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Transient thermal stresses of orthotropic functionally graded thick strip due to nonuniform heat supply

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Abstract. This paper is concerned with the theoretical treatment of transient thermal stresses involving an orthotropic functionally graded thick strip due to nonuniform heat supply in the width direction. The thermal and thermoelastic constants of the strip are assumed to possess orthotropy and vary exponentially in the thickness direction. The transient two-dimensional temperature is analyzed by the methods of Laplace and finite sine transformations. We obtain the exact solution for the simply supported strip under the state of plane strain. Some numerical results for the temperature change, the displacement and the stress distributions are shown in figures. Furthermore, the influence of the orthotropy and nonhomogeneity of the material is investigated.

Key words: thermoelasticity; functionally graded material; material orthotropy; strip; transient state; exact solution.

1. Introduction

Functionally graded materials (FGM) have become of major interest as new material that is adaptable for a super-high-temperature environment. Therefore, there are many analytical studies concerned with the thermoelastic problems for the functionally graded materials. As the exact treatments for thermoelastic problems of the functionally graded materials, several analysis were done. Lutz and Zimmerman presented the exact solutions for one-dimensional thermal stresses of functionally graded cylinder and sphere whose elastic moduli and coefficient of linear thermal expansion vary linearly with the radius (Lutz and Zimmerman 1996, Zimmerman and Lutz 1999). Jabbari *et al*. presented the exact solution for one- or two-dimensional thermal stresses of functionally graded hollow cylinder whose material properties vary with the power product form of radial cordinate variable (Jabbari *et al*. 2002, 2003). Assuming that the shear modulus of elasticity, the thermal conductivity and the coefficient of linear thermal expansion vary with the power product form of axial cordinate variable, the axisymmetrical thermoelastic probrem of nonhomogeneous slab (Jeon *et al*. 1997) and three-dimensional thermoelastic probrems of semi-infinite body (Tanigawa *et al*. 1999) were analyzed by analytical methods. The exact solutions for two-dimensional or three-dimensional thermal stress of functionally graded beam whose thermoelastic constants vary

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exponentially through the thickness (Sankar and Tzeng 2002) and functionally graded rectangular plate whose material properties vary with the power product form through the thickness (Vel and Batra 2002) were reported. These papers, however, treated only the thermoelastic problems under the uniform heating or the steady temperature distribution. As a transient thermoelastic problem, Sugano had analyzed exactly one-dimensional thermal stresses of nonhomogeneous plate where the thermal conductivity and Young's modulus vary exponentially, whereas Poisson's ratio and the coefficient of linear thermal expansion vary arbitrarily in the thickness direction (Sugano 1987). Vel and Batra analyzed the three-dimensional transient thermal stresses of the functionally graded rectangular plate (Vel and Batra 2003) by extending the analytical technique reported in Vel and Batra (2002). A transient thermal stress problem of functionally graded thick strip due to nonuniform heat supply is analyzed by Ootao and Tanigawa (2004).

On the other hand, few analysis for thermoelastic problems of orthotropic functionally graded materials can be found in Ye *et al.* (2001), Kawamura *et al.* (2001), Ding *et al.* (2003). To the author's knowledge, the transient thermoelastic analysis of the orthotropic functionally graded materials has not been reported.

In the present article, we analyzed exactly the transient problem of thermoelasticity involving an orthotropic functionally graded thick strip due to nonuniform heat supply in the width direction as a plane strain problem by modify the method which was reported in Ootao and Tanigawa (2004).

2. Analysis

2.1 Heat conduction problem

We consider an orthotropic functionally graded strip that has nonhomogeneous thermal and mechanical properties in the thickness direction as shown in Fig. 1. The thickness and length of the strip are represented by B and L_x, respectively. The coordinate axes x and z are chosen as shown in Fig. 1. The strip is assumed to be initially at zero temperature and is suddenly heated from the lower and upper surfaces by surrounding media with relative heat transfer coefficients h_a and h_b. We denote the temperatures of the surrounding media by the functions $T_a f_a(x)$ and $T_b f_b(x)$ and assume its end surfaces ($x = 0, L_x$) are at zero temperature. The thermal conductivity is assumed to take the following form

$$\lambda_x(z) = \lambda_{x0}\exp(az/B), \quad \lambda_z(z) = \lambda_{z0}\exp(az/B), \quad a \neq 0 \tag{1}$$

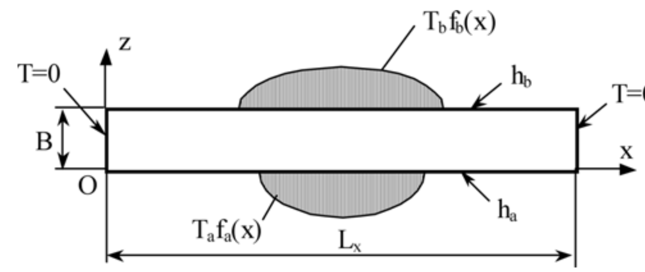


Fig. 1 Analytical model and coordinate system

while the specific heat c and density ρ are constant. In Eq. (1), a is an arbitrary constant which is not zero. Then the temperature distribution shows a two-dimensional distribution in $x-z$ plane, and the transient heat conduction equation is taken in the following form:

$$\frac{\partial}{\partial x}\left\{\lambda_x(z)\frac{\partial T}{\partial x}\right\}+\frac{\partial}{\partial z}\left\{\lambda_z(z)\frac{\partial T}{\partial z}\right\} = c\rho\frac{\partial T}{\partial t} \tag{2}$$

Substituting the Eq. (1) into the Eq. (2), the transient heat conduction equation in dimensionless form is

$$\bar{\lambda}_{x0}\frac{\partial^2 \bar{T}}{\partial \bar{x}^2}+a\frac{\partial \bar{T}}{\partial \bar{z}}+\frac{\partial^2 \bar{T}}{\partial \bar{z}^2} = e^{-a\bar{z}}\frac{\partial \bar{T}}{\partial \tau} \tag{3}$$

The initial and thermal boundary conditions in dimensionless form are

$$\tau = 0; \quad \bar{T} = 0 \tag{4}$$

$$\bar{z} = 0; \quad \frac{\partial \bar{T}}{\partial \bar{z}}-H_a\bar{T} = -H_a\bar{T}_a f_a(\bar{x}) \tag{5}$$

$$\bar{z} = 1; \quad \frac{\partial \bar{T}}{\partial \bar{z}}+H_b\bar{T} = H_b\bar{T}_b f_b(\bar{x}) \tag{6}$$

$$\bar{x} = 0, \bar{L}_x; \quad \bar{T} = 0 \tag{7}$$

In expressions (3)-(7), we have introduced the following dimensionless values:

$$(\bar{T}, \bar{T}_a, \bar{T}_b) = \frac{(T, T_a, T_b)}{T_0}, \quad (\bar{x}, \bar{z}, \bar{L}_x) = \frac{(x, z, L_x)}{B}, \quad \bar{\lambda}_{x0} = \frac{\lambda_{x0}}{\lambda_{z0}}$$

$$\tau = \frac{\kappa_{z0}t}{B^2}, \quad \kappa_{z0} = \frac{\lambda_{z0}}{c\rho}, \quad (H_a, H_b) = (h_a, h_b)B \tag{8}$$

where T is the temperature change; t is time; T_0 is reference temperature; and κ_{z0} is typical value of thermal diffusivity.

