loop system are obtained and listed in Tables 2, 3. In the Tables, k is the mode number; $|\underline{S}_{kR}|$ is the lower bound of the real part of kth complex eigenvalue; $\underline{S}_{kI}$ the lower bound of the imaginary part; S_{kI}^c and $|S_{kR}^c|$ the middle value of the imaginary part and the real part; $|\overline{S}_{kR}|$ the upper bound of the real part; $\overline{S}_{kI}$ the upper bound of the imaginary part; The results in Tables 2 and 3 show that if $\alpha_1^I = [-0.03, 0.03]$ and $\alpha_{21}^I = [-0.04, 0.04]$, we have $|S_{kR}| > 0$ ($k = 1, 3$). This indicates that the stability condition (76) can be satisfied.

Table 2 The lower and upper bounds of complex eigenvalues ($\alpha_1^I = [-0.03, 0.03]$)

k	$\underline{S_{KR}}$	$\underline{S_{KI}}$	S_{KR}^C	S_{KI}^C	$\overline{S_{KR}}$	$\overline{S_{KI}}$	$\left\lvert\frac{\Delta S_{kR}}{S_{kR}^c}\right\rvert\%$	$\frac{\Delta S_{kI}}{S_{kI}^c}\%$
1	–0.50728	1.50525	–0.50000	1.51022	–0.49272	1.51519	2.9	0.66
3	–0.50728	0.45219	–0.50000	0.46821	–0.49272	0.48423	2.9	6.84

Table 3 The lower and upper bounds of complex eigenvalues ($\alpha_2^I = [-0.04, 0.04]$)

k	$\underline{S_{KR}}$	$\underline{S_{KI}}$	S_{KR}^C	S_{KI}^C	$\overline{S_{KR}}$	$\overline{S_{KI}}$	$\left\lvert\frac{\Delta S_{kR}}{S_{kR}^c}\right\rvert\%$	$\frac{\Delta S_{kI}}{S_{kI}^c}\%$
1	–0.50485	1.48543	–0.50000	1.51022	–0.49515	1.53501	1.94	3.28
3	–0.50485	0.45233	–0.50000	0.46821	–0.49515	0.50508	1.94	11.7

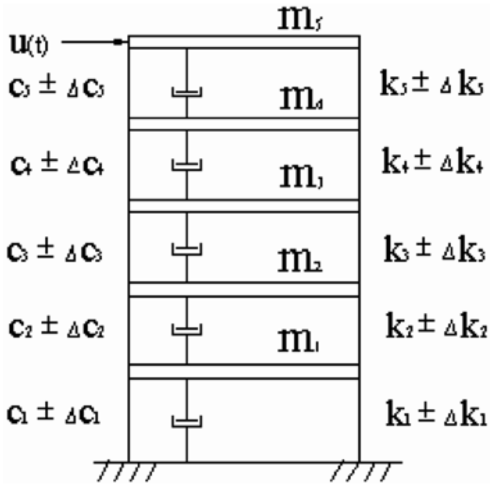

Fig. 2 The frame structure

8.2 Example 2

Consider a vibration control problem of a frame structure shown in Fig. 2. Assume mass (kg), stiffness (N/m) and damping (N/m·s^{-1}) are given as follows:

$m_1 = 29$, $m_2 = 26$, $m_3 = 26$, $m_4 = 24$, $m_5 = 17$,
$k_{10} = 2000$, $k_{20} = 1800$, $k_{30} = 1600$, $k_{40} = 1400$, $k_{50} = 1200$
$c_{10} = 40$, $c_{20} = 40$, $c_{30} = 60$, $c_{40} = 80$, $c_{50} = 80$
$k_i^I = k_{i0} + \Delta k_i^I \quad \Delta k_i^I = \alpha_i^I k_{i0} \quad c_i^I = c_{i0} + \Delta c_i^I \quad \Delta c_i^I = \beta_i^I c_{i0} \quad (i = 1, 2, 3, 4, 5)$

where the mass parameters are assumed to be deterministic; stiffness parameters, k_{i0} and $c_{i0}(i = 1, 2, 3, 4, 5)$ are deterministic, and Δk_i^I and $\Delta c_i^I(i = 1, 2, 3, 4, 5)$ are uncertain parts. The coefficients α_i^I and $\beta_i^I(i = 1, 2, 3, 4, 5)$ are uncertain interval parameters. Assume that a control force $\mathbf{u}(t)$ is input to m_5. Thus the mass matrix is

$$\mathbf{M} = diag(m_1 \ m_2 \ m_3 \ m_4 \ m_5) = diag(29 \ 26 \ 26 \ 24 \ 17)$$

