Structural Engineering and Mechanics, Vol. 20, No. 6 (2005) 713-726

DOI: http://dx.doi.org/10.12989/sem.2005.20.6.713

An improved interval analysis method for uncertain structures

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(Received August 31, 2004, Accepted April 26, 2005)

Abstract. Based on the improved first order Taylor interval expansion, a new interval analysis method for the static or dynamic response of the structures with interval parameters is presented. In the improved first order Taylor interval expansion, the first order derivative terms of the function are also considered to be intervals. Combining the improved first order Taylor series expansion and the interval extension of function, the new interval analysis method is derived. The present method is implemented for a continuous beam and a frame structure. The numerical results show that the method is more accurate than the one based on the conventional first order Taylor expansion.

Key words: interval parameter; interval parameter structure; interval analysis; interval extension of function.

1. Introduction

The numerical analysis of structural behavior is usually performed for specified structural parameters and loading conditions. However, in most practical situations, the structural parameters and loads are uncertain, for example, there may be measurement inaccuracy or errors in the manufacturing process. Therefore, the concept of uncertainty plays an important role in the investigation of various engineering problems, and it is very necessary to predict the errors resulted from the above mentioned uncertainties in structural analysis and design. The most common approach to problems of uncertainty is to model the structural parameters as random variables or fields. Under the circumstances, all information about the structural parameters is provided by the joint probability density function (or distribution function) of the structural parameters. Unfortunately, probabilistic model is not the only one that can be used to described the uncertainty, and uncertainty is not tantamount to randomness. Indeed, probabilistic methods are not able to deliver reliable results at the required precision without sufficient experimental data to validate the

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assumptions made regarding the joint probability densities of the random variables or functions involved. In addition to the probabilistic models, convex models have been used for modeling uncertainty phenomena in a wide range of engineering applications (Ben-Haim and Elishakoff 1990, Qiu 2003).

Since the mid-1960s, a new method called the interval analysis has appeared. Moore (1979) and his co-workers, Alefeld and Herzberger (1983) have done the pioneering work. Mathematically, linear interval equations, nonlinear interval equations and interval eigenvalue problems have been resolved partly. But because of the complexity of the algorithms, it is difficult to apply these results to deal with practical engineering problems. Recently, the interval analysis method has been used to deal with the static displacement, eigenvalue, and dynamic response analysis of the uncertain structures with interval parameters (Qiu and Wang 2003, Chen and Lian 2002, Chen and Yang 2000). However, these results are all based on the conventional first order Taylor expansion. In fact, the exact interval values of the structural behavior don't lie in the intervals obtained by Chen, Lian and Yang, especially when the interval relative uncertainties of the structural interval parameters are fairly large. Hence, it is necessary to develop a more accurate method to solve the uncertain problems of structures with interval parameters. This paper presents an improved interval analysis method based on the improved first order Taylor expansion.

The paper will start with a brief review of the interval analysis technique based on the first order Taylor expansion presented by Chen and Yang (2000), and then discuss the improved first order Taylor interval expansion. Using the improved first order Taylor series expansion and interval extension of function, the new interval analysis method can be obtained. In the improved first order Taylor expansion, the first order derivative terms of the function are also considered to be intervals so that the exact interval values of the function with interval parameters are included in the approximate intervals obtained by the new interval analysis method. The present method is implemented for a continuous beam and a frame structure. The numerical results are compared with those obtained by the method presented by Chen and Yang.

2. The Taylor expansion of the interval functions

2.1 The first order Taylor expansion of the interval functions

It's well known that typical structural analysis and design problems resort to Finite Element analysis, in which the structural behaviors might not be analytic. So it is difficult to get the exact interval solutions of the structural behaviors. We can resort to the first order Taylor expansion to obtain the rational approximation of a complex function and then apply the natural interval extension to the rational approximation to get its interval solutions. Thus the rational approximation of a complex function is a linear function of the variables, and each variable appears only once, so the interval solution of the rational approximation is unique (Hansen 1992).

In general, an interval-valued function can be described as

$$f(\mathbf{X}^I) = f(X_1^I, \ldots, X_n^I) \tag{1}$$

where $\mathbf{X}^I = (X_1^I, \ldots, X_n^I)^T$ is the interval parameter vector of the function.

Using the first order Taylor expansion to expand $f(\mathbf{X}^I)$ about the mid-point of the interval vector $\mathbf{X}^I$, one can obtain

$$\begin{aligned}
f(\mathbf{X}^I) &\approx f(\mathbf{X}^0) + (\mathbf{X}^I - \mathbf{X}^0)^T g(x_1^0, \dots, x_n^0) \\
&= f(\mathbf{X}^0) + \sum_{i=1}^{n} \frac{\partial f(\mathbf{X}^0)}{\partial x_i}(X_i^I - x_i^0) \\
&= f(\mathbf{X}^0) + \sum_{i=1}^{n} \frac{\partial f(\mathbf{X}^0)}{\partial x_i}\Delta X_i e_\Delta \\
&= f(\mathbf{X}^0) + \sum_{i=1}^{n} \left| \frac{\partial f(\mathbf{X}^0)}{\partial x_i}\Delta X_i \right| e_\Delta \\
&= f(\mathbf{X}^0) + \left(\sum_{i=1}^{n} \left| \frac{\partial f(\mathbf{X}^0)}{\partial x_i}\Delta X_i \right| \right) e_\Delta \\
&= [f(\mathbf{X}^0), f(\mathbf{X}^0)] + \left[-\sum_{i=1}^{n} \left| \frac{\partial f(\mathbf{X}^0)}{\partial x_i}\Delta X_i \right|, \sum_{i=1}^{n} \left| \frac{\partial f(\mathbf{X}^0)}{\partial x_i}\Delta X_i \right| \right] \\
&= \left[f(\mathbf{X}^0) - \sum_{i=1}^{n} \left| \frac{\partial f(\mathbf{X}^0)}{\partial x_i}\Delta X_i \right|, f(\mathbf{X}^0) + \sum_{i=1}^{n} \left| \frac{\partial f(\mathbf{X}^0)}{\partial x_i}\Delta X_i \right| \right]
\end{aligned} \tag{2}$$

where $g(x_1^0, \dots, x_n^0) = \left[\frac{\partial f(\mathbf{X}^0)}{\partial x_1}, \dots, \frac{\partial f(\mathbf{X}^0)}{\partial x_n} \right]^T$ is the gradient of f, $\mathbf{X}^I = (X_1^I, \dots, X_n^I)^T$, $\mathbf{X}^0 = (x_1^0, \dots, x_n^0)^T$, $e_\Delta = [-1, 1]$.