To solve the fundamental Eq. (3), we introduce the finite sine transformation with respect to the variable $\bar{x}$ and Laplace transformation with respect to the variable τ. Performing these integral transformations under the conditions (4) and (7), we obtain

$$\frac{d^2\hat{\bar{T}}^*}{d\bar{z}^2}+a\frac{d\hat{\bar{T}}^*}{d\bar{z}}-[\bar{\lambda}_{x0}q^2+p\exp(-a\bar{z})]\hat{\bar{T}}^* = 0 \tag{9}$$

where the symbols (^) and (*) mean the integral transformation with respect to the variable $\bar{x}$ and τ, and the parameters of the transformations are denoted by q and $p(=-a^2\mu^2/4)$, respectively. And q represents the root of the equation

$$\sin q\bar{L}_x = 0 \tag{10}$$

To solve the fundamental Eq. (9), we introduce the following variable and the auxiliary function $\hat{\bar{U}}^*$ as

$$\xi = \mu \exp\left(-\frac{a}{2}\bar{z}\right) \tag{11}$$

$$\hat{\bar{T}}^* = \xi \hat{\bar{U}}^* \tag{12}$$

Taking into account Eqs. (11) and (12), we obtain the next fundamental equation for $\hat{\bar{U}}^*$

$$\frac{d^2\hat{\bar{U}}^*}{d\xi^2} + \frac{1}{\xi}\frac{d\hat{\bar{U}}^*}{d\xi} + \left(1 - \frac{\gamma^2}{\xi^2}\right)\hat{\bar{U}}^* = 0 \tag{13}$$

where

$$\gamma = \sqrt{1 + \frac{4q^2\bar{\lambda}_{x0}}{a^2}} \tag{14}$$

The solution of Eq. (13) is

$$\hat{\bar{U}}^* = AJ_\gamma(\xi) + BY_\gamma(\xi) \tag{15}$$

From Eqs. (11), (12) and (15), we obtain the next equation for $\hat{\bar{T}}^*$

$$\hat{\bar{T}}^* = \mu\exp\left(-\frac{a}{2}\bar{z}\right)\left\{AJ_\gamma\left[\mu\exp\left(-\frac{a}{2}\bar{z}\right)\right] + BY_\gamma\left[\mu\exp\left(-\frac{a}{2}\bar{z}\right)\right]\right\} \tag{16}$$

where $J_\gamma(\ )$ and $Y_\gamma(\ )$ are the Bessel functions of the first and second kind of order γ, respectively. Terms A and B are unknown constants. Accomplishing the inverse Laplace transformation and the inverse finite sine transformation on Eq. (16), the temperature solution is shown as follows:

$$\bar{T} = \sum_{k=1}^{\infty} \bar{T}_k(\bar{z}, \tau)\sin q_k\bar{x} \tag{17}$$

where

$$\bar{T}_k(\bar{z}, \tau) = \frac{2}{\bar{L}_x}\left[\frac{1}{F}\left\{\bar{A}'\exp\left[-\frac{a}{2}(1-\gamma)\bar{z}\right] + \bar{B}'\exp\left[-\frac{a}{2}(1+\gamma)\bar{z}\right]\right\}\right.$$
$$\left. -\sum_{j=1}^{\infty}\frac{2\exp\left(-\frac{a}{2}\bar{z} - \frac{a^2\mu_j^2}{4}\tau\right)}{\mu_j\Delta'(\mu_j)}\left\{\bar{A}J_\gamma\left[\mu_j\exp\left(-\frac{a}{2}\bar{z}\right)\right] + \bar{B}Y_\gamma\left[\mu_j\exp\left(-\frac{a}{2}\bar{z}\right)\right]\right\}\right] \tag{18}$$

where Δ and F are the determinants of 2×2 matrix $[a_{ij}]$ and $[e_{ij}]$, respectively; the coefficients $\bar{A}$ and $\bar{B}$ are defined as the determinant of the matrix similar to the coefficient matrix $[a_{ij}]$, in which the first column or second column is replaced by the constant vector $\{c_i\}$, respectively. Similarly, the coefficients $\bar{A}'$ and $\bar{B}'$ are defined as the determinant of the matrix similar to the coefficient matrix $[e_{ij}]$, in which the first column or second column is replaced by the constant vector $\{c_i\}$, respectively. The elements of the coefficient matrices $[a_{ij}]$, $[e_{ij}]$ and the constant vector $\{c_i\}$ are given in Appendix A. In Eqs. (17) and (18), $\Delta'(\mu_j)$ and q_k are

$$\Delta'(\mu_j) = \left.\frac{d\Delta}{d\mu}\right|_{\mu = \mu_j}, \quad q_k = \frac{k\pi}{L_x} \tag{19}$$

and μ_j represents the jth positive roots of the following transcendental equation

$$\Delta(\mu) = 0 \tag{20}$$

2.2 Thermal stress analysis

We now analyze the transient thermal stress of an orthotropic functionally graded thick strip with simply supported edges as a plane strain problem. The displacement-strain relations are expressed in dimensionless form as follows:

$$\bar{\varepsilon}_{xx} = \bar{u}_{,\bar{x}}, \quad \bar{\varepsilon}_{zz} = \bar{w}_{,\bar{z}}, \quad \bar{\gamma}_{zx} = \bar{u}_{,\bar{z}} + \bar{w}_{,\bar{x}}, \quad \bar{\varepsilon}_{yy} = \bar{\gamma}_{xy} = \bar{\gamma}_{yz} = 0 \tag{21}$$

where a comma denotes partial differentiation with respect to the variable that follows. Stress-strain relation in dimensionless form is given by the following relations:

$$\begin{aligned} \bar{\sigma}_{xx} &= \bar{C}_{11}\bar{\varepsilon}_{xx} + \bar{C}_{13}\bar{\varepsilon}_{zz} - \bar{\beta}_x\bar{T} \\ \bar{\sigma}_{yy} &= \bar{C}_{12}\bar{\varepsilon}_{xx} + \bar{C}_{23}\bar{\varepsilon}_{zz} - \bar{\beta}_y\bar{T} \\ \bar{\sigma}_{zz} &= \bar{C}_{13}\bar{\varepsilon}_{xx} + \bar{C}_{33}\bar{\varepsilon}_{zz} - \bar{\beta}_z\bar{T} \\ \bar{\sigma}_{zx} &= \bar{C}_{55}\bar{\gamma}_{zx} \end{aligned} \tag{22}$$

where

$$\begin{aligned} \bar{\beta}_x &= \bar{C}_{11}\bar{\alpha}_x + \bar{C}_{12}\bar{\alpha}_y + \bar{C}_{13}\bar{\alpha}_z \\ \bar{\beta}_y &= \bar{C}_{12}\bar{\alpha}_x + \bar{C}_{22}\bar{\alpha}_y + \bar{C}_{23}\bar{\alpha}_z \\ \bar{\beta}_z &= \bar{C}_{13}\bar{\alpha}_x + \bar{C}_{23}\bar{\alpha}_y + \bar{C}_{33}\bar{\alpha}_z \end{aligned} \tag{23}$$

