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Free vibration of orthotropic functionally graded beams with various end conditions

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Abstract. Free vibration of orthotropic functionally graded beams, whose material properties can vary arbitrarily along the thickness direction, is investigated based on the two-dimensional theory of elasticity. A hybrid state-space/differential quadrature method is employed along with an approximate laminate model, which allows us to obtain the semi-analytical solution easily. With the introduction of continuity conditions at each fictitious interface and boundary conditions at the top and bottom surfaces, the frequency equation for an inhomogeneous beam is derived. A completely exact solution of an FGM beam with material constants varying in exponential way through the thickness is also presented, which serves a benchmark to verify the present method. Numerical results are performed and discussed.

Key words: functionally graded beams; differential quadrature; state space method; approximate laminate model; exact solution.

1. Introduction

Functionally graded materials (FGMs) are known for the nature that they possess the properties varying with location continuously. For their perfect performance of material and mechanics, FGMs are finding increasing applications in the fields of aerospace, electromagnetics, optics, biomedicine and nuclear etc. (Suresh and Mortensen 1998). Hence, the activities of developing new and efficient analysis methods are of great practical significance in the structural design and mechanical behavior investigation.

In the past decade, behaviors of FGM plates and shells, such as thermal stresses, thermal deformations and vibrations etc., have been posing intensive research focuses (Reddy *et al.* 1999, Chen *et al.* 2003a). As regards FGM beams, Wetherhold *et al.* (1996) analyzed the thermal deformation by the beam theory and showed that the deformation could be controlled by selecting proper material variations along the thickness. Sankar (2001) derived an exact elasticity static

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solution for a simply-supported isotropic FGM beam with the Young's modulus being exponential function of the thickness coordinate while the Poisson's ratio keeping constant. He also proposed a simple Euler-Bernoulli type beam theory for slender FGM beams on the basis of the hypothesis that cross sections remain plane and normal to the beam axis after deformation. Recent works on FGM beams can also be found in the papers of Chakraborty *et al.* (2003), Chakraborty and Gopalakrishnan (2003).

In this study, a newly developed hybrid method, i.e., the state-space-based differential quadrature method (SSDQM) (Chen *et al.* 2003b), is employed to investigate the free vibration of orthotropic FGM beams. Details on the applications of differential quadrature method (DQM) in science and engineering fields can be found in Bert and Malik's review paper (1996) and Shu's work (2000). Due to its excellent properties, DQM has continuously received extensive applications in the new century (Redekop and Makhoul 2000, Jiang and Redekop 2002, Civalek and Ülker 2004). In the current work, the conventional state equation, derived from the fundamental elasticity equations of plane stress problem, is discretized along the beam axis by virtue of the differential quadrature (DQ) technique. The proper form of discrete state equation is obtained by involving the exact end conditions. For the sake of analysis, the approximate laminate model (Wang *et al.* 1999, Ootao and Tanigawa 2000, Chen and Ding 2002) is employed to turn the discrete state equation with variable coefficients into the one with constant coefficients within each separate layer. Use of the continuity conditions at each hypothetic interface leads to a transfer relationship between the unknown state variables at the upper and lower surfaces. The eigen-equation is then derived by taking account of the boundary conditions at the two surfaces.

Using the above method, the elastic constants and mass density can vary arbitrarily through the thickness direction of the beam. However, for the purpose of validating the effectiveness of the present approach, exact solution is deduced for the case of a simply-supported FGM beam with the material constants varying along the thickness direction in the following exponential way (Sankar 2001),

$$\textit{Type I}:\ \kappa(y) = \kappa^0 e^{\alpha_1 y} \tag{1}$$

where y is the thickness coordinate of the beam, and κ^0 is the material constant. This solution is then utilized to check the validity and effectiveness of the proposed hybrid method. Then, numerical examples are performed for beams composed of two material phases with the material constants varying through the thickness direction in a power way (Reddy *et al.* 1999) as

$$\textit{Type II}:\ \kappa(y) = \kappa_1^0\left(1-\frac{y}{h}\right)^{\alpha_2} + \kappa_2^0\left[1-\left(1-\frac{h}{y}\right)^{\alpha_2}\right] \tag{2}$$

where κ_1^0 and κ_2^0 denote the material constants of the two phases in a composite beam, and h is the beam thickness. In the above two types of materials, α_1 and α_2 are known as the gradient indexes, and $\kappa(y)$ represents an arbitrary material constant of the FGM depending on the thickness coordinate y. It should be pointed out that for all the cases, the Poisson's ratios remain constant along the thickness direction.

