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An analytical solution of the annular plate on elastic foundation

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Abstract. A new method for deriving analytical solution of the annular elastic plate on elastic foundation under axisymmetric loading is presented. The formulation is based on application of Hankel integral transforms and Bessel functions' properties in the corresponding boundary-value problem. A representative example is studied and the obtained solution is compared with published numerical results indicating excellent agreement.

Key words: annular plate; elastic foundation; Hankel transforms.

1. Introduction

The model of the plate on elastic support is used in a wide range of civil or mechanical engineering problems, such as tanks' or silos' foundations, aerospace engineering, building infrastructures etc. The reaction of the foundation in these problems can be considered as a linear function of the plate's deflection w at each point. These problems often belong to the following type of boundary-value problems:

$$Lw(x, y) + p(x, y)w(x, y) = q(x, y), \quad a_1 \le x \le b_1, a_2 \le y \le b_2 \tag{1}$$

Especially, for simply supported or fixed circular (or annular) plates with axisymmetric loading, where $w = w(r)$, $p = p(r)$, $q = q(r)$, the boundary conditions have the following form:

$$w(r_i) = c_1 \quad \text{and} \quad w'(r_i) = c_2 \quad \text{or} \quad w''(r_i) = c_3 \tag{2a}$$

$$w(r_o) = d_1 \quad \text{and} \quad w'(r_o) = d_2 \quad \text{or} \quad w''(r_o) = d_3 \tag{2b}$$

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where L is a fourth order linear differential operator, $p(r)$ and $q(r)$ are variable coefficients, $w(r)$ is the plate's vertical deflection and r_i, r_o are the inner and outer radius of the annular plate respectively. The analytical solution of the above problem cannot be found for all $p(r)$ and $q(r)$ (e.g. Papakaliatakis and Simos 1997). Numerical procedures to solve such boundary-value problems are mostly based on finite elements (Cheung and Zienkiewicz 1965), finite differences (Long and Alturi 2002, Krysl and Belytschko 1995) or meshless methods (Van Daele *et al.* 1994, Melerski 1989). An interesting hybrid procedure combining finite elements and analytical method to analyze annular plate-soil interaction is presented by Chandrashekhara and Antony (1997). Hankel integral transforms have also been used by Wang and Ishikawa (2001) to analyze thick or multi-layered plates resting on rigid foundation. An alternative numerical procedure for the thin circular plate on elastic foundation developed by Utku *et al.* (2000) represents the considered plate as a series of simply supported annular plates resting on support springs along their common edges and obtains the stiffness coefficients by the classical thin plate theory.

Analytical solutions of the Eq. (1) have been formulated e.g. by Timoshenko and Woinowsky (1959) in terms of infinite series. However, they are limited to the case of a free infinite circular plate on an elastic foundation loaded by a concentrated load acting on the center. Concerning the rectangular plates, the method of double series representation (Navier method) and the method of single series representation (Levy method) (e.g. Naruoka 1981) can be used in order to solve the basic differential equation $\nabla^2\nabla^2 w(x,y) = p(x,y)/D$. Although extension of these solutions to a variety cases of loading types and boundary conditions is possible, it depends on each case.

In the present work a unified analytical solution based on Hankel integral transforms and Bessel functions' properties is presented to solve annular elastic plates on elastic foundation. Since most of the generalized functions (e.g. δ-function and Heaviside function) and several typical algebraic functions are Hankel integral transformable, the solution of an infinite plate on an elastic foundation is performed herein in closed form for several axisymmetric loading types (e.g. step wised loading, concentrated line loading, exponential loading, diminishing harmonic loading etc.). Using the derived analytical solution for an infinite plate, a technique based on the superposition of solutions which produce the real boundary conditions at the end points of a finite annular plate embedded within an infinite plate on elastic foundation is used to solve relevant boundary-value problems. To this end, the basic differential equation is transformed into a linear algebraic system incorporating the real boundary conditions. The required solutions are obtained using inverse Hankel transforms.