The elastic stiffness constants and the coefficients of linear thermal expansion in dimensionless form are assumed to take the following forms

$$\bar{C}_{kl} = \bar{C}^0_{kl}\exp(l\bar{z}), \quad \bar{\alpha}_k = \bar{\alpha}^0_k\exp(b\bar{z}), \quad l \neq 0 \tag{24}$$

where l and b are arbitrary constants. In Eq. (24), $\bar{C}^0_{kl}$ are represented by the Young's moduli of elasticity and the Poisson's ratios as

$$\bar{C}^0_{11} = \frac{1}{C_D}(\bar{E}^0_x)^2(\bar{E}^0_y - \bar{E}^0_z\nu^2_{yz}), \qquad \bar{C}^0_{12} = \frac{1}{C_D}\bar{E}^0_x\bar{E}^0_y(\bar{E}^0_y\nu_{xy} + \bar{E}^0_z\nu_{xz}\nu_{yz})$$

$$\bar{C}^0_{13} = \frac{1}{C_D}\bar{E}^0_x\bar{E}^0_y\bar{E}^0_z(\nu_{xz} + \nu_{xy}\nu_{yz}), \qquad \bar{C}^0_{22} = -\frac{1}{C_D}(\bar{E}^0_y)^2(\bar{E}^0_z\nu^2_{xz} - \bar{E}^0_x)$$

$$\bar{C}_{23}^{0} = \frac{1}{C_D}\bar{E}_y^0\bar{E}_z^0(\bar{E}_y^0\nu_{xy}\nu_{xz} + \bar{E}_x^0\nu_{yz}), \qquad \bar{C}_{33}^{0} = -\frac{1}{C_D}\bar{E}_y^0\bar{E}_z^0(\bar{E}_y^0\nu_{xy}^2 - \bar{E}_x^0)$$

$$C_D = \bar{E}_x^0(\bar{E}_y^0 - \bar{E}_z^0\nu_{yz}^2) - \bar{E}_y^0[\bar{E}_y^0\nu_{xz}^2 + \bar{E}_z^0\nu_{xz}(\nu_{xz} + 2\nu_{xy}\nu_{yz})] \tag{25}$$

where the Poisson's ratios ν_{ij} are assumed to be constant. Substituting Eq. (24) into Eq. (23), we obtain the next equations.

$$(\bar{\beta}_x, \bar{\beta}_y, \bar{\beta}_z) = (\bar{\beta}_x^0, \bar{\beta}_y^0, \bar{\beta}_z^0)\exp[(l+b)\bar{z}] \tag{26}$$

where

$$\begin{aligned}
\bar{\beta}_x^0 &= \bar{C}_{11}^0\bar{\alpha}_x^0 + \bar{C}_{12}^0\bar{\alpha}_y^0 + \bar{C}_{13}^0\bar{\alpha}_z^0 \\
\bar{\beta}_y^0 &= \bar{C}_{12}^0\bar{\alpha}_x^0 + \bar{C}_{22}^0\bar{\alpha}_y^0 + \bar{C}_{23}^0\bar{\alpha}_z^0 \\
\bar{\beta}_z^0 &= \bar{C}_{13}^0\bar{\alpha}_x^0 + \bar{C}_{23}^0\bar{\alpha}_y^0 + \bar{C}_{33}^0\bar{\alpha}_z^0
\end{aligned} \tag{27}$$

In expressions (21)-(27), the following dimensionless values are introduced:

$$\bar{\sigma}_{ij} = \frac{\sigma_{ij}}{\alpha_0 E_0 T_0}, \quad (\bar{\varepsilon}_{ij}, \bar{\gamma}_{ij}) = \frac{(\varepsilon_{ij}, \gamma_{ij})}{\alpha_0 T_0}, \quad (\bar{u}, \bar{v}, \bar{w}) = \frac{(u, v, w)}{\alpha_0 T_0 B}$$

$$(\bar{C}_{ij}, \bar{C}_{ij}^0, \bar{E}_i^0) = \frac{(C_{ij}, C_{ij}^0, E_i^0)}{E_0}, \qquad (\bar{\alpha}_i, \bar{\alpha}_i^0) = \frac{(\alpha_i, \alpha_i^0)}{\alpha_0} \tag{28}$$

where σ_{ij} are the stress components, ε_{ij} are the normal strain components, γ_{ij} are the shearing strain components, (u, v, w) are the displacement components, and α_0 and E_0 are the typical values of the coefficient of linear thermal expansion and the Young's modulus of elasticity, respectively.

Substituting Eqs. (21), (22), (24) and (26) into the equilibrium equations, the displacement equations of equilibrium are written as

$$\bar{C}_{11}^0\bar{u}_{,\bar{x}\bar{x}} + \bar{C}_{55}^0(\bar{u}_{,\bar{z}\bar{z}} + l\bar{u}_{,\bar{z}}) + (\bar{C}_{13}^0 + \bar{C}_{55}^0)\bar{w}_{,\bar{x}\bar{z}} + l\bar{C}_{55}^0\bar{w}_{,\bar{x}} = \bar{\beta}_x^0 e^{b\bar{z}}\bar{T}_{,\bar{x}} \tag{29}$$

$$(\bar{C}_{13}^0 + \bar{C}_{55}^0)\bar{u}_{,\bar{x}\bar{z}} + l\bar{C}_{13}^0\bar{u}_{,\bar{x}} + \bar{C}_{55}^0\bar{w}_{,\bar{x}\bar{x}} + \bar{C}_{33}^0(\bar{w}_{,\bar{z}\bar{z}} + l\bar{w}_{,\bar{z}}) = \bar{\beta}_z^0 e^{b\bar{z}}[(l+b)\bar{T} + \bar{T}_{,\bar{z}}] \tag{30}$$