2. Theoretical formulations

Consider an orthotropic functionally graded material composite beam of small width b, depth h

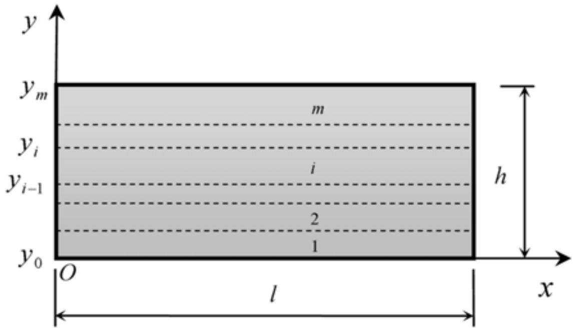

Fig. 1 An FGM beam divided into m layers

and length l placed in a referred system of Cartesian coordinates, which is originating at the bottom plane of the beam with the x axis coincident with the beam axial direction and the y axis with the thickness-wise direction (Fig. 1). For each of the material component, the principle material axes are assumed to coincide with the coordinate axes.

According to the theory of two-dimensional elasticity, for a beam with small width in the state of plane stress ($\sigma_z = \tau_{yz} = \tau_{xz} = 0$), the constitutive relations read (Timoshenko and Goodier 1970)

$$\sigma_x = c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y}, \quad \sigma_y = c_{12}\frac{\partial u}{\partial x} + c_{22}\frac{\partial v}{\partial y}, \quad \tau_{xy} = c_{66}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{3}$$

where σ_x and σ_y are two normal stresses, τ_{xy} is the shear stress, u and v are respectively the displacement components in x and y directions, and c_{rs} (r, s = 1, 2, 6) are the elastic constants.

In the absence of body forces, the differential equilibrium equations for free vibration of an elastic body are written as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = \rho\frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = \rho\frac{\partial^2 v}{\partial t^2} \tag{4}$$

in which ρ is the mass density of the elastic body and t is the time. In this paper, we assume that the elastic constants as well as the mass density are all functions of the coordinate variable y.

Following a routine method of derivation (Das and Setlur 1970, Bahar 1975), the following state equation can be derived from Eqs. (3) and (4),

$$\begin{aligned} &\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} + \frac{1}{c_{66}}\tau_{xy}, \qquad \frac{\partial \sigma_y}{\partial y} = -\rho\omega^2 v - \frac{\partial \tau_{xy}}{\partial x} \\ &\frac{\partial v}{\partial y} = -\frac{c_{12}}{c_{22}}\frac{\partial u}{\partial x} + \frac{1}{c_{22}}\sigma_y, \qquad \frac{\partial \tau_{xy}}{\partial y} = -\rho\omega^2 u + \left(\frac{c_{12}^2}{c_{22}} - c_{11}\right)\frac{\partial^2 u}{\partial x^2} - \frac{c_{12}}{c_{22}}\frac{\partial \sigma_y}{\partial x} \end{aligned} \tag{5}$$

where ω is the circular frequency, u, σ_y, v and τ_{xy} are the state variables, and the induced variable σ_x is obtained as

$$\sigma_x = \left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)\frac{\partial u}{\partial x} + \frac{c_{12}}{c_{22}}\sigma_y \tag{6}$$

The end supporting conditions for a special problem are expressed in terms of the state variables or induced variable. Attentions herein are confined to some important practical representations of different supports, including simply supported (S), clamped (C) and free (F) ends:

$$v = 0,\ \sigma_x = 0, \text{ at a simply supported end} \tag{7a}$$

$$u = 0,\ v = 0, \text{ at a clamped end} \tag{7b}$$

$$\sigma_x = 0,\ \tau_{xy} = 0, \text{ at a free end} \tag{7c}$$

Hereafter, for brevity, S-S, C-C or C-F beam signifies the beam with simply supported-simply supported, clamped-clamped or clamped-free ends.