Suppose the state vector is

$$\mathbf{x}(t) = [q_1(t) \ q_2(t) \ q_3(t) \ q_4(t) \ q_5(t) \ \dot{q}_1(t) \ \dot{q}_2(t) \ \dot{q}_3(t) \ \dot{q}_4(t) \ \dot{q}_5(t)]^T$$

Then the state matrix of the system is

$$\mathbf{A}^I = \mathbf{A}_0 + \sum_{i=1}^{5} \alpha_i^I \mathbf{A}_i + \sum_{i=1}^{5} \beta_i^I \mathbf{B}_i$$

where $\mathbf{A}_0$ is the state matrix with deterministic parameters and $\alpha_i^I \mathbf{A}_i$ and $\beta_i^I \mathbf{B}_i(i = 1, 2, 3, 4, 5)$ are the uncertain parts. Assume that the input coefficient matrix $\mathbf{B}$ is deterministic, $\mathbf{B} = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1/17]$.

The eigenvalues of $\mathbf{A}_0$ are

$$
\begin{array}{ll}
S_{01} = -0.0963 + 2.5275i & S_{02} = -0.0963 - 2.5275i \\
S_{03} = -1.0267 + 6.7292i & S_{04} = -1.0267 - 6.7292i \\
S_{05} = -2.3488 + 10.2718i & S_{06} = -2.3488 - 10.2718i \\
S_{07} = -5.0397 + 12.5008i & S_{08} = -5.0397 - 12.5008i \\
S_{09} = -3.1695 + 13.8673i & S_{010} = -3.1695 - 13.8673i
\end{array}
$$

To guarantee the stability of the control system, it is only necessary to impart the eigenvalues larger negative real parts, it is not necessary to alter the frequencies. To this end, the real parts can be assigned. Assume that the assigned eigenvalues are

$$
\begin{array}{ll}
S_1^* = -1.0110 + 2.5275i & S_2^* = -1.0110 - 2.5275i \\
S_3^* = -2.6917 + 6.7292i & S_4^* = -2.6917 - 6.7292i \\
S_5^* = -4.1087 + 10.2718i & S_6^* = -4.1087 - 10.2718i \\
S_7^* = -5.0003 + 12.5008i & S_8^* = -5.0003 - 12.5008i \\
S_9^* = -5.5469 + 13.8673i & S_{10}^* = -5.5469 - 13.8673i
\end{array}
$$

Using Eq. (22) the state feedback gain matrix $\mathbf{G}_0$ for the system with deterministic parameters can be obtained

$$
\mathbf{G}_0 = [4382.7060 \quad -5104.3220 \quad 1720.9740 \quad 1991.4040 \quad -1566.5190 \quad 177.6021 \quad -316.7099 \quad 384.1323 \quad -107.0294 \quad -227.0397]
$$

Table 4 The lower and upper bounds of complex eigenvalues ($\alpha_i^I = [-0.05, 0.05]$, $i = 1, 2, 3, 4, 5$)

k	$\underline{S_{KR}}$	$\underline{S_{KI}}$	S_{KR}^C	S_{KI}^C	$\overline{S_{KR}}$	$\overline{S_{KI}}$	$\left\lvert\frac{\Delta S_{kR}}{S_{kR}^c}\right\rvert\%$	$\frac{\Delta S_{kI}}{S_{kI}^c}\%$
1	−1.01522	2.43268	−1.01099	2.52747	−1.00676	2.62226	0.84	7.50
3	−2.73756	6.47444	−2.69170	6.72924	−2.64583	6.98403	3.41	7.57
5	−4.27110	9.87896	−4.10870	10.27176	−3.94632	10.66459	7.90	7.65
7	−5.23483	11.86659	−5.00031	12.50077	−4.76578	13.13494	9.38	10.0
9	−5.78113	13.18894	−5.54691	13.86728	−5.31270	14.54563	8.44	9.78