2. Formulation of the problem

An annular elastic plate with thickness t, inner radius r_i, and outer radius r_o rests on Winkler type foundation (Fig. 1). The plate is loaded by the axisymmetric loading $q(r)$ and by the foundation reaction $q^*(r)$ assumed to be proportional to the vertical deflection $w(r)$ of the plate, i.e.,

$$q^*(r) = k_s w(r) \tag{3}$$

where k_s is the modulus of the Winkler foundation.

It is well known that the basic differential equation describing the circular plate on elastic foundation under axisymmetric loading has the form:

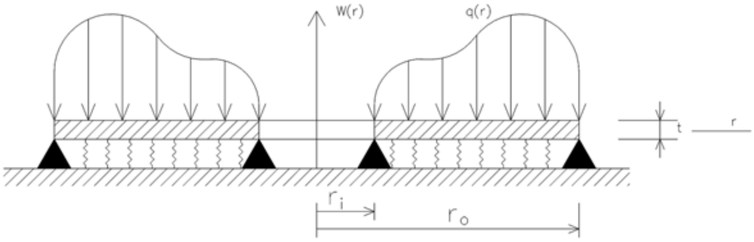


Fig. 1 Annular plate on elastic foundation under axisymmetric loading

$$D\nabla^4 w(r) = q(r) - q^*(r) \tag{4}$$

where ∇^4 is the differential operator given by

$$\nabla^4 w(r) = \nabla^2\nabla^2 w(r) = \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2 w(r)}{dr^2} + \frac{1}{r}\frac{dw(r)}{dr}\right] \tag{5}$$

and D is the flexural rigidity of the plate depending on the modulus of elasticity E and Poisson ratio ν:

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{6}$$

At the locations of inner and outer radius, the annular plate shown in Fig. 1 is simply supported. Then, the boundary conditions of the problem can be written as:

$$w(r_i) = 0 \tag{7}$$

$$M_{rr}(r_i) = 0 \tag{8}$$

$$w(r_o) = 0 \tag{9}$$

$$M_{rr}(r_o) = 0 \tag{10}$$

where M_{rr} is the bending moment in the direction $r - r$.

3. Analytical solution

3.1 Solution for the infinite circular plate on elastic foundation under arbitrary load q(r) lying on the infinite area 0 < r < ∞

To solve the differential Eq. (4), the Hankel integral transform will be used. The definition for this transform and its inverse form employed in the proposed solution is written:

$$\bar{f}_n(\xi) = H_n\{f(r);\xi\} = \int_0^\infty r f(r) J_n(\xi r)dr \tag{11}$$

and

$$f(r) = H_n^{-1}\{\bar{f}_n(\xi);r\} = \int_0^\infty \xi \bar{f}_n(\xi) J_n(\xi r) d\xi \tag{12}$$

where $J_n(x)$ is the n-th Bessel function and $H_n\{f(r);\xi\}$, $H_n^{-1}\{\bar{f}_n(\xi);r\}$ are the Hankel and inverse Hankel transform operator, respectively.

Taking the operator H_0 to Eq. (4) it can be written:

$$H_0\{\nabla^4 w(r);\xi\} + \frac{k_s}{D} H_0\{w(r);\xi\} = \frac{1}{D} H_0\{q(r);\xi\} \tag{13}$$

Considering the substitution

$$\nabla^2 w(r) = f(r) \tag{14}$$

the transformation $H_0\{\nabla^4 w(r);\xi\}$ can be written:

$$H_0\{\nabla^4 w(r);\xi\} = H_0\{\nabla^2 f(r);\xi\} \tag{15}$$

or

$$H_0\{\nabla^4 w(r);\xi\} = H_0\left\{\frac{d^2 f(r)}{dr^2} + \frac{1}{r}\frac{df(r)}{dr};\xi\right\} \tag{16}$$