3. Exact solution of a particular orthotropic FGM beam

Provided that an S-S FGM beam is composed of the material with the mechanical properties dependent on the thickness coordinate in an exponential way (Sankar 2001) as described in Eq. (1), an exact solution can be derived using a method similar to that employed by Zhong and Shang (2003). Herein, it is assumed that

$$\begin{Bmatrix} u \\ \sigma_y \\ v \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} h\bar{u}(\eta)\cos(n\pi\xi) \\ c_{66}^0 e^{\alpha_1 h\eta}\bar{\sigma}_\eta(\eta)\sin(n\pi\xi) \\ h\bar{v}(\eta)\sin(n\pi\xi) \\ c_{66}^0 e^{\alpha_1 h\eta}\bar{\tau}(\eta)\cos(n\pi\xi) \end{Bmatrix} e^{i\omega t} \tag{8}$$

where $\xi = x/l$, $\eta = y/h$, n is the half-wave number along the beam axis, an over-bar represents the non-dimensional variable, and the superscript '0' of elastic constants c_{ij}^0 denotes the value at $y = 0$. Note that the state variables given by Eq. (8) automatically satisfy the simply-supported conditions at $\xi = 0$ and $\xi = 1$. Substitution of Eq. (8) into Eqs. (3) and (4) yields

$$\frac{d}{d\eta}\bar{\delta}(\eta) = \mathbf{A}\bar{\delta}(\eta) \tag{9}$$

where $\bar{\delta}(\eta) = [\bar{u}(\eta)\ \ \bar{\sigma}_\eta(\eta)\ \ \bar{v}(\eta)\ \ \bar{\tau}(\eta)]^T$, and the constant coefficient matrix $\mathbf{A}$ is obtained as

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -k_n & 1 \\ 0 & -\alpha_1 h & -\Omega^2 & k_n \\ \dfrac{c_{12}^0}{c_{22}^0}k_n & \dfrac{c_{66}^0}{c_{22}^0} & 0 & 0 \\ \left(c_{11}^0 - \dfrac{(c_{12}^0)^2}{c_{22}^0}\right)\dfrac{k_n^2}{c_{66}^0} - \Omega^2 & -\dfrac{c_{12}^0}{c_{22}^0}k_n & 0 & -\alpha_1 h \end{bmatrix} \quad (k_n = n\pi h/l) \tag{10}$$

where $\Omega = \omega h\sqrt{\rho^0/c_{66}^0}$ is the non-dimensional frequency parameter. According to the matrix theorem, the solution of $\bar{\delta}(\eta)$ in Eq. (9) is derived as

$$\bar{\delta}(\eta) = e^{\mathbf{A}\eta}\bar{\delta}(0) \qquad (0 \le \eta \le 1) \tag{11}$$

One can get the relationship between the state vectors at upper and lower surfaces by setting $\eta = 1$, that is

$$\bar{\delta}(1) = \mathbf{T}\bar{\delta}(0) \tag{12}$$

where $\mathbf{T} = \exp(\mathbf{A})$ is known as the global transfer matrix. By incorporating the tractions-free boundary conditions at the lateral surfaces ($\eta = 0$ and $\eta = 1$) into Eq. (12), one gets

$$\begin{bmatrix} T_{21} & T_{23} \\ T_{41} & T_{43} \end{bmatrix} \begin{Bmatrix} \bar{u}(0) \\ \bar{v}(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{13}$$

Eq. (13) has nontrivial solutions for $\bar{u}(0)$ and $\bar{v}(0)$ only if the determinant of the coefficient matrix vanishes, giving rise to the following frequency equation for the beam,

$$\begin{vmatrix} T_{21} & T_{23} \\ T_{41} & T_{43} \end{vmatrix} = 0 \tag{14}$$

in which T_{ij} are elements of matrix $\mathbf{T}$. From Eq. (14) one can calculate the natural frequency of the beam. Different from that described in various conventional beam theories or numerical methodologies, Eq. (14) is not a polynomial equation but a transcendental one about ω^2. Actually, the number of solutions to this equation is infinite for every mode number n, that is, one can obtain innumerable natural frequencies from this equation.

4. Approximate solutions

Recently, Chen *et al.* utilized the DQ technique to solve the partial differential state equations based on the theory of elasticity for static and vibration problems of laminated composite beams and plates (Chen *et al.* 2003b, 2004, Chen and Lü 2005). In this section, the authors will apply this technique in the analysis of FGM beams coupled with the approximate laminate model.