Take a function as a simple example, that is, $g(x,a) = ax/(x-1)$, $x \neq 1$, $a \neq 0$. The exact interval solutions for different interval variables are easy to calculated. Now we use the first order Taylor series expansion to expand the function about the mid-points of the interval variables to get the approximation of the interval value. In Table 1, the comparison is given for the interval value of the exact solutions and the approximate solutions for different interval variables, where δ is the relative uncertainty of an interval variable which is defined early in this paper. Suppose the mid-point of the exact solution is denoted as f^C. Similarly, denote the mid-point of the approximate solution as g^C. The error of the mid-point is the value of $|(g^C - f^C)/f^C|$.

Table 1 Comparison for the interval value of $g(x, a)$ based on the first order Taylor expansion

Interval variables	δ	Exact solution	Approximate solution	Error of mid-point
$x^I = [2.4, 2.6]$ $a^I = [0.4, 0.6]$	0.04 0.2	f^I: [0.65, 1.029] f^C: 0.8393	g^I: [0.64, 1.02] g^C: 0.8333	0.71%
$x^I = [2.3, 2.7]$ $a^I = [0.3, 0.7]$	0.08 0.4	f^I: [0.476, 1.238] f^C: 0.8575	g^I: [0.46, 1.21] g^C: 0.8333	2.82%
$x^I = [2.2, 2.8]$ $a^I = [0.2, 0.8]$	0.12 0.6	f^I: [0.311, 1.467] f^C: 0.8889	g^I: [0.27, 1.4] g^C: 0.8333	6.25%

It should be noted that in the first order Taylor expansion Eq. (2), the gradient of f is a constant $g(x_1^0,\ldots,x_n^0)$ which is the gradient at mid-point of the interval vector $\mathbf{X}^I$. From Table 1, it can be seen that the interval values of the function obtained by this method do not actually include the exact interval values of the function. To improve the accuracy of the interval value of the function, the gradient of f can be also considered to be interval.

2.2 The improved first order Taylor expansion of the interval functions

Let f be a function of a single variable. Expanding $f(x)$ about x^0 and neglecting the higher-order terms, one can obtain

$$f(x) = f(x^0) + (x - x^0)f'(\xi) \tag{3}$$

ξ lies between x^0 and x. Hence, if x^0 and x are in an interval X^I, ξ must be in X^I also, therefore $f'(\xi) \in f'(X^I)$,

$$(x - x^0)f'(\xi) \in (x - x^0)f'(X^I), \quad \forall x^0 \in X^I, \forall x \in X^I \tag{4}$$

and

$$f(x) \in f(x^0) + (x - x^0)f'(X^I) \subset f(x^0) + (X^I - x^0)f'(X^I) \tag{5}$$

Since this relation holds for all $x \in X^I$, therefore

$$f(X^I) \approx f(x^0) + (X^I - x^0)f'(X^I) \tag{6}$$

Now suppose f is a function of n variables. Let $\mathbf{X}^0$ and $\mathbf{X}$ be vectors of n components and α be a scalar. Then, $f[\mathbf{X}^0 + \alpha(\mathbf{X} - \mathbf{X}^0)]$ can be regarded as a function of the single variable α, using Eq. (3) and expanding $f[\mathbf{X}^0 + \alpha(\mathbf{X} - \mathbf{X}^0)]$ about $\alpha = 0$ and setting $\alpha = 1$, one can have

$$f(\mathbf{X}) = f(\mathbf{X}^0) + (\mathbf{X} - \mathbf{X}^0)^T g[\mathbf{X}^0 + \xi(\mathbf{X} - \mathbf{X}^0)] \tag{7}$$

where $0 \le \xi \le 1$, g is the gradient of f. If X_i^I is an interval containing x_i^0 and $x_i (i = 1, \ldots, n)$, then $x_i^0 + \xi(x_i - x_i^0) \in X_i^I$, therefore

$$f(\mathbf{X}) \in f(\mathbf{X}^0) + (\mathbf{X} - \mathbf{X}^0)^T g(X_1^I, \ldots, X_n^I) \subset f(\mathbf{X}^0) + (\mathbf{X}^I - \mathbf{X}^0)^T g(X_1^I, \ldots, X_n^I) \tag{8}$$

Since this relation holds for all $x_i \in X_i^I (i = 1, \ldots, n)$, one can have

$$f(\mathbf{X}) \approx f(\mathbf{X}^0) + (\mathbf{X}^I - \mathbf{X}^0)^T g(X_1^I, \ldots, X_n^I) \tag{9}$$

where $\mathbf{X}^I = (X_1^I, \ldots, X_n^I)^T$, $\mathbf{X}^0 = (x_1^0, \ldots, x_n^0)^T$.

Note that all the arguments of g are intervals. In the following, a method due to Hansen (1992) which shows that some of the arguments can be replaced by real quantities will be described. This means that a sharper bound on $f(\mathbf{X}^I)$ can be obtained, in general.