According to Sneddon (1972a), the following property of the Hankel transform can be used:

$$H_n\{B_n f;\xi\} = -\xi^2 H_n\{f;\xi\} \tag{17}$$

where

$$B_n = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} \tag{18}$$

Taking into account Eqs. (17), (18) and putting $n = 0$, it can be written:

$$H_0\{\nabla^2 f(r);\xi\} = -\xi^2 H_0\{f(r);\xi\} \tag{19}$$

In the above equation, if $w(r)$ is inserted instead of $f(r)$ the following form can be obtained:

$$H_0\{\nabla^2 w(r);\xi\} = -\xi^2 H_0\{w(r);\xi\} \tag{20}$$

Considering Eq. (14), the Eq. (19) takes the form:

$$H_0\{\nabla^2\nabla^2 w(r);\xi\} = -\xi^2 H_0\{\nabla^2 w(r);\xi\} \tag{21}$$

This equation with the aid of Eq. (20) results:

$$H_0\{\nabla^4 w(r);\xi\} = \xi^4 H_0\{w(r);\xi\} \tag{22}$$

Then, Eq. (13) gives:

$$\xi^4 H_0\{w(r);\xi\} + \frac{k_s}{D} H_0\{w(r);\xi\} = \frac{1}{D} H_0\{q(r);\xi\} \tag{23}$$

or

$$H_0\{w(r);\xi\} = \frac{\bar{q}_0(\xi)}{\xi^4 + \lambda} \tag{24}$$

where

$$\bar{q}_0(\xi) = \frac{1}{D} H_0\{q(r);\xi\} \tag{25}$$

$$\lambda = \frac{k_s}{D} \tag{26}$$

Considering Eq. (24), the analytical solution of the differential Eq. (4) can be written:

$$w(r) = H_0^{-1}\left\{\frac{\bar{q}_0(\xi)}{\xi^4 + \lambda};r\right\} \tag{27}$$

Eq. (27) represents the deflection distribution of an infinite circular plate on elastic foundation under arbitrary axisymmetric loading $q(r)$ lying on the infinite area $0 < r < \infty$

3.2 Solution of infinite circular plate on elastic foundation under step wised distributed loading q_o lying in the finite area $r_i < r < r_0$

For the case of uniform loading q_o lying in the finite area $r_i < r < r_0$, the function $q(r)$ can be written:

$$q(r) = q_0(H(r - r_i) - H(r - r_0)) \tag{28}$$

where $H(r - a)$ is the Heaviside function. To determine the Hankel transform for this loading case, the following property (Sneddon 1995b) is used:

$$H_0\{H(r - a);\xi\} = J_1(\xi a) \tag{29}$$

Then, according to Eqs. (28), (29), the Eq. (25) can be written:

$$\bar{q}_0(\xi) = \frac{1}{D} q_0(J_1(\xi r_i) - J_1(\xi r_0)) \tag{30}$$

and Eq. (27) takes the form:

$$w^{q_0}(r) = \frac{q_0}{D} H_0^{-1}\left\{\frac{J_1(\xi r_i) - J_1(\xi r_0)}{\xi^4 + \lambda};r\right\} \tag{31}$$

or

$$w^{q_0}(r) = \frac{q_0}{D} H_0^{-1}\{f_1(\xi);r\} \tag{32}$$

where

$$f_1(\xi) = \frac{J_1(\xi r_i) - J_1(\xi r_0)}{\xi^4 + \lambda} \tag{33}$$

Consequently, the distribution of bending moment $M_{rr}^{q_0}(r)$ due to q_o can be obtained:

$$M_{rr}^{q_0}(r) = -D\left(\nu \frac{1}{r}\frac{dw^{q_0}(r)}{dr} + \frac{d^2 w^{q_0}(r)}{dr^2}\right) \tag{34}$$

Taking into account the formula

$$\frac{dJ_p(x)}{dx} = \frac{1}{x}(pJ_p(x) - xJ_{p+1}(x)) \tag{35}$$

it can be written:

$$M_{rr}^{q_0}(r) = q_0\left((\nu+1)\frac{1}{r}H_1^{-1}\{\xi f_1(\xi);r\} - H_2^{-1}\{\xi^2 f_1(\xi);r\}\right) \tag{36}$$

3.3 Solution of infinite circular plate on elastic foundation under concentrated line load P acting along a circle with radius r = a