4.1 Application of DQM

As it comes to a beam with other supporting conditions, say clamped or free end condition, it is rather difficult to seek an exact solution to Eq. (5). According to the main point of differential quadrature, an arbitrary partial differentiation about x of a continuous function $f(x, y)$ at a given point x_i can be approximated as a linear sum of weighted function values at all the discrete points in the domain of x. If N sampling points are adopted, one gets

$$\left.\frac{\partial^n f(x, y)}{\partial x^n}\right|_{x = x_i} = \sum_{j=1}^{N} g_{ij}^{(n)} f(x_j, y) \qquad (i = 1, 2, \ldots, N;\ n = 1, 2, \ldots, N-1) \tag{15}$$

where $g_{ij}^{(n)}$ are the weighted coefficients that can be thoroughly determined by the coordinates of all discrete points (Shu and Richards 1992).

By virtue of Eq. (15), one can obtain the discrete form of Eqs. (5) and (6) as

$$\frac{d}{dy}u_i = -\sum_{j=1}^{N} g_{ij}^{(1)} v_j + \frac{1}{c_{66}}\tau_i$$

$$\frac{d}{dy}\sigma_{yi} = -\rho\omega^2 v_i - \sum_{j=1}^{N} g_{ij}^{(1)}\tau_j$$

$$\frac{d}{dy}v_i = -\frac{c_{12}}{c_{22}}\sum_{j=1}^{N} g_{ij}^{(1)} u_j + \frac{1}{c_{22}}\sigma_{yi} \tag{16}$$

$$\frac{d}{dy}\tau_i = \left(\frac{c_{12}^2}{c_{22}} - c_{11}\right)\sum_{j=1}^{N} g_{ij}^{(2)} u_j - \rho\omega^2 u_i - \frac{c_{12}}{c_{22}}\sum_{j=1}^{N} g_{ij}^{(1)}\sigma_{yj}$$

$$\sigma_{xi} = \left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)\sum_{j=1}^{N} g_{ij}^{(1)} u_j + \frac{c_{12}}{c_{22}}\sigma_{yi} \tag{17}$$

where u_i, σ_{yi}, v_i, τ_i and σ_{xi} are continuous functions about the independent variable y at the given sampling point x_i.

It should be noted that, when $i = 1$ or $i = N$, the end conditions should be incorporated into the discrete state equation, Eq. (16) (Chen *et al.* 2003b). In this paper, three representative end conditions in Eq. (7) are considered. It can be seen that among the three types of boundary conditions, only $\sigma_x = 0$ is not expressed directly in terms of state variables, for which use should be made of Eq. (17) to get

$$\sigma_{yi} = -\frac{c_{22}}{c_{12}}\left(c_{11} - \frac{c_{12}^2}{c_{22}}\right)\sum_{j=1}^{N} g_{ij}^{(1)} u_j \qquad (i = 1 \text{ and } i = N) \tag{18}$$

Then the appropriate form of discrete state equation that is suitable for solving practical problem can be derived. With the incorporation of any type of end conditions in Eq. (7) into Eq. (16), we assemble all the discrete equations at all sampling points into a global one, and rewrite it in the following matrix form,

$$\frac{d}{dy}\boldsymbol{\delta}(y) = \mathbf{M}\boldsymbol{\delta}(y) \tag{19}$$

where $\boldsymbol{\delta}(y) = [\mathbf{u}^T\ \boldsymbol{\sigma}_y^T\ \mathbf{v}^T\ \boldsymbol{\tau}^T]^T$ is the state vector at an arbitrary coordinate of y, with the superscript T denoting the transpose of a matrix and $\mathbf{u}$, $\boldsymbol{\sigma}_y$, $\mathbf{v}$ and $\boldsymbol{\tau}$ column vectors composed of all unknown discrete state variables at y, and $\mathbf{M}$ is the global coefficient matrix.

4.2 Approximate laminate model and frequency equation

As stated earlier in this paper, the elastic constants c_{rs} (r, $s = 1, 2, 6$) and the mass density ρ in the coefficient matrix $\mathbf{M}$ are not constant; actually they can vary along the thickness direction in arbitrary ways. Thus, it is difficult to obtain the exact solution to Eq. (19). Hereby, we adopt the approximate laminate model (Wang *et al.* 1999, Ootao and Tanigawa 2000, Chen and Ding 2002) to transform Eq. (19) into the one with constant coefficients.