For simplicity, let $n = 2$. First consider $f(x_1, x_2)$ as a function of x_2 only. Expanding $f(x_1, x_2)$ about x_2^0, one can obtain

$$f(x_1, x_2) = f(x_1, x_2^0) + (x_2 - x_2^0)g_2(x_1, \xi_2) \tag{10}$$

then expanding $f(x_1, x_2^0)$ about x_1^0 as a function of x_1, one has

$$f(x_1, x_2^0) = f(x_1^0, x_2^0) + (x_1 - x_1^0)g_1(\xi_1, x_2^0) \tag{11}$$

Combining Eq. (10) and Eq. (11), one can have

$$f(x_1, x_2) = f(x_1^0, x_2^0) + (x_1 - x_1^0)g_1(\xi_1, x_2^0) + (x_2 - x_2^0)g_2(x_1, \xi_2) \tag{12}$$

If $x_i^0 \in X_i^I$ and $x_i \in X_i^I$, then $\xi_1 \in X_i^I (i = 1, 2)$.

Replacing x in the arguments of the components of g by X^I and $\xi_i (i = 1, 2)$ by the bounding interval X_i^I, one can obtain

$$f(x_1, x_2) \in f(x_1^0, x_2^0) + (x_1 - x_1^0)g_1(X_i^I, x_2^0) + (x_2 - x_2^0)g_2(X_1^I, X_2^I) \tag{13}$$

Since this relation holds for all $x_i \in X_i^I (i = 1, 2)$, one can have

$$f(X_1^I, X_2^I) \approx f(x_1^0, x_2^0) + (X_1^I - x_1^0)g_1(X_1^I, x_2^0) + (X_2^I - x_2^0)g_2(X_1^I, X_2^I) \tag{14}$$

For n variables, the corresponding expression is

$$f(\mathbf{X}^I) \approx f(\mathbf{X}^0) + \sum_{i=1}^{n} (X_i^I - x_i^0)g_i(X_1^I, \ldots, X_i^I, x_{i+1}^0, \ldots, x_n^0) \tag{15}$$

where $\mathbf{X}^I = (X_1^I, \ldots, X_n^I)^T$, $\mathbf{X}^0 = (x_1^0, \ldots, x_n^0)^T$.

Note that all the arguments of all components of g are intervals in Eq. (9), while in Eq. (15), some of them are real. Thus, the bound obtained by Eq. (15) is sharper than that obtained by Eq. (9).

From Eq. (7), a complex function can be approximated by a rational function

$$f(\mathbf{X}) = f(\mathbf{X}^0) + (\mathbf{X} - \mathbf{X}^0)^T g[\mathbf{X}^0 + \xi(\mathbf{X} - \mathbf{X}^0)] \tag{16}$$

that is,

$$f(\mathbf{X}) = f(\mathbf{X}^0) + \sum_{i=1}^{n} \frac{\partial f[\mathbf{X}^0 + \xi(\mathbf{X} - \mathbf{X}^0)]}{\partial x_i}(x_i - x_i^0) \tag{17}$$

where $\mathbf{X}^0 = (x_1^0, \ldots, x_n^0)^T$.

Thus, the derivatives of $f(\mathbf{X})$ with respect to x_i are

$$\frac{\partial f(\mathbf{X})}{\partial x_i} \approx \frac{f(\mathbf{X}^0)}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial^2 f(\mathbf{X}^0)}{\partial x_i \partial x_j}(x_j - x_j^0) \quad (i = 1, \ldots, n) \tag{18}$$

Using the interval extension of function, Eq. (18) becomes

$$\frac{\partial f(\mathbf{X}^I)}{\partial x_i} \approx \frac{f(\mathbf{X}^0)}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial^2 f(\mathbf{X}^0)}{\partial x_i \partial x_j}(X_j^I - x_j^0) \quad (i = 1, \ldots, n) \tag{19}$$

From Eq. (15) and Eq. (19), one can have

$$f(\mathbf{X}^I) \approx f(\mathbf{X}^0) + \sum_{i=1}^{n} (X_i^I - x_i^0)\left[\frac{\partial f(\mathbf{X}^0)}{\partial x_i} + \sum_{j=1}^{i} \frac{\partial^2 f(\mathbf{X}^0)}{\partial x_i \partial x_j}(X_j^I - x_j^0)\right] \tag{20}$$

For simplicity, let $n = 2$, one can obtain

$$\begin{aligned}
f(X_1^I, X_2^I) &\approx f(x_1^0, x_2^0) + (X_1^I - x_1^0)\left[\frac{\partial f(\mathbf{X}^0)}{\partial x_1} + \frac{\partial^2 f(\mathbf{X}^0)}{\partial x_1^2}(X_1^I - x_1^0)\right] + \\
&\quad (X_2^I - x_2^0)\left[\frac{\partial f(\mathbf{X}^0)}{\partial x_2} + \frac{\partial^2 f(\mathbf{X}^0)}{\partial x_2 \partial x_1}(X_1^I - x_1^0) + \frac{\partial^2 f(\mathbf{X}^0)}{\partial x_2^2}(X_2^I - x_2^0)\right] \\
&= f(x_1^0, x_2^0) + (\Delta X_1 e_\Delta)\left[\frac{\partial f(\mathbf{X}^0)}{\partial x_1} + \frac{\partial^2 f(\mathbf{X}^0)}{\partial x_1^2}(\Delta X_1 e_\Delta)\right] + \\
&\quad (\Delta X_2 e_\Delta)\left[\frac{\partial f(\mathbf{X}^0)}{\partial x_2} + \frac{\partial^2 f(\mathbf{X}^0)}{\partial x_2 \partial x_1}(\Delta X_1 e_\Delta) + \frac{\partial^2 f(\mathbf{X}^0)}{\partial x_2^2}(\Delta X_2 e_\Delta)\right] \\
&= f(x_1^0, x_2^0) + \left|\frac{\partial f(\mathbf{X}^0)}{\partial x_1}\Delta X_1\right| e_\Delta + \left|\frac{\partial^2 f(\mathbf{X}^0)}{\partial x_1^2}\Delta X_1 \Delta X_1\right| e_\Delta + \\
&\quad \left|\frac{\partial f(\mathbf{X}^0)}{\partial x_2}\Delta X_2\right| e_\Delta + \left|\frac{\partial^2 f(\mathbf{X}^0)}{\partial x_2 \partial x_1}\Delta X_1 \Delta X_2\right| e_\Delta + \left|\frac{\partial^2 f(\mathbf{X}^0)}{\partial x_2^2}\Delta X_2 \Delta X_2\right| e_\Delta \\
&= f(x_1^0, x_2^0) + \left[\left|\frac{\partial f(\mathbf{X}^0)}{\partial x_1}\Delta X_1\right| + \left|\frac{\partial^2 f(\mathbf{X}^0)}{\partial x_1^2}\Delta X_1 \Delta X_1\right| + \right. \\
&\quad \left.\left|\frac{\partial f(\mathbf{X}^0)}{\partial x_2}\Delta X_2\right| + \left|\frac{\partial^2 f(\mathbf{X}^0)}{\partial x_2 \partial x_1}\Delta X_1 \Delta X_2\right| + \left|\frac{\partial^2 f(\mathbf{X}^0)}{\partial x_2^2}\Delta X_2 \Delta X_2\right|\right] e_\Delta \\
&= f(x_1^0, x_2^0) + \Delta f e_\Delta \\
&= [f(x_1^0, x_2^0) - \Delta f, f(x_1^0, x_2^0) + \Delta f]
\end{aligned} \tag{21}$$