Table 5 The lower and upper bounds of complex eigenvalues ($\beta_i^I = [-0.05, 0.05]$, $i = 1, 2, 3, 4, 5$)

k	$\underline{S_{KR}}$	$\underline{S_{KI}}$	S_{KR}^C	S_{KI}^C	$\overline{S_{KR}}$	$\overline{S_{KI}}$	$\left\lvert\frac{\Delta S_{kR}}{S_{kR}^c}\right\rvert\%$	$\frac{\Delta S_{kI}}{S_{kI}^c}\%$
1	−1.06252	2.52710	−1.01099	2.52747	−0.95946	2.52785	10.2	0.0
3	−2.83748	6.71810	−2.69170	6.72924	−2.54592	6.74037	10.8	0.3
5	−4.32880	10.21347	−4.10870	10.27176	−3.88860	10.33007	10.7	1.14
7	−5.26144	12.30299	−5.00031	12.50077	−4.73918	12.69854	10.4	3.17
9	−5.84124	13.75911	−5.54691	13.86728	−5.25258	13.97546	10.8	1.56

If $\mathbf{G}_0$ is applied to the actual uncertain system with interval parameters given by $\alpha_i^I = \Delta\alpha e_i$, $\beta_i^I = \Delta\beta e_i (i = 1, 2, \ldots, 5)$, the closed-loop eigenvalues will have some errors. The lower and upper bounds and the mean value of the eigenvalues of the closed-loop system are obtained and listed in Tables 4, 5. In the Tables, k is the mode number; $|\underline{S}_{kR}|$ is the lower bound of the real part of complex eigenvalue; $\underline{S}_{kI}$ is the lower bound of the imaginary part; S_{kI}^c and $|S_{kR}^c|$ are the middle values of the imaginary part and the real part; $|\overline{S}_{kR}|$ is the upper bound of the real part; $\overline{S}_{kI}$ is the upper bound of the imaginary part. $\left|\dfrac{\Delta S_{kR}}{\Delta S_{kR}^c}\right|$ and $\dfrac{\Delta S_{kI}}{S_{kI}^c}$ are the relative uncertainties of the real and imaginary parts, respectively.

The curves of upper and lower bounds of the 1st eigenvalue are shown in Figs. 3-6 for example 2, where Fig. 3 is the upper and lower bounds of the real part of the 1st eigenvalue obtained by the changes of the stiffness coefficients; Fig. 4 is the upper and lower bounds of the imaginary part of the 1st eigenvalue obtained by the changes of the stiffness coefficients; Figs. 5 and 6 are the corresponding quantities obtained by changes of the damping coefficients, respectively. From Tables 2-5 and Figs. 3-6, it can been seen that the relative uncertainties will be come large as the uncertain parameters, α and β, go up; for example, if $\alpha_i^I = [-0.05, 0.05] (i = 1, 2, 3, 4, 5)$, the max relative uncertainty is 10% at the imaginary part of the 7-th eigenvalue (Table 4) ; if $\beta_i^I = [-0.05, 0.05]$ $(i = 1, 2, 3, 4, 5)$, the max relative uncertainty is 10.8% at the real part of the 3-rd eigenvalue. From Table 5 it is shown that the effects of the damping coefficients on the real parts of eigenvalues are larger than that on the imaginary parts of eigenvalues. And the stability condition (76) can be satistied.

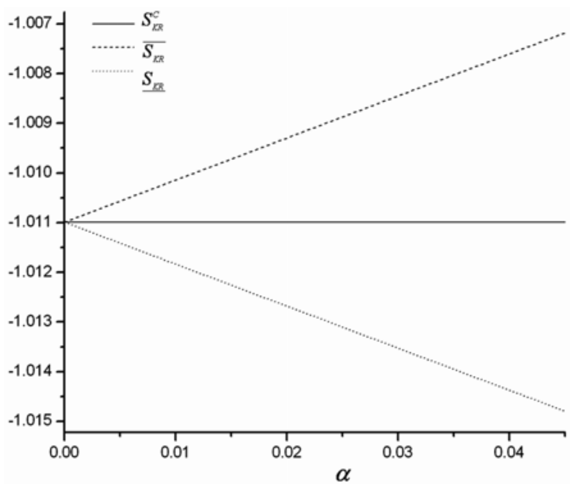

Fig. 3 The upper and lower bounds of the real part of the first eigenvalue obtained by the changes of the stiffness coefficients