In detail, the FGM beam is divided into m layers (see Fig. 1), each with a sufficiently small thickness of $h_k = y_k - y_{k-1}$, in which y_k is the coordinate of the fictitious interface between the k-th and the $(k+1)$-th layers. In this circumstance, the material constants within the k-th layer can be assumed invariable, and their values at the mediate plane of that layer are to be taken, that is,

$$\begin{aligned} c_{rs}^{(k)} &= c_{rs}\big|_{y = (y_k + y_{k-1})/2} \qquad (r, s = 1, 2, 6) \\ \rho^{(k)} &= \rho\big|_{y = (y_k + y_{k-1})/2} \end{aligned} \tag{20}$$

for $k = 1, 2, \ldots, m$. Thus, the coefficient matrix $\mathbf{M}$ in Eq. (19) becomes constant within the k-th layer and is denoted as $\mathbf{M}_k$ hereafter. Once the beam is treated as a laminated one, the following relation between the state vectors $\delta(h)$ and $\delta(0)$ is then obtained similar to the procedure outlined in our previous papers (Chen *et al.* 2003b, Chen and Lü 2005),

$$\delta(h) = \mathbf{S}\delta(0) = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} & \mathbf{S}_{13} & \mathbf{S}_{14} \\ \mathbf{S}_{21} & \mathbf{S}_{22} & \mathbf{S}_{23} & \mathbf{S}_{24} \\ \mathbf{S}_{31} & \mathbf{S}_{32} & \mathbf{S}_{33} & \mathbf{S}_{34} \\ \mathbf{S}_{41} & \mathbf{S}_{42} & \mathbf{S}_{43} & \mathbf{S}_{44} \end{bmatrix} \begin{Bmatrix} \mathbf{u}(0) \\ \sigma_y(0) \\ \mathbf{v}(0) \\ \tau(0) \end{Bmatrix} \tag{21}$$

where $\mathbf{S} = \prod_{k=m}^{1} e^{h_k \mathbf{M}_k}$ is the global transfer matrix for the discrete state variables, and $\mathbf{S}_{ij}$ are the corresponding partitioned matrices. Similarly, for free vibration problem, we can derive the frequency equation as

$$\begin{vmatrix} \mathbf{S}_{21} & \mathbf{S}_{23} \\ \mathbf{S}_{41} & \mathbf{S}_{43} \end{vmatrix} = 0 \tag{22}$$

From this equation, the natural frequency of the beam can easily be calculated. Similarly, this equation is also transcendental with respect to ω^2, and can yield an infinite number of natural frequencies.

5. Numerical illustrations

It should be noted that the use of uniformly distributed discrete points in DQM usually could not precisely describe the deformation near the supporting ends, which leads to a bad convergence and accuracy when predicting higher-order natural frequencies (Bert and Malik 1996). Hence, the following unequally spaced sampling points in cosine pattern

$$x_i = \frac{1 - \cos[(i-1)\pi/(N-1)]}{2} l \qquad (i = 1, 2, \ldots, N) \tag{23}$$

are adopted, which is known as the Chebyshev-Gauss-Lobatto points (Sherbourne and Pandey 1991).

Now, we consider an S-S orthotropic FGM beam of *Type I*, for which a completely exact elasticity solution can be derived, as described in Section 3. This solution can serve as a benchmark for checking the validity of the present method. The gradient index is taken to be $\alpha_1 = 0.5$, and the

Table 1 Material constants

Material	Constants (c_{rs}^0 – 10^{10} N/m^2, ρ^0 – kg/m^3)
$\mathscr{A}$	$c_{11}^0 = 13.9,\ c_{12}^0 = 1.4,\ c_{22}^0 = 33.64,\ c_{66}^0 = 16.25,\ \rho^0 = 7.5\times10^3$
$\mathscr{B}$	$c_{11}^0 = 20.97,\ c_{12}^0 = 10.51,\ c_{22}^0 = 21.09,\ c_{66}^0 = 4.25,\ \rho^0 = 5.676\times10^3$

Table 2 Comparisons of Ω of the present method with the exact solution for an S-S FGM beam (*Type I*, $\alpha_1 = 0.5$)