where $\Delta f = \left|\frac{\partial f(\mathbf{X}^0)}{\partial x_1}\Delta X_1\right| + \left|\frac{\partial^2 f(\mathbf{X}^0)}{\partial x_1^2}\Delta X_1 \Delta X_1\right| + \left|\frac{\partial f(\mathbf{X}^0)}{\partial x_2}\Delta X_2\right| + \left|\frac{\partial^2 f(\mathbf{X}^0)}{\partial x_2 \partial x_1}\Delta X_1 \Delta X_2\right| + \left|\frac{\partial^2 f(\mathbf{X}^0)}{\partial x_2^2}\Delta X_2 \Delta X_2\right|$.

Thus the lower and upper bounds of the function $f(X_1^I, X_2^I)$ will be

$$\underline{f}(X_1^I, X_2^I) = f(x_1^0, x_2^0) - \Delta f, \quad \bar{f}(X_1^I, X_2^I) = f(x_1^0, x_2^0) + \Delta f \tag{22}$$

Table 2 Comparison for the interval value of $g(x, a)$ based on the improved first order Taylor expansion

Interval variables	δ	Exact solution	Approximate solution	Error of mid-point
$x^I = [2.4, 2.6]$ $a^I = [0.4, 0.6]$	0.04 0.2	f^I: [0.65, 1.029] f^C: 0.8393	g^I: [0.637, 1.0296] g^C: 0.8333	0.71%
$x^I = [2.3, 2.7]$ $a^I = [0.3, 0.7]$	0.08 0.4	f^I: [0.476, 1.238] f^C: 0.8575	g^I: [0.426, 1.2407] g^C: 0.8333	2.82%
$x^I = [2.2, 2.8]$ $a^I = [0.2, 0.8]$	0.12 0.6	f^I: [0.311, 1.467] f^C: 0.8889	g^I: [0.2, 1.4667] g^C: 0.8333	6.25%

Take the same function considered early in Sec.2.1 as a simple example, that is $g(x, a) = ax/(x-1)$, $x \neq 1$, $a \neq 0$. In Table 2, the comparison is given for the interval value of the exact solution and the approximate solution based on the improved Taylor expansion for different interval variables.

From Table 2, it can be seen that the interval values of the function obtained by the improved method include the exact interval values of the function, and the error of the mid-point goes up as the relative uncertainties of the interval variables increase. In fact, the relative uncertainties of the interval variables are small in practical engineering problems, so the approximate approach based on the improved first order Taylor expansion is more accurate that that obtained by the original one.

3. Numerical examples

In order to explain the concepts and the method presented, two examples are given as follows.

3.1 Example 1

Fig. 1 is a continuous beam which is modeled with 4 nodes and 3 elements. Suppose the height and width of the cross-section of all beam elements are $H^C = 0.3$ m and $B^C = 0.2$ m, respectively. Young's modulus of the first element is $E_1^C = 5 \times 10^6$ kN/m^2 while in elements 2 and 3, the Young's moduli are $E_2^C = E_3^C = 10^7$ kN/m^2. The results for the interval static displacements and their uncertainties are listed in Tables 3-6 where i is the node number; j is the displacement number; U_{ij}

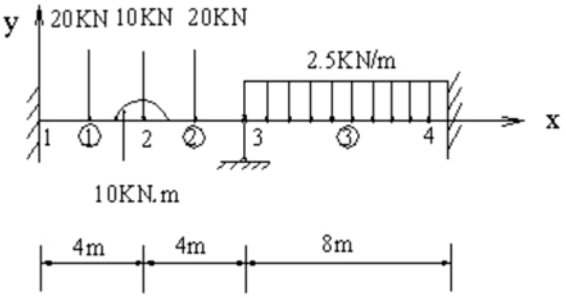

Fig. 1 A continuous beam system

Table 3 Comparison of displacements and uncertainties obtained by the present method and the method presented by Chen ($\Delta E_i = (2/1000)E_i^C$)

i	j	Exact values	U_{ij}^C (Chen's method)	ΔU_{ij} (Chen's method)	U_{ij}^C (present)	ΔU_{ij} (present)
2	2	−0.36578E-01	−0.35743E-01	0.21976E-03	-0.35743E-01	0.41573E-03
2	3	0.35634E-02	0.35273E-02	0.40567E-04	0.35273E-02	0.62577E-04
3	3	0.71434E-02	0.72663E-02	0.94700E-04	0.72663E-02	1.14302E-04