The boundary conditions of lower and upper surfaces can be represented as follows:

$$\bar{z} = 0, 1; \quad \bar{\sigma}_{zz} = 0, \quad \bar{\sigma}_{zx} = 0 \tag{31}$$

We now consider the case of a simply supported strip given by the following relations:

$$\bar{x} = 0, \bar{L}_x; \quad \bar{\sigma}_{xx} = 0, \quad \bar{w} = 0 \tag{32}$$

We assume the solutions of Eqs. (29) and (30) in order to satisfy Eq. (32) in the following form.

$$\bar{u} = \sum_{k=1}^{\infty}[U_{ck}(\bar{z}) + U_{pk}(\bar{z})]\sin q_k\bar{x}$$

$$\bar{w} = \sum_{k=1}^{\infty} [W_{ck}(\bar{z}) + W_{pk}(\bar{z})]\cos q_k \bar{x} \tag{33}$$

In expressions (33), the first term on the right-hand side gives the homogeneous solution and the second term of right-hand side gives the particular solution. We now consider the homogeneous solution. We express $U_{ck}(\bar{z})$ and $W_{ck}(\bar{z})$ as follows:

$$[U_{ck}(\bar{z}), W_{ck}(\bar{z})] = (U_{ck}^0, W_{ck}^0)\exp(s\bar{z}) \tag{34}$$

Substituting the first term on the right-hand side of Eq. (33) and Eq. (34) into the homogeneous equations of Eqs. (29) and (30), the condition that non-trivial solutions of (U_{ck}^0, W_{ck}^0) exist leads to the following equation.

$$s^4 + A_1 s^3 + A_2 s^2 + A_3 s + A_4 = 0 \tag{35}$$

where

$$A_1 = 2l$$

$$A_2 = \frac{1}{\bar{C}_{33}^0 \bar{C}_{55}^0}\{[(\bar{C}_{13}^0)^2 + 2\bar{C}_{13}^0\bar{C}_{55}^0 - \bar{C}_{11}^0\bar{C}_{33}^0]q_k^2 + \bar{C}_{33}^0\bar{C}_{55}^0 l^2\}$$

$$A_3 = \frac{lq_k^2}{\bar{C}_{33}^0\bar{C}_{55}^0}[\bar{C}_{13}^0(\bar{C}_{13}^0 + 2\bar{C}_{55}^0) - \bar{C}_{11}^0\bar{C}_{33}^0]$$

$$A_4 = \frac{q_k^2}{\bar{C}_{33}^0}(\bar{C}_{13}^0 l^2 + \bar{C}_{11}^0 q_k^2) \tag{36}$$

From Eq. (35), there might be four real roots, two real roots and one pair of conjugate complex roots, or two pairs of conjugate complex roots.

The case of real roots for s:

Given J_R real roots for s, $U_{ck}(\bar{z})$ and $W_{ck}(\bar{z})$ are given by the following expressions:

$$U_{ck}(\bar{z}) = \sum_{J=1}^{J_R} F_{kJ}\exp(s_J\bar{z}), \quad W_{ck}(\bar{z}) = \sum_{J=1}^{J_R} M_{kJ}(s_J)F_{kJ}\exp(s_J\bar{z}) \tag{37}$$

where

$$M_{kJ}(s_J) = \frac{[(\bar{C}_{13}^0 + \bar{C}_{55}^0)s_J + \bar{C}_{13}^0 l]q_k}{-\bar{C}_{55}^0 q_k^2 + \bar{C}_{33}^0 s_J(s_J + l)} \tag{38}$$

Here, F_{kJ} are unknown constants.

The case of complex roots for s:

If the complex root for s is expressed by $s_J = \alpha_J \pm j\beta_J$, and given J_I pairs of complex roots for s, $U_{ck}(\bar{z})$ and $W_{ck}(\bar{z})$ are given by the following expressions:

$$U_{ck}(\bar{z}) = \sum_{J=1}^{J_I} [C_{1J}\exp(\alpha_J\bar{z})\cos(\beta_J\bar{z}) + C_{2J}\exp(\alpha_J\bar{z})\sin(\beta_J\bar{z})]$$

$$W_{ck}(\bar{z}) = \sum_{J=1}^{J_I} \{C_{1J}\exp(\alpha_J\bar{z})[\Gamma_J\cos(\beta_J\bar{z}) - \Omega_J\sin(\beta_J\bar{z})]$$

$$+ C_{2J}\exp(\alpha_J\bar{z})[\Omega_J\cos(\beta_J\bar{z}) + \Gamma_J\sin(\beta_J\bar{z})]\} \tag{39}$$

where

$$\Gamma_J = R_e[M_{kJ}|_{s_J = \alpha_J + j\beta_J}], \quad \Omega_J = I_m[M_{kJ}|_{s_J = \alpha_J + j\beta_J}] \tag{40}$$

In Eq. (40), j, $R_e[\ ]$ and $I_m[\ ]$ are imaginary unit $j = \sqrt{-1}$, real part and imaginary part, respectively. Furthermore, in Eq. (39), C_{1J} and C_{2J} are unknown constants.

In order to obtain the particular solution, we use the series expansions of the Bessel functions as follows:

$$J_\gamma(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+\gamma}}{n!\Gamma(\gamma+n+1)} \tag{41}$$

$$Y_\gamma(x) = \frac{1}{\sin\gamma\pi}[\cos\gamma\pi J_\gamma(x) - J_{-\gamma}(x)] \text{ if } \gamma \neq \text{ integer} \tag{42}$$

Since the order γ of the Bessel function in Eq. (18) is not integer in general, Eq. (18) can be written as the following expression using Eqs. (41) and (42).

$$\bar{T}_k(\bar{z}, \tau) = \sum_{n=0}^{\infty} \left\{ a_n(\tau)\exp\left[-\frac{a}{2}(2n+1+\gamma)\bar{z}\right] + b_n(\tau)\exp\left[-\frac{a}{2}(2n+1-\gamma)\bar{z}\right] \right\} \tag{43}$$

Expressions for the functions $a_n(\tau)$ and $b_n(\tau)$ in Eq. (43) are omitted here for the sake of brevity. We assume $U_{pk}(\bar{z})$ and $W_{pk}(\bar{z})$ as follows:

$$U_{pk}(\bar{z}) = \sum_{n=0}^{\infty} [A_{1n}\exp(d_{n1}\bar{z}) + A_{2n}\exp(d_{n2}\bar{z})]$$

$$W_{pk}(\bar{z}) = \sum_{n=0}^{\infty} [B_{1n}\exp(d_{n1}\bar{z}) + B_{2n}\exp(d_{n2}\bar{z})] \tag{44}$$

where

$$d_{n1} = b - \frac{a}{2}(2n+1+\gamma), \quad d_{n2} = b - \frac{a}{2}(2n+1-\gamma) \tag{45}$$

Substituting Eqs. (44) into Eqs. (29) and (30), and comparing the coefficients of functions with regard to $\bar{z}$, the constants A_{1n}, A_{2n}, B_{1n} and B_{2n} are obtained.