$\frac{l}{h}$	Mode	$N = 11$	$N = 12$	$N = 13$	$N = 14$	$N = 15$	$N = 16$			Exact
		$m = 5$					$m = 4$	$m = 5$	$m = 6$	
15	1	0.011573	0.011573	0.011573	0.011573	0.011573	0.011573	0.011573	0.011573	0.011572
	2	0.045802	0.045802	0.045802	0.045802	0.045802	0.045802	0.045802	0.045802	0.045801
	3	0.101317	0.101319	0.101320	0.101320	0.101320	0.101320	0.101320	0.101320	0.101319
	4	0.176313	0.176104	0.176128	0.176130	0.176130	0.176129	0.176130	0.176131	0.176130
	5	*0.193298	*0.193298	*0.193298	*0.193298	*0.193298	*0.193299	*0.193298	*0.193297	*0.193297
	6	0.264284	0.268781	0.267747	0.267916	0.267914	0.267912	0.267914	0.267917	0.267916
	7	-	0.364344	0.377290	0.373560	0.374365	0.374290	0.374295	0.374299	0.374308
	8	-	*0.386593	*0.386593	*0.386593	*0.386593	*0.386594	*0.386593	*0.386591	*0.386591
	9	-	-	0.473157	0.501127	0.490590	0.493408	0.493418	0.493424	0.493076
	10	-	-	*0.579880	*0.579880	*0.579880	*0.579882	*0.579880	*0.579877	*0.579877
7	1	0.052471	0.052471	0.052471	0.052471	0.052471	0.052471	0.052471	0.052471	0.052470
	2	0.200726	0.200726	0.200726	0.200726	0.200726	0.200725	0.200726	0.200728	0.200727
	3	*0.414206	*0.414206	*0.414206	*0.414206	*0.414206	*0.414207	*0.414206	*0.414204	*0.414204
	4	0.423795	0.423803	0.423805	0.423805	0.423805	0.423798	0.423805	0.423810	0.423816
	5	0.700641	0.699899	0.699984	0.699992	0.699992	0.699973	0.699991	0.700002	0.700022
	6	*0.828374	*0.828375	*0.828375	*0.828375	*0.828375	*0.828377	*0.828375	*0.828370	*0.828370
	7	1.000475	1.015241	1.011850	1.012403	1.012397	1.012362	1.012398	1.012418	1.012457
	8	*1.242446	*1.242459	*1.242462	*1.242462	*1.242462	*1.242466	*1.242462	*1.242455	*1.242456
	9	-	1.318731	1.358519	1.347085	1.349553	1.349285	1.349340	1.349370	1.349458
	10	-	*1.656278	*1.645163	*1.656409	*1.656408	*1.656412	*1.656407	*1.656398	*1.656399

Note: '-' denotes the frequency not obtainable, and that marked with an asterisk correspond to the modes that longitudinal modes are more dominating.

material constants of material $\mathscr{A}$ listed in Table 1 are employed. The first ten non-dimensional frequencies $\Omega = \omega h\sqrt{(\rho^0/c_{66}^0)_{\mathscr{A}}}$ for different length-to-depth ratios are presented in Table 2, where m and N represent the layer number and sampling point number, respectively.

It can easily be seen that the results obtained by the present method, either for the slender thin or for the short thick beams, agree well with the exact ones. In particular, the difference between results for three different layer division schemes with a given sampling point number, say $N = 16$, is completely negligible. This illustrates that the present approximate laminate model has a very good convergence characteristics. It should be pointed out that although the analysis based on beam theories is simpler, it can only predict several lowest frequencies, which will further suffer from inaccuracy when the length-to-depth ratio decreases. On the other hand, the present method can predict any higher-order frequencies, from the theoretical point of view, by using large layer number and sampling point number.

The frequencies labeled by an asterisk in Table 2 correspond to those vibration modes in which the longitudinal displacement is more dominating (Further investigation based on Eqs. (3) and (4) show that pure longitudinal vibration can not exist due to the inhomogeneity of the beam). These frequencies usually can not be predicted by beam theories based on the bending hypothesis. The orders of those modes (referred to longitudinal mode hereafter for simplicity; but we should bear in mind that they are not purely longitudinal) are also different for different length-to-depth ratios. For example, the first longitudinal mode corresponds to the fifth frequency for $l/h = 15$, while it corresponds to the third frequency for $l/h = 7$.