Table 4 Comparison of displacements and uncertainties obtained by the present method and the method presented by Chen ($\Delta B_i = (2/1000)B_i^C$)

i	j	Exact values	U_{ij}^C (Chen's method)	ΔU_{ij} (Chen's method)	U_{ij}^C (present)	ΔU_{ij} (present)
2	2	−0.36578E-01	−0.35743E-01	0.21976E-03	-0.35743E-01	0.40867E-03
2	3	0.35634E-02	0.35273E-02	0.40567E-04	0.35273E-02	0.59688E-04
3	3	0.71434E-02	0.72663E-02	0.94700E-04	0.72663E-02	1.08776E-04

Table 5 Comparison of displacements and uncertainties obtained by the present method and the method presented by Chen ($\Delta H_i = (2/1000)H_i^C$)

i	j	Exact values	U_{ij}^C (Chen's method)	ΔU_{ij} (Chen's method)	U_{ij}^C (present)	ΔU_{ij} (present)
2	2	−0.36578E-01	−0.35743E-01	0.66061E-03	−0.35743E-01	0.87573E-03
2	3	0.35634E-02	0.35273E-02	0.12194E-03	0.35273E-02	0.32678E-03
3	3	0.71434E-02	0.72663E-02	0.28467E-03	0.72663E-02	0.49302E-03

Table 6 Comparison of displacements and uncertainties obtained by the present method and the method presented by Chen ($\Delta E_i = (2/1000)E_i^C$, $\Delta B_i = (2/1000)B_i^C$, $\Delta H_i = (2/1000)H_i^C$)

i	j	Exact values	U_{ij}^C (Chen's method)	ΔU_{ij} (Chen's method)	U_{ij}^C (present)	ΔU_{ij} (present)
2	2	−0.36578E-01	−0.35743E-01	0.11032E-02	−0.35743E-01	0.21374E-02
2	3	0.35634E-02	0.35273E-02	0.20365E-03	0.35273E-02	0.32574E-03
3	3	0.71434E-02	0.72663E-02	0.47540E-03	0.72663E-02	0.56332E-03

is the jth displacement of the ith node; and ΔU_{ij} is the uncertainty of the jth displacement of the ith node. For comparison, the results obtained by the method presented by Chen and Yang are also listed in the Tables.

From Tables 3-6, it can be seen that the displacement uncertainties obtained by the improved method are a little larger than those obtained by the one presented by Chen and Yang, but the approximate displacement intervals obtained by the improved method include the exact displacement intervals. In many cases, the structural parameter errors or uncertainties are small, but their integration will yield a larger error, which can be seen from Table 6. Also, one can see that the nonlinear parameters have more effect on the results than the linear ones. So, we suggest that the nonlinear parameter errors should be minimized during the design or manufacture process.

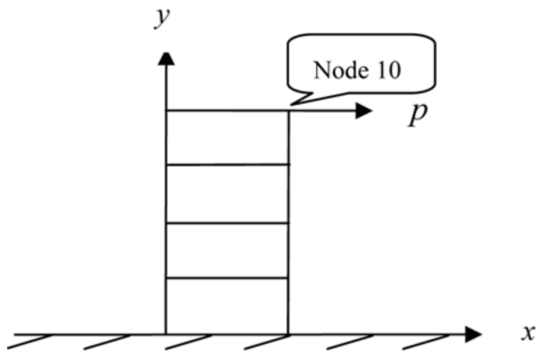

Fig. 2 A frame structure

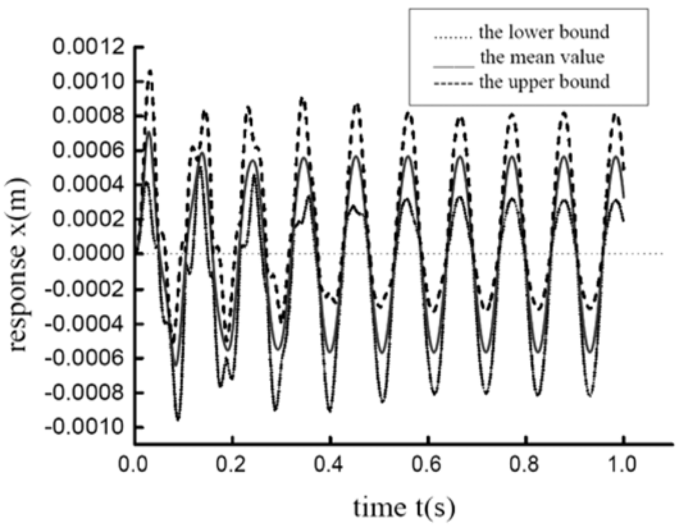

Fig. 3 The interval response for example 2

3.2 Example 2

Fig. 2 shows a frame structure with 10 nodes and 12 elements. Suppose the sine excitation with frequency ($\omega = 100\ \mathrm{s}^{-1}$) is applied to node 10 along the x-positive direction and the amplitude of the load is 3000 N. Suppose that Young's modulus of all beam elements is $E = 2.1 \times 10^{11}\ \mathrm{N/m^2}$, mass density of all beam elements is $\rho = 7800\ \mathrm{kg/m^3}$. The width and height of all beam elements are $B^C = 5.8 \times 10^{-2}$ m and $H^C = 7.83 \times 10^{-2}$ m, respectively; the initial conditions are $\mathbf{x} = 0$ and $\mathbf{x} = 0$.

When the parameters of B and H are interval parameters, the interval response based on the improved analysis method at $\Delta B = 5\% B^C$ and $\Delta H = 5\% H^C$ is given in Fig. 3, which is similar for other interval parameters. For comparison, some results obtained by the present method and the method presented by Chen and Yang are listed in Table 7.