To derive the solution of the infinite circular plate on elastic foundation under a concentrated line load P acting along a circle with radius $r = \alpha$, the uniform distributed load q^* lying in the area $a < r < a + \varepsilon$ is considered (Fig. 2). By the equilibrium between P and q^* it can be written:

$$q^* \pi[(a+\varepsilon)^2 - a^2] = 2\pi r_m P \tag{37}$$

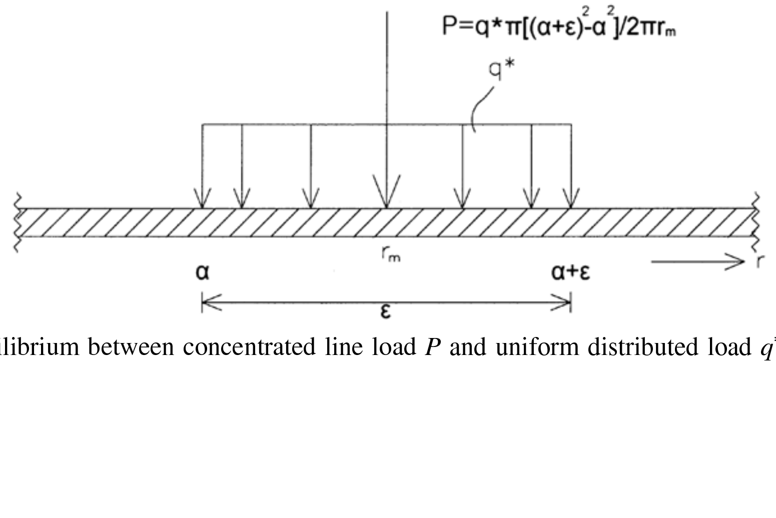


Fig. 2 Equilibrium between concentrated line load P and uniform distributed load q^* for $\varepsilon \to 0$

where P is a concentrated line load acting along a circle with radius

$$r_m = a + \frac{\varepsilon}{2} \tag{38}$$

Then

$$q^*(r) = \frac{2\pi(a + \varepsilon/2)P}{\pi((a + \varepsilon)^2 - a^2)}[H(r-a) - H(r-(a+\varepsilon))] \tag{39}$$

According to Eq. (25) it follows:

$$\bar{q}_0{}^*(\xi) = \frac{1}{D}H_0\{q^*(r);\xi\} = \frac{2(a + \varepsilon/2)P}{D((a+\varepsilon)^2 - a^2)}[J_1(\xi a) - J_1(\xi(a+\varepsilon))] \tag{40}$$

Considering the solution (27) and Eq. (40), the deflection distribution $w^P(r)$ due to a concentrated load P acting along a circle with radius $r = \alpha$ can be obtained:

$$w^P(r) = H_0^{-1}\left\{\frac{2P}{D}\lim_{\varepsilon\to 0}\frac{a+\varepsilon/2}{(a+\varepsilon)^2 - a^2}\frac{J_1(\xi a) - J_1(\xi(a+\varepsilon))}{\xi^4 + \lambda};r\right\} \tag{41}$$

or

$$w^P(r) = \frac{P}{2D}H_0^{-1}\left\{\frac{\xi}{\xi^4 + \lambda}(J_2(\xi a) - J_0(\xi a));r\right\} \tag{42}$$

With the aid of Eqs. (34), (35) the corresponding bending moment $M_{rr}^P(r)$ can be obtained:

$$M_{rr}^P(r) = \frac{P}{\pi}\left((\nu+1)\frac{1}{r}H_1^{-1}\{\xi f_2(\xi);r\} - H_2^{-1}\{\xi^2 f_2(\xi);r\}\right) \tag{43}$$

where

$$f_2(\xi) = \frac{\xi}{\xi^4 + \lambda}\left(\frac{J_2(\xi a) - J_0(\xi a)}{4a}\right) \tag{44}$$

3.4 Solution of infinite plate on an elastic foundation under exponential and diminishing harmonic loading types

For the case of an exponentially distributed loading described by the expression

$$q(r) = e^{-q_o r} \tag{45}$$

the function $\bar{q}_o(\xi)$ given by the Eq. (25) can be determined with the aid of a relevant table of Hankel transforms