Now we consider an orthotropic FGM beam of *Type II*, with κ_1^0 and κ_2^0 respectively corresponding to materials $\mathcal{A}$ and $\mathcal{B}$ listed in Table 1. The free vibrations of C-F and C-C beams are studied and the numerical results are tabulated in Tables 3 and 4. The gradient index is taken

Table 3 Non-dimensional frequency Ω of a C-F beam (*Type II*, $\alpha_2 = 1.0$)

Mode	$l/h = 15$			$l/h = 7$		
	$N = 15$, $m = 7$	$N = 15$, $m = 8$	$N = 16$, $m = 8$	$N = 15$, $m = 7$	$N = 15$, $m = 8$	$N = 16$, $m = 8$
1	0.00467	0.00467	0.00467	0.02127	0.02127	0.02132
2	0.02884	0.02883	0.02883	0.12529	0.12525	0.12527
3	0.07898	0.07895	0.07896	*0.23573	*0.23571	*0.23576
4	*0.10990	*0.10989	*0.10990	0.32341	0.32332	0.32336
5	0.15019	0.15014	0.15015	0.57609	0.57596	0.57597
6	0.23951	0.23944	0.23949	*0.70638	*0.70633	*0.70633
7	*0.32961	*0.32959	*0.32961	0.86432	0.86415	0.86427
8	0.34414	0.34404	0.34374	1.16500	1.16483	1.16456
9	0.46251	0.46240	0.46093	*1.18456	*1.18446	*1.18414
10	*0.54902	*0.54898	*0.54905	1.50292	1.50266	1.49682

Note: Frequencies marked with an asterisk correspond to the modes that longitudinal modes are more dominating

Table 4 Non-dimensional frequency Ω of a C-C beam (*Type II*, $\alpha_2 = 1.0$)

Mode	$l/h = 15$			$l/h = 7$		
	$N = 15$, $m = 7$	$N = 15$, $m = 8$	$N = 16$, $m = 8$	$N = 15$, $m = 7$	$N = 15$, $m = 8$	$N = 16$, $m = 8$
1	0.02927	0.02928	0.02922	0.12676	0.12673	0.12638
2	0.07882	0.07886	0.07864	0.31930	0.31922	0.31839
3	0.14981	0.14988	0.14951	*0.47268	*0.47265	*0.47191
4	*0.22321	*0.22320	*0.22054	0.56825	0.56814	0.56747
5	0.23945	0.23953	0.23843	0.85334	0.85321	0.85145
6	0.34320	0.34332	0.34243	*0.94347	*0.94341	*0.94247
7	*0.44362	*0.44365	*0.44116	1.16092	1.16077	1.16038
8	0.45820	0.45835	0.45853	*1.41044	*1.41034	*1.40835
9	0.58425	0.58442	0.58288	1.47977	1.47960	1.48428
10	*0.66684	*0.66687	*0.66099	1.80176	1.80156	1.80282

Note: Frequencies marked with an asterisk correspond to the modes that longitudinal modes are more dominating.

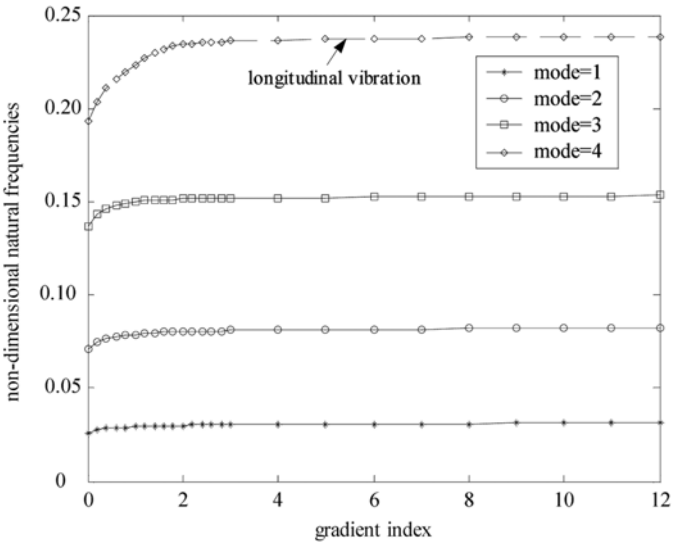

Fig. 2 Effect of gradient index α_2 on non-dimensional frequency Ω of a C-C beam (*Type II*, $l/h = 15$)

to be $\alpha_2 = 1.0$. From the tables, it is seen that although the convergence of our method behaves slightly differently for various parameters (end condition, length-to-depth ratio, layer number as well as sampling point number), the results are very accurate even for the high modes, from the engineering point of view. Further numerical investigations show that although for the beam with the same end conditions, yet different material properties result in different convergence behaviors of the present method. It can be expected that when the material property varies sharply along the thickness direction, then more separate layers are needed to yield an accurate result. It is known from beam theory that the fastener the constraint of the beam, the higher the frequency will be. The present method also predicts this character as shown in Tables 3 and 4. We also note that the order of the longitudinal modes further depends on the end conditions as well as the material properties.