Table 7 Comparison of dynamic responses and uncertainties of node 10 at x direction obtained by the present method and the method presented by Chen ($\Delta B_i = (5/100)B_i^C$, $\Delta H_i = (5/100)H_i^C$)

t (s)	Exact values	$x^C(t)$ (Chen's method)	$\Delta x^C(t)$ (Chen's method)	$x^C(t)$ (present)	$\Delta x^C(t)$ (present)
0.413655	0.000575	0.000565	0.000259	0.000565	0.000382
0.414764	0.000561	0.000559	0.000266	0.000559	0.000401
0.415872	0.000536	0.000549	0.000269	0.000549	0.000402
0.416981	0.000537	0.000535	0.000269	0.000535	0.000402
0.418089	0.000509	0.000516	0.000265	0.000516	0.000394
0.419197	0.000487	0.000493	0.000258	0.000493	0.000385
0.420306	0.000474	0.000466	0.000230	0.000466	0.000376
0.421414	0.000435	0.000436	0.000212	0.000436	0.000364
0.422523	0.000411	0.000402	0.000189	0.000402	0.000342
0.423631	0.000376	0.000365	0.000165	0.000365	0.000284
0.424739	0.000335	0.000325	0.000138	0.000325	0.000265
0.425848	0.000287	0.000283	0.000109	0.000283	0.000241
0.426956	0.000250	0.000239	0.000082	0.000239	0.000235
0.428065	0.000195	0.000192	0.000052	0.000192	0.000187
0.429173	0.000156	0.000145	2.68E-05	0.000145	0.000103
0.430281	9.65E-05	9.62E-05	9.80E-06	9.62E-05	4.12E-05
0.431390	4.68E-05	4.69E-05	2.75E-05	4.69E-05	1.50E-05
0.432498	−2.9E-06	−2.8E-06	5.04E-05	−2.8E-06	4.15E-05
0.433607	−5.3E-05	−5.2E-05	0.00007	−5.2E-05	8.02E-05
0.434715	−0.00011	−0.00010	0.00009	−0.00010	0.000120
0.435823	−0.00016	−0.00015	0.00011	−0.00015	0.000150
0.436932	−0.00022	−0.00020	0.00012	−0.00020	0.000180
0.438040	−0.00023	−0.00024	0.00014	−0.00024	0.000190
0.439149	−0.00025	−0.00029	0.00015	−0.00029	0.000210
0.440257	−0.00030	−0.00033	0.00016	−0.00033	0.000260
0.441365	−0.00035	−0.00037	0.00016	−0.00037	0.000270
0.442474	−0.00042	−0.00040	0.00018	−0.00040	0.000270
0.443582	−0.00046	−0.00044	0.00019	−0.00044	0.000310
0.444691	−0.00043	−0.00047	0.00019	−0.00047	0.000310
0.445799	−0.00045	−0.00049	0.00021	−0.00049	0.000320
0.446907	−0.00050	−0.00052	0.00021	−0.00052	0.000350
0.448016	−0.00051	−0.00053	0.00022	−0.00053	0.000350
0.449124	−0.00052	−0.00055	0.00022	−0.00055	0.000360
0.450233	−0.00053	−0.00056	0.00023	−0.00056	0.000380
0.451341	−0.00053	−0.00056	0.00023	−0.00056	0.000380
0.452523	−0.00054	−0.00057	0.00025	−0.00057	0.000410

4. Conclusions

In this paper, a new interval analysis method based on the improved first order Taylor interval expansion is proposed for the response of structures with interval parameters. The improved first order Taylor interval expansion is developed in which the first order derivative terms of the function are considered to be intervals. It can be seen that, using the improved method, one can obtain more accurate interval response than that obtained by the method based on the conventional first order Taylor expansion. The technique in this paper is based on Taylor series expansion, therefore the conditions under which the technique may be applied can be explained as: 1) The structural response can be approximated by the Taylor series expansion; 2) The ranges of the structural parameters can be obtained and the interval uncertainties are small compared with the mean values of the interval parameters. The most typical numerical examples are given to illustrate the validity of the proposed technique, and the method can also be applied to more complex problems subject to the conditions given above. If the interval relative uncertainties of the interval parameters are fairly large, in order to obtain higher computational accuracy, the second order Taylor expansion should be considered.

Acknowledgements

This work is supported by Jiangsu Planned Projects for Postdoctoral Research Funds and Innovation Funds of Scientific Research of Nanjing University of Aeronautics and Astronautics.

References

Alefeld, G. and Herzberger, J. (1983), *Introductions to Interval Computations*, Academic Press, New York.

Ben-Haim, Y. and Elishakoff, I. (1990), *Convex Models of Uncertainty in Applied Mechanics*, Elsevier Science Publishers, Amsterdam.

Chen, S.H., Lian H.D. and Yang, X.W. (2002), "Interval displacement analysis for structures with interval parameters", *Int. J. Numer. Meth. Eng.*, **53**(2), 393-407.

Chen, S.H. and Yang, X.W. (2000), "Interval finite element method for beam structures", *Finite Element in Analysis and Design,* **34**(1), 75-88.

Chen, S.H. and Lian H.D. (2002), "Dynamic response analysis for structures with interval parameters", *Struct. Eng. Mech.*, **13**(3), 299-312.

Hansen, E. (1992), *Global Optimization Using Interval Analysis*, Marcel Dekker, New York.

Moore, R.E. (1979), *Methods and Applications of Interval Analysis*, SIAM, Philadelphia.

Qiu, Z.P. and Wang, X.J. (2003), "Comparison of dynamic response of structures with uncertain-but-bounded parameters using non-probabilistic interval analysis method and probabilistic approach", *Int. J. Solids Struct.*, **40**, 5423-5439.

Qiu, Z.P. (2003), "Comparison of static response of structures using convex models and interval analysis method", *Int. J. Numer. Meth. Eng.*, **56**(12), 1735-1753.