The stress components are evaluated by substituting Eq. (33) into Eq. (21), and later into Eq. (22). The unknown constants in Eqs. (37) and (39) are determined so as to satisfy the boundary condition (31).

3. Numerical results

To illustrate the foregoing analysis, numerical parameters of heat conduction and the geometry of the strip are presented as follows:

$$H_a = H_b = 1.0, \quad \bar{T}_a = 1, \quad \bar{T}_b = 0, \quad \bar{L}_x = 6.0$$

$$f_a(\bar{x}) = \begin{cases} 1 - \bar{x}'^2/\bar{x}_0^2 : -\bar{x}_0 \le \bar{x}' \le \bar{x}_0 \\ 0 : \bar{x}' \le -\bar{x}_0, \bar{x}_0 \le \bar{x}' \end{cases}, \quad \bar{x}_0 = 1.0, \bar{x}' = \bar{x} - \bar{L}_x/2 \tag{46}$$

where H_a and H_b are the Biot numbers as defined in Eq. (8), $\bar{L}_x$ is the aspect ratio as defined in Eq. (8), $\bar{x}'$ is the dimensionless local coordinate, and $\bar{x}_0$ is half of the dimensionless heating length, respectively. We assume that the strip is heated from the lower surface by surrounding media, the temperature of which is denoted by the symmetric function with respect to the center of strip $\bar{x} = \bar{L}_x/2$. The Biot numbers of the lower and upper surfaces are assumed to be same values. The end surfaces ($\bar{x} = 0, \bar{L}_x$) of the strip are at zero temperature as shown in Eq. (7). The mechanical boundary conditions have been shown in Eqs. (31) and (32).

The orthotropic and nonhomogeneous parameters adopted for the numerical calculations are shown in Table 1. In Table 1, a is a nonhomogeneous parameter of the thermal conductivity, b is a nonhomogeneous parameter of the coefficient of linear thermal expansion, and l is a nonhomogeneous parameter of the elastic stiffness constant, respectively. Values given for Case 1 correspond to the results for orthotropy and nonhomogeneity of the thermal conductivity; values given for Case 2 correspond to the results for orthotropy and nonhomogeneity of the coefficient of linear thermal expansion; and values given for Case 3 correspond to the results for orthotropy and nonhomogeneity of the elastic stiffness constant. The case $b = 0$ shows that the coefficient of linear thermal expansions are constant, and the case $l = 0$ shows that the elastic stiffness constants are constant. The case $\bar{E}_x^0 = \bar{E}_y^0 = \bar{E}_z^0 = 1$ shows that the Young's modulus of elasticity is isotropic. The case $\bar{\alpha}_x^0 = \bar{\alpha}_y^0 = \bar{\alpha}_z^0 = 1$ shows that the coefficient of linear thermal expansion is isotropic. The case $\bar{\lambda}_x^0 = 1$ shows that the thermal conductivity is isotropic. As the derived solution breaks down if $l = 0$ or $\bar{E}_x^0 = 1$, the nonhomogeneous parameter l is taken to be 0.01, and the Young's modulus of elasticity $\bar{E}_x^0$ is taken to be 1.01.

Table 1 Orthotropy and nonhomogeneous parameters

	a	b	l	$\bar{E}_x^0$	$\bar{E}_y^0$	$\bar{E}_z^0$	$\bar{\alpha}_x^0$	$\bar{\alpha}_y^0$	$\bar{\alpha}_z^0$	$\bar{\lambda}_x^0$
Case 1	1	0	0.01	1.01	1	1	1	1	1	0.5
	1	0	0.01	1.01	1	1	1	1	1	2
	−1	0	0.01	1.01	1	1	1	1	1	0.5
	−1	0	0.01	1.01	1	1	1	1	1	2
Case 2	1	1	0.01	1.01	1	1	0.5	1	1	1
	1	1	0.01	1.01	1	1	2	1	1	1
	1	−1	0.01	1.01	1	1	0.5	1	1	1
	1	−1	0.01	1.01	1	1	2	1	1	1
Case 3	1	0	1	0.5	1	1	1	1	1	1
	1	0	1	2	1	1	1	1	1	1
	1	0	−1	0.5	1	1	1	1	1	1
	1	0	−1	2	1	1	1	1	1	1

In Figs. 2-11, the results for $\tau = \infty$ correspond to the case for steady state condition. In order to examine the influence of orthotropy and nonhomogeneity of the thermal conductivities on the temperature, thermal displacement, and thermal stress distributions, Figs. 2-5 show the results for Case 1, that is, for two values of nonhomogeneous parameter a and two values of the dimensionless value $\bar{\lambda}_{x0}$. Fig. 2 shows the variation in the thickness direction at the midpoint $\bar{x} = \bar{L}_x/2$ of the strip. From Fig. 2, it can be seen that the temperature change on the heated surface increases when

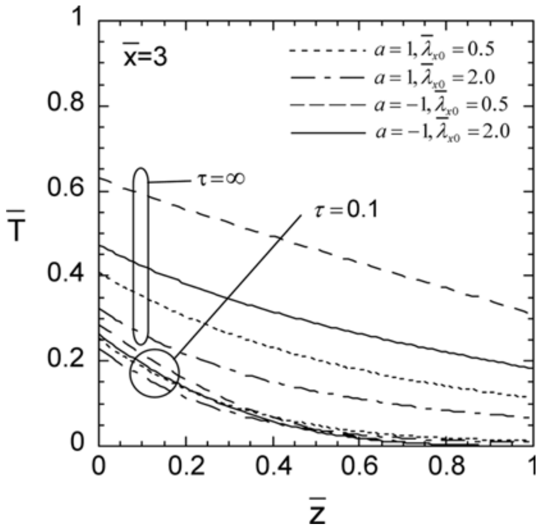


Fig. 2 Variation of temperature change in the thickness direction (Case 1, $\bar{x} = \bar{L}_x/2$)

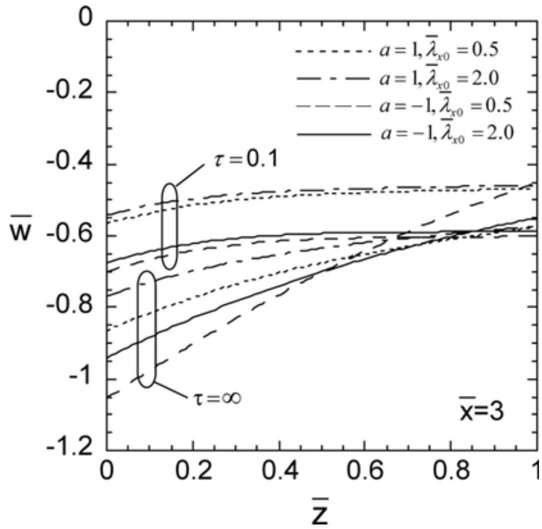


Fig. 3 Variation of thermal displacement $\bar{w}$ in the thickness direction (Case 1, $\bar{x} = \bar{L}_x/2$)