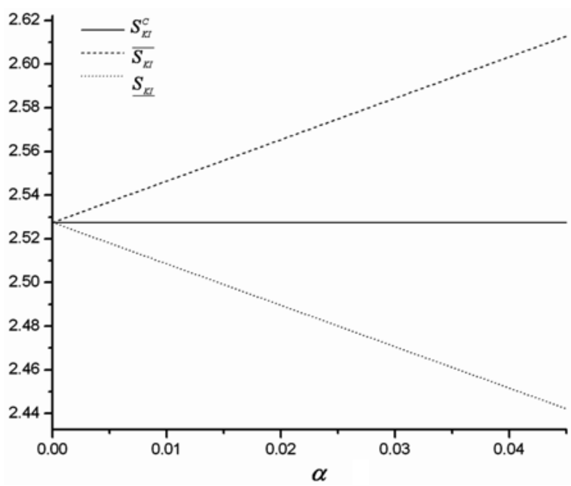

Fig. 4 The upper and lower bounds of the imaginary part of the first eigenvalue obtained by the changes of the stiffness coefficients

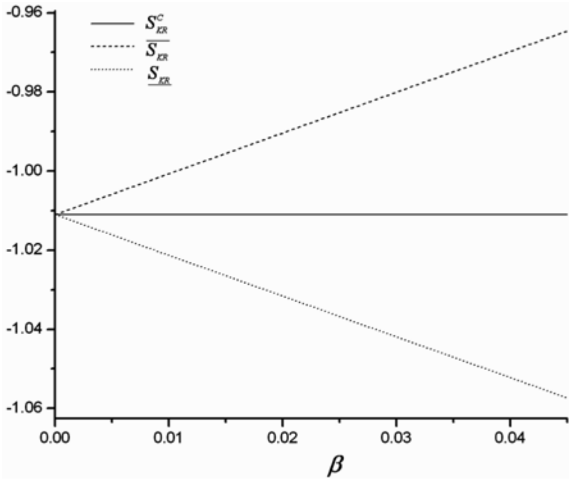

Fig. 5 The upper and lower bounds of the real part of the first eigenvalue obtained by the changes of the damping coefficients

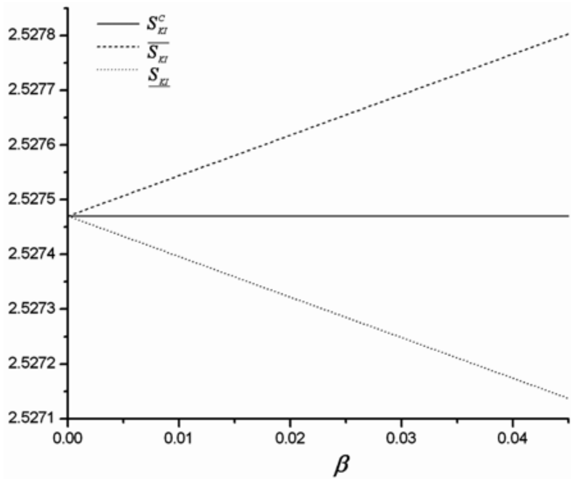

Fig. 6 The upper and lower bounds of the imaginary part of the first eigenvalue obtained by the changes of the damping coefficients

9. Conclusions

The vibration control problem of structures with uncertain parameters was discussed in this paper. The control problem was approximated with the corresponding deterministic one. The uncertain parameters are modeled to be a interval set rather than a probabilistic set. This does not require the probabilistic distribution descriptions of the uncertain parameters. The method for building the state matrices with system physical parameters was presented. With the matrix perturbation of the complex eigenvalues and the complex circle plate extension in interval analysis, a new method for evaluating the lower and upper bounds of eigenvalues of closed-loop systems with uncertain parameters has been presented. The results can be used to estimate the stability robustness of the uncertain controlled systems, two numerical examples are given to illustrate the applications. From the numerial results it can be seen that it the assigned real parts of eigenvalues of the closed-loop system are large enough, the stability robustness of the uncertain close-loop system will be remained.

It should be noted that the distributed structures are infinite-dimensional systems, the present method for discrete systems can not be directly applied to distributed systems. However, if we consider either the classical Rayleigh-Ritz method or the finite element method, the discrete system can be obtained. At this point, the present method for discrete systems can be used to deal with the control problem of the distributed structure.

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