Appendix. Basic interval mathematics

In interval mathematics, the errors or uncertainties are always denoted by intervals. So it is necessary to introduce some results in interval analysis (Moore 1979, Hansen 1992).

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T$ be a structural parameter vector with bound errors or uncertainties, where

$$\alpha_i \in \alpha_i^I = [\alpha_i^C - \Delta\alpha_i, \alpha_i^C + \Delta\alpha_i]$$

then

$$\alpha \in \alpha^I = [\alpha^C - \Delta\alpha, \alpha^C + \Delta\alpha]$$

where

$$\alpha^C = (\alpha_1^C, \alpha_2^C, \ldots, \alpha_m^C)^T$$

and

$$\Delta\alpha = (\Delta\alpha_1, \Delta\alpha_2, \ldots, \Delta\alpha_m)^T$$

In interval mathematics, a subset of real numbers R of the form $[a_1, a_2] = \{t | a_1 \le t \le a_2, a_1, a_2 \in R\}$ is called a closed real interval denoted by $X^I = [\underline{X}, \overline{X}]$, where $\underline{X}$ and $\overline{X}$ are the lower and upper bounds, respectively. The set of all the closed real intervals is denoted by $I(R)$.

The mid-point and uncertainties of an interval X^I are defined as

$$X^C = m(X^I) = (\underline{X} + \overline{X})/2 \tag{A.1}$$

and

$$\Delta X = rad(X^I) = (\overline{X} - \underline{X})/2 \tag{A.2}$$

Respectively.

A symmetric interval means an interval X^I in which $\underline{X} = -\overline{X}$.

Let $X^I = [\underline{X}, \overline{X}] \in I(R)$, $Y^I = [\underline{Y}, \overline{Y}] \in I(R)$ be any intervals, the relative uncertainty of X^I is defined as $\delta(X^I) = \Delta X / |X^C| = (\overline{X} - \underline{X}) / |\underline{X} + \overline{X}|$, we say $X^I = Y^I$ if and only if $\underline{X} = \underline{Y}$ and $\overline{X} = \overline{Y}$, X^I is called point interval or degenerate interval if $\underline{X} = \overline{X}$.

An n-dimensional interval vector is represented as

$$\mathbf{X}^I = (X_1^I, X_2^I, \ldots, X_n^I)^T \tag{A.3}$$

the set of all n-dimensional interval vectors is denoted by $I(R^n)$.

Similarly, the mid-vector and uncertainty of an interval vector can be defined as

$$\mathbf{X}^C = (X_1^C, X_2^C, \ldots, X_n^C)^T \tag{A.4}$$

and

$$\Delta\mathbf{X} = (\Delta X_1, \Delta X_2, \ldots, \Delta X_n)^T \tag{A.5}$$

where X_i^C and ΔX_i are given by Eq. (A.1) and Eq. (A.2), respectively.

A matrix whose elements are intervals is called an interval matrix and denoted by $\mathbf{A}^I = [\underline{\mathbf{A}}, \overline{\mathbf{A}}]$, where $\underline{\mathbf{A}}$ is a matrix composed of the lower bounds of the intervals and $\overline{\mathbf{A}}$ is a matrix composed of the upper bounds of the intervals. The set of all interval matrices is denoted by $I(R^{m \times n})$. The mid-matrix and uncertainty of an interval matrix $\mathbf{A}^I$ are defined as

$$\mathbf{A}^C = \frac{\overline{\mathbf{A}} + \underline{\mathbf{A}}}{2} \quad \text{or} \quad a_{ij}^C = \frac{\overline{a_{ij}} + \underline{a_{ij}}}{2} \tag{A.6}$$

and

$$\Delta\mathbf{A} = \frac{\overline{\mathbf{A}} - \underline{\mathbf{A}}}{2} \quad \text{or} \quad \Delta a_{ij}^C = \frac{\overline{a_{ij}} - \underline{a_{ij}}}{2} \tag{A.7}$$

where $\mathbf{A}^C = (a_{ij}^C)$ and $\Delta\mathbf{A} = (\Delta a_{ij})$.

An arbitrary interval $X^I \in I(R)$ can be written as the following form

$$X^I = X^C + \Delta X^I = X^C + \Delta X e_\Delta = [X^C - \Delta X, X^C + \Delta X] \tag{A.8}$$

where $\Delta X^I = [-\Delta X, \Delta X]$ and $e_\Delta = [-1, 1]$.

Similar expressions exist for the interval vector and interval matrix. For $\mathbf{A}^I \in I(R^{m \times n})$, one can have

$$\mathbf{A}^I = \mathbf{A}^C + \Delta\mathbf{A}^I = \mathbf{A}^C + \Delta\mathbf{A} e_\Delta = [\mathbf{A}^C - \Delta\mathbf{A}, \mathbf{A}^C + \Delta\mathbf{A}] \tag{A.9}$$

where $\Delta\mathbf{A}^I = [-\Delta\mathbf{A}, \Delta\mathbf{A}]$.

These basic quantities will play an important role in the following discussion.

Let $X^I = [\underline{x}, \bar{x}]$ and $Y^I = [\underline{y}, \bar{y}]$ be the interval numbers, respectively, then $X^I + Y^I, X^I - Y^I, X^I \times Y^I, X^I/Y^I$ are defined by the following formulas

$$X^I + Y^I = [\underline{x}, \bar{x}] + [\underline{y}, \bar{y}] = [\underline{x} + \underline{y}, \bar{x} + \bar{y}] \tag{A.10}$$

$$X^I - Y^I = [\underline{x}, \bar{x}] - [\underline{y}, \bar{y}] = [\underline{x} - \bar{y}, \bar{x} - \underline{y}] \tag{A.11}$$

$$X^I \times Y^I = [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] = [\min(\underline{x}\,\underline{y}, \underline{x}\,\bar{y}, \bar{x}\,\underline{y}, \bar{x}\bar{y}), \max(\underline{x}\,\underline{y}, \underline{x}\,\bar{y}, \bar{x}\,\underline{y}, \bar{x}\bar{y})] \tag{A.12}$$

$$\frac{X^I}{Y^I} = \frac{[\underline{x}, \bar{x}]}{[\underline{y}, \bar{y}]} = [\underline{x}, \bar{x}] \times \left[\frac{1}{\bar{y}}, \frac{1}{\underline{y}}\right], \quad 0 \notin Y^I \tag{A.13}$$

$$X^I \cap Y^I = [\max(\underline{x}, \underline{y}), \min(\bar{x}, \bar{y})] \tag{A.14}$$

$$X^I \cup Y^I = [\min(\underline{x}, \underline{y}), \max(\bar{x}, \bar{y})] \tag{A.15}$$