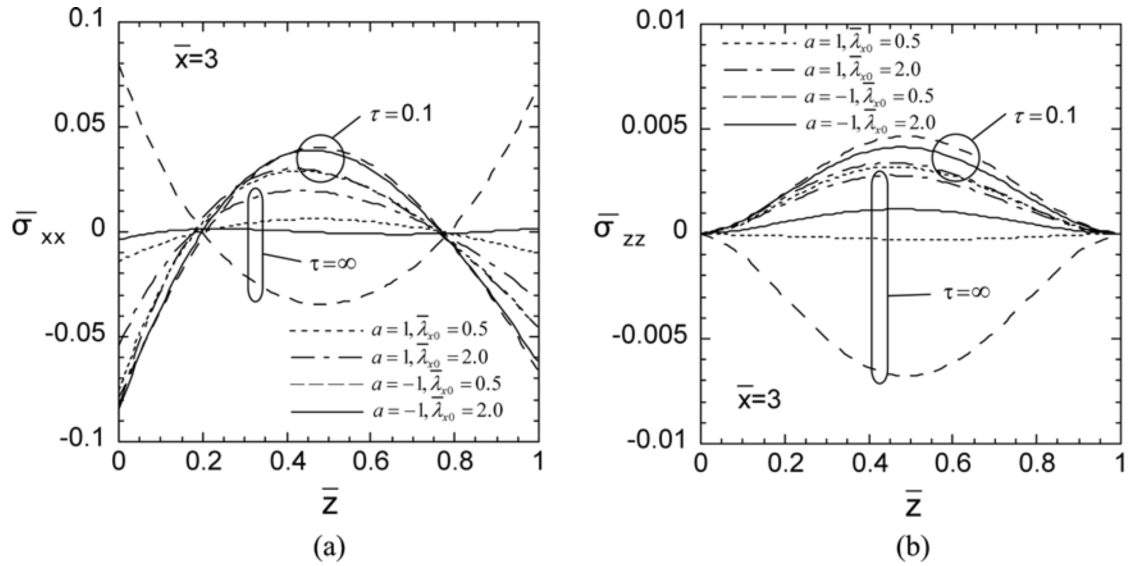


Fig. 4 Variation of thermal stress in the thickness direction (Case 1, $\bar{x} = \bar{L}_x/2$); (a) Normal stress $\bar{\sigma}_{xx}$, (b) Normal stress $\bar{\sigma}_{zz}$

the parameter a or $\overline{\lambda}_x^0$ decreases. Fig. 3 shows the variation of thermal displacement $\overline{w}$ in the thickness direction at the midpoint of the strip. As shown in Fig. 3, the displacement $\overline{w}$ changes along the thickness direction and shows the maximum absolute value on the heated surface. It can be seen from Fig. 3 that the absolute values of thermal displacement $\overline{w}$ increase when the parameter a or $\overline{\lambda}_x^0$ decreases. Fig. 4 shows the variation of thermal stresses in the thickness direction at the midpoint of the strip. The variations of normal stresses $\overline{\sigma}_{xx}$ and $\overline{\sigma}_{zz}$ are shown in Figs. 4(a) and 4(b), respectively. From Figs. 4(a) and 4(b), the tensile stress occurs inside the strip in a transient state without distinction of parameter a or $\overline{\lambda}_x^0$, while the distributions in a steady state substantially change when the parameter a or $\overline{\lambda}_x^0$ changes. Fig. 5 shows the variation of shearing stress $\overline{\sigma}_{zx}$ in the thickness direction at the edge ($\overline{x} = 2.0$) of the heated region, because the maximum stress occurs near $\overline{x} = 2.0$ inside of strip. As shown in Fig. 5, it can be seen that the large stress occurs near the heated surface in a transient state without distinction of parameter a or $\overline{\lambda}_x^0$, while the distribution for the parameters $a = -1$ and $\overline{\lambda}_x^0 = 0.5$ differs considerably from those for other parameters in the steady state condition.

In order to examine the influence of orthotropy and nonhomogeneity of the coefficient of linear thermal expansions on the thermal displacement and thermal stress distributions, Figs. 6-8 show the results for Case 2, that is, for two values of nonhomogeneous parameter b and two values of the dimensionless value $\overline{\alpha}_{x0}$. Fig. 6 shows the variation of thermal displacement $\overline{w}$ in the thickness direction at the midpoint of the strip. From Fig. 6, the influence of the nonhomogeneity on the thermal displacement $\overline{w}$ in a transient state is small. While the influence of orthotropy or nonhomogeneity on the thermal displacement $\overline{w}$ in a steady state is large. Fig. 7 shows the variation of thermal stresses in the thickness direction at the midpoint of the strip. The variations of normal stresses $\overline{\sigma}_{xx}$ and $\overline{\sigma}_{zz}$ are shown in Figs. 7(a) and 7(b), respectively. Fig. 8 shows the variation of shearing stress $\overline{\sigma}_{zx}$ in the thickness direction at the edge ($\overline{x} = 2.0$) of the heated region. From Figs. 7 and 8, it can be seen that the absolute values of thermal stresses $\overline{\sigma}_{xx}$, $\overline{\sigma}_{zz}$ and $\overline{\sigma}_{zx}$ in a transient state increase when the parameter b decreases or $\overline{\alpha}_x^0$ increases, and the

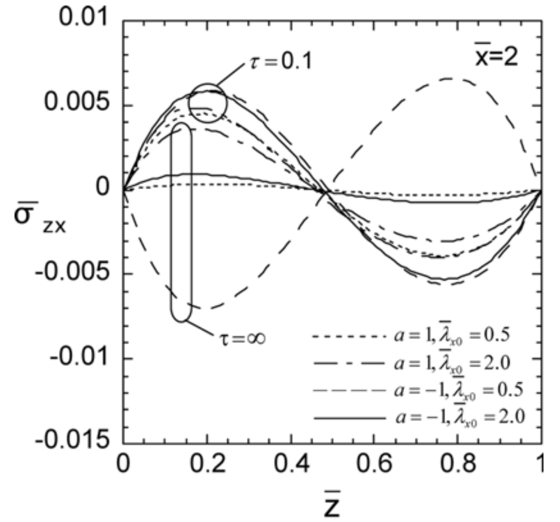


Fig. 5 Variation of shearing stress $\overline{\sigma}_{zx}$ in the thickness direction (Case 1, $\overline{x} = 2.0$)