Fig. 2 shows the curves of the first four non-dimensional frequencies Ω versus the gradient index α_2, of a C-C beam with $l/h = 15$. The layer number and sampling point number are taken to be $N = 15$ and $m = 8$, respectively. From the figure, it is seen that Ω increases with the increasing of α_2 for all the modes in consideration. Furthermore, the variation of Ω with α_2 is quite acute when $\alpha_2 \leq 2$, while it becomes gentle when $\alpha_2 > 2$. Note that when $\alpha_2 = 0$ the beam is a homogenous one made of material $\mathcal{A}$, while it becomes the one made of material $\mathcal{B}$ when $\alpha_2 \to \infty$. Thus the frequency will gradually approaches the one of a homogeneous beam made of material B when α_2 becomes larger and larger.

Fig. 3 displays the curves of the first four non-dimensional frequencies Ω of a C-C beam with $\alpha_2 = 1.0$ versus the length-to-depth ratio l/h. The layer number and sampling point number are the same as that taken in the above example. It is shown that the decreasing of Ω is significant when $l/h \leq 40$, but when $l/h > 40$ all curves tend to be horizontal indicating that Ω becomes almost invariant. This particular phenomenon for the slender thin FGM beam agrees with that predicated by the classical beam theory for homogeneous beams.

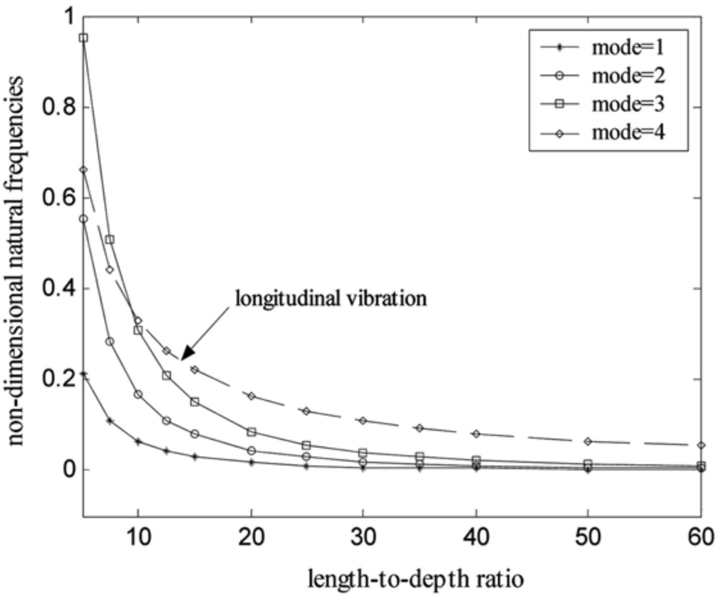

Fig. 3 Effect of length-to-depth ratio (l/h) on non-dimensional frequency Ω of a C-C beam (*Type II*, $\alpha_2 = 1.0$)

6. Conclusions

In this paper, the recently developed SSDQM (Chen *et al*. 2003b) is combined with the approximate laminate model to study the free vibration of FGM beams directly based on the orthotropic elasticity theory of plane stress problem. The validity and efficiency of the current methodology is clarified by considering several numerical examples. The following advantages of the present method can be concluded:

(1) By comparing the numerical results to the exact ones for an S-S FGM beam with material constants varying exponentially through the thickness coordinate, it is seen that the present method has a perfect performance even for higher vibration modes of thick beams. The present method can deal with arbitrary end conditions, breaking through the limitation of the conventional state-space approach, and can provide a benchmark for clarifying various beam theories or numerical methods.

(2) The employment of approximate laminate model makes it possible for us to deal with arbitrary material inhomogeneity easily. The thickness of each layer also can be different to match the variations of the material constants. In particular, in the area where the material constants vary rapidly, the layers should be sufficiently thin. It is obvious that when the number of layers increases, the solution based on the laminate model will gradually approach the solution of the original FGM beam.

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