Let $\alpha \in R$ be any real number and $X^I = [\underline{x}, \bar{x}] = X^C + \Delta X e_\Delta \in I(R)$ be any real interval, then

$$\alpha X^I = X^I \alpha = X^C \alpha + \Delta X |\alpha| e_\Delta = \alpha X^C + |\alpha| \Delta X e_\Delta \tag{A.16}$$

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T \in R^n$ be any real vector and $\mathbf{A}^I = [\underline{\mathbf{A}}, \overline{\mathbf{A}}] = \mathbf{A}^C + \Delta\mathbf{A} e_\Delta \in I(R^{m \times n})$ be any real interval matrix, then

$$\mathbf{A}^I \mathbf{u} = \mathbf{A}^C \mathbf{u} + \Delta\mathbf{A} |\mathbf{u}| e_\Delta \tag{A.17}$$

$$\mathbf{u}^T \mathbf{A}^I = \mathbf{u}^T \mathbf{A}^C + |\mathbf{u}|^T \Delta\mathbf{A} e_\Delta \tag{A.18}$$

where $|\mathbf{u}| = (|u_1|, |u_2|, \ldots, |u_n|)^T$, $e_\Delta = [-1, 1]$.

Let f be a real-valued function of n real variables $x_1, x_2, \ldots, x_n$. An interval extension of f means that an interval-valued function F of n interval variables $X_1^I, X_2^I, \ldots, X_n^I$, for all $x_i \in X_i^I (i = 1, 2, \ldots, n)$, possesses the following property

$$F([x_1, x_1], [x_2, x_2], \ldots, [x_n, x_n]) = f(x_1, x_2, \ldots, x_n) \tag{A.19}$$

Given a rational function of real variables, one can replace the real variables by the corresponding interval variables and replace the real arithmetic operations by the corresponding interval arithmetic operations to obtain a rational interval function called a natural interval extension of the real rational function.

An interval function F is said to be inclusion monotonic if $X_i^I \subset Y_i^I (i = 1, 2, \ldots, n)$ implies

$$F(X_1^I, X_2^I, \ldots, X_n^I) \subset F(Y_1^I, Y_2^I, \ldots, Y_n^I) \tag{A.20}$$

It is obvious that interval arithmetic is inclusion monotonic. That is, if *op* denotes +, −, ×, /, then $X_i^I \subset Y_i^I$ $(i = 1, 2)$ implies

$$(X_1^I \; op \; X_2^I) \subset (Y_1^I \; op \; Y_2^I) \tag{A.21}$$

The interval extensions of a given function f are not unique. For example, two expressions for function g are given by

$$g^{(1)}(x, a) = \frac{ax}{x-1}, \quad x \neq 1, a \neq 0 \tag{A.22}$$

$$g^{(2)}(x, a) = \frac{ax}{1-\frac{1}{x}}, \quad x \neq 1, a \neq 0 \tag{A.23}$$

Using $A^I = [0,1]$ and $X^I = [2,3]$ replace a and x, two possible evaluations can be obtained

$$g^{(1)}([2, 3], [0, 1]) = \frac{[0, 1], [2, 3]}{[2, 3]-1} = [0, 3]$$

$$g^{(2)}([2, 3], [0, 1]) = \frac{[0, 1]}{1-\frac{1}{[2, 3]}} = [0, 2] \neq g^{(1)}([2, 3], [0, 1])$$

Both interval results contain the exact result of f for $x \in [2, 3]$ and $a \in [0, 1]$, which is [0, 2]. The result for $g^{(2)}$ is precisely the range of g over the given sets, because X^I and A^I occur only once in the expression in $g^{(2)}$ (Hansen 1992). It shows one important rule in interval calculation, that is, the least times the interval parameters appear, the sharper the interval is, which is important in interval calculations.

Irrational functions are treated as follows. Let f be a real irrational function of a real vector $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$. Assume that a rational approximation $r(\mathbf{X})$ is known such that $|f(\mathbf{X}) - r(\mathbf{X})| < \varepsilon$ for all x such that $a_i \leq x_i \leq b_i$ $(i = 1, \ldots, n)$ for some constants a_i and b_i. Then $f(X_1^I, \ldots, X_n^I) \subset r(X_1^I, \ldots, X_n^I) + [-\varepsilon, \varepsilon]$ for any intervals $X_i^I \subset [a_i, b_i] (i = 1, \ldots, n)$. Thus the range of f over the region with $x_i \in X_i^I (i = 1, 2, \ldots, n)$ can be bounded by evaluating $r(X_1^I, \ldots, X_n^I)$ using interval arithmetic and adding the error bound $[-\varepsilon, \varepsilon]$.

This "interval evaluation" of the irrational function f is inclusion monotonic if the interval evaluation of r is inclusion monotonic. The result is an interval extension of f.

Then one can obtain the general conclusion. Let $F(X_1^I, \ldots, X_n^I)$ be an inclusion monotonic interval extension of a real function $f(x_1, \ldots, x_n)$. Then $F(X_1^I, \ldots, X_n^I)$ contains the range of values of $f(x_1, \ldots, x_n)$ for all $x_i \in X_i^I (i = 1, 2, \ldots, n)$.