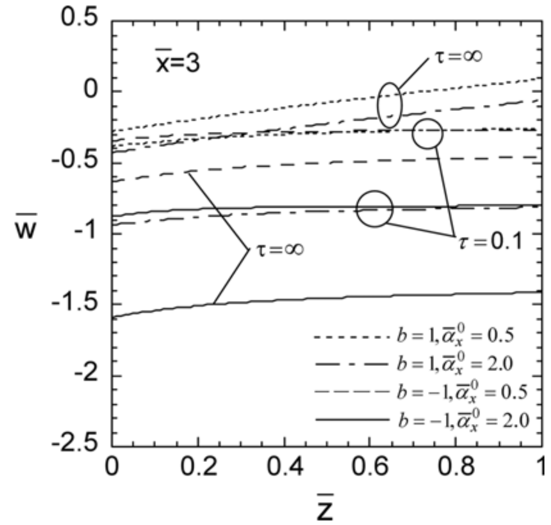
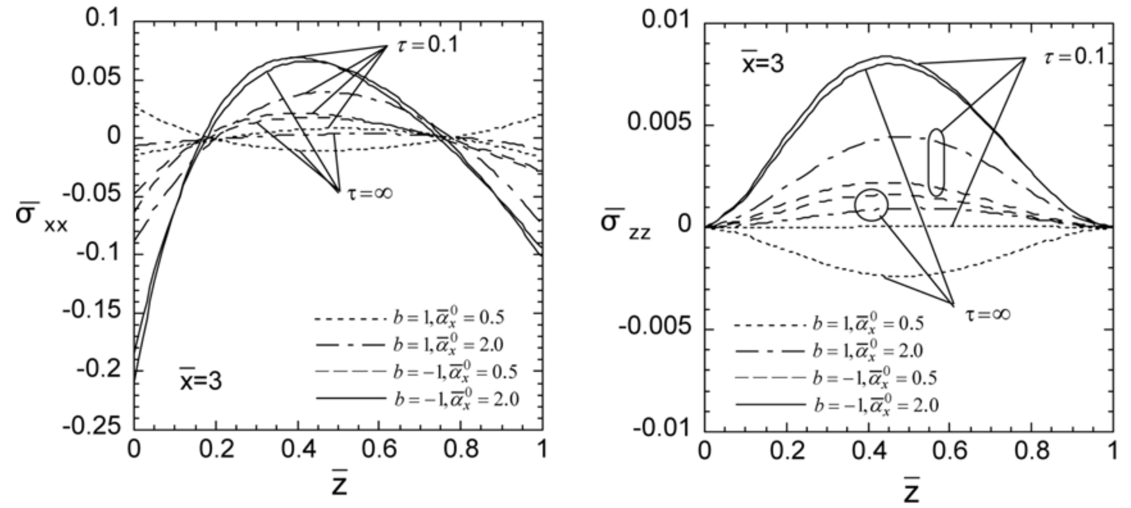


Fig. 6 Variation of thermal displacement $\overline{w}$ in the thickness direction (Case 2, $\overline{x} = L_x/2$)



(a) (b)

Fig. 7 Variation of thermal stress in the thickness direction (Case 2, $\bar{x} = \bar{L}_x/2$); (a) Normal stress $\bar{\sigma}_{xx}$, (b) Normal stress $\bar{\sigma}_{zz}$

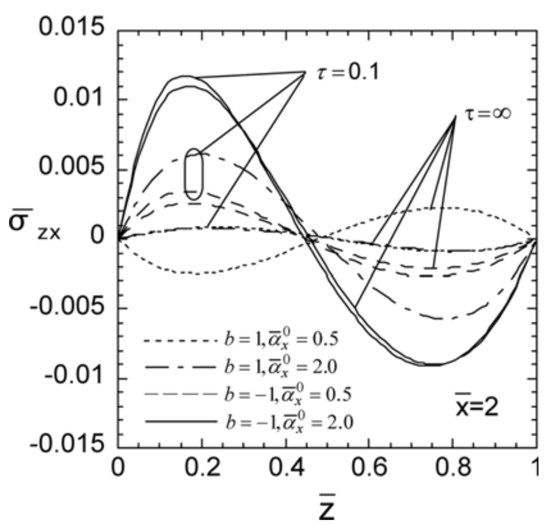


Fig. 8 Variation of shearing stress $\bar{\sigma}_{zx}$ in the thickness direction (Case 2, $\bar{x} = 2.0$)

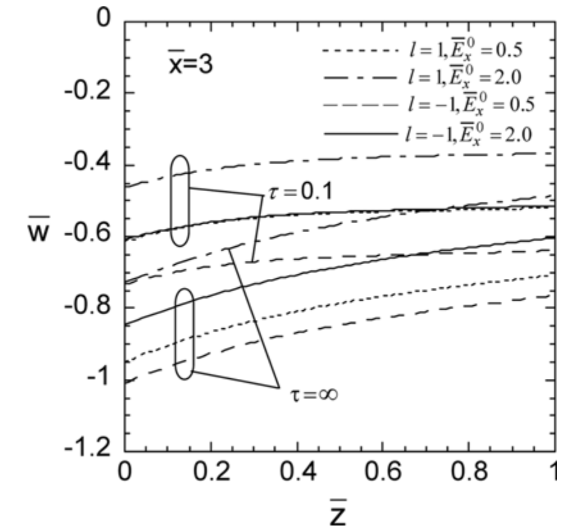
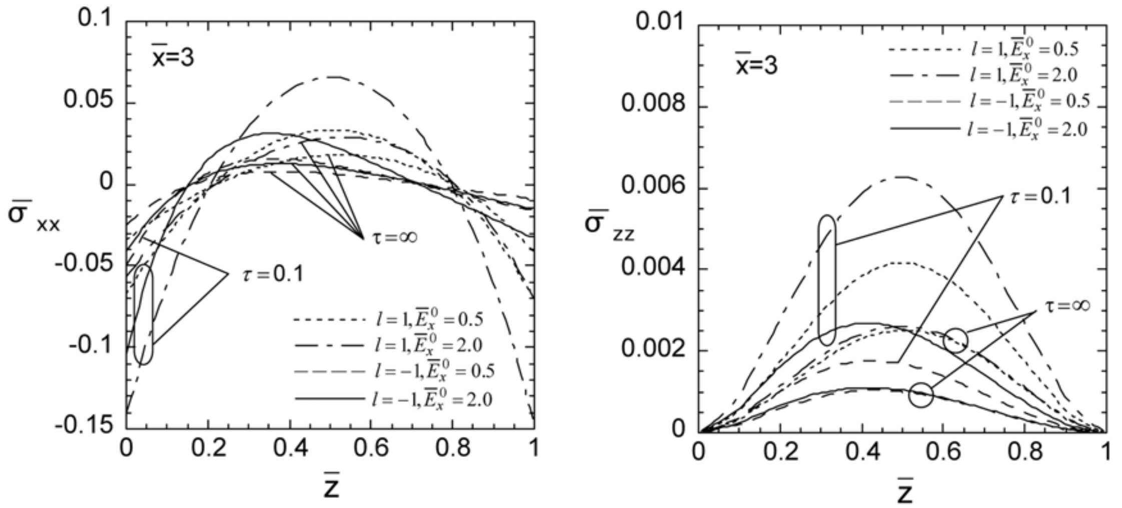


Fig. 9 Variation of thermal displacement $\bar{w}$ in the thickness direction (Case 3, $\bar{x} = \bar{L}_x/2$)

distributions in the steady state condition substantially change when the parameters b or $\bar{\alpha}_x^0$ change. As shown in Figs. 7 and 8, the thermal stresses $\bar{\sigma}_{xx}$, $\bar{\sigma}_{zz}$ and $\bar{\sigma}_{zx}$ for the parameters $b = -1$ and $\bar{\alpha}_x^0 = 2.0$ show the maximum values.

In order to examine the influence of orthotropy and nonhomogeneity of the Young's modulus of elasticity on the thermal displacement and thermal stress distributions, Figs. 9-11 show the results for Case 3, that is, for two values of nonhomogeneous parameter l and two values of the dimensionless value $\bar{E}_{x0}$. Fig. 9 shows the variation of thermal displacement $\bar{w}$ in the thickness



(a) (b)

Fig. 10 Variation of thermal stress in the thickness direction (Case 3, $\bar{x} = \bar{L}_x/2$); (a) Normal stress $\bar{\sigma}_{xx}$, (b) Normal stress $\bar{\sigma}_{zz}$

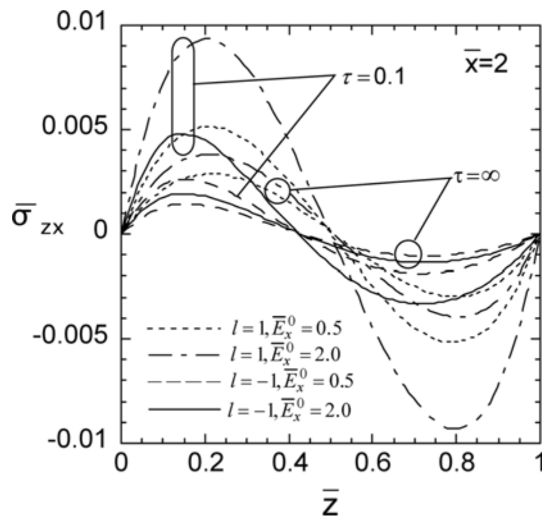


Fig. 11 Variation of shearing stress $\bar{\sigma}_{zx}$ in the thickness direction (Case 3, $\bar{x} = 2.0$)

direction at the midpoint of the strip. From Fig. 9, the absolute values of thermal displacement $\bar{w}$ increase when the parameters l or $\bar{E}_x^0$ decreases. Fig. 10 shows the variation of thermal stresses in the thickness direction at the midpoint of the strip. The variations of normal stresses $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{zz}$ are shown in Figs. 10(a) and 10(b), respectively. Fig. 11 shows the variation of shearing stress $\bar{\sigma}_{zx}$ in the thickness direction at the edge ($\bar{x} = 2.0$) of the heated region. From Figs. 10 and 11, it can be seen that the absolute values of thermal stresses $\bar{\sigma}_{xx}$, $\bar{\sigma}_{zz}$ and $\bar{\sigma}_{zx}$ increase when the parameters l or $\bar{E}_x^0$ increase. As shown in Figs. 10 and 11, the thermal stresses $\bar{\sigma}_{xx}$, $\bar{\sigma}_{zz}$ and $\bar{\sigma}_{zx}$ for the parameters $l = 1$ and $\bar{E}_x^0 = 2.0$ show the maximum values.

4. Conclusions

In the present article, we analyzed the thermoelastic problem involving an orthotropic functionally graded thick strip that has nonhomogeneous thermal and mechanical properties in the thickness direction.

In the analysis, the following conditions are assumed:

(1) Thermal conductivities, the elastic stiffness constants and the coefficients of linear thermal expansion vary exponentially in the thickness direction.
(2) The specific heat and density are constant.
(3) The edges of the strip are at zero temperature.
(4) The mechanical boundary conditions of the edges are given by Eq. (32).

In the analysis of the heat conduction problem, the methods of Laplace and finite sine transformations were used. We obtained the exact solution for the transient temperature and transient thermal stresses of an orthotropic functionally graded strip with simply supported edges due to a nonuniform heat supply in the width direction under the plane strain condition. We conclude that the transverse shearing stress and normal stress in the thickness direction are evaluated precisely in a transient state.

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Appendix A

$$a_{11} = \left[\frac{a}{2}(1+\gamma) + H_a\right] J_\gamma(\mu) - \frac{a}{2}\mu J_{\gamma+1}(\mu)$$

$$a_{12} = \left[\frac{a}{2}(1+\gamma) + H_a\right] Y_\gamma(\mu) - \frac{a}{2}\mu Y_{\gamma+1}(\mu)$$

$$a_{21} = e^{-(a/2)}\left\{\left[\frac{a}{2}(1+\gamma) - H_b\right] J_\gamma(\mu e^{-(a/2)}) - \frac{a}{2}\mu e^{-(a/2)} J_{\gamma+1}(\mu e^{-(a/2)})\right\}$$

$$a_{22} = e^{-(a/2)}\left\{\left[\frac{a}{2}(1+\gamma) - H_b\right] Y_\gamma(\mu e^{-(a/2)}) - \frac{a}{2}\mu e^{-(a/2)} Y_{\gamma+1}(\mu e^{-(a/2)})\right\} \tag{A1}$$

$$e_{11} = -\frac{a}{2}(1-\gamma) - H_a, \quad e_{12} = -\frac{a}{2}(1+\gamma) - H_a$$

$$e_{21} = \left[H_b - \frac{a}{2}(1-\gamma)\right]\exp\left[-\frac{a}{2}(1-\gamma)\right]$$

$$e_{22} = \left[H_b - \frac{a}{2}(1+\gamma)\right]\exp\left[-\frac{a}{2}(1+\gamma)\right] \tag{A2}$$

$$c_1 = -H_a \overline{T}_a \hat{f}_a(q), \quad c_2 = H_b \overline{T}_b \hat{f}_b(q) \tag{A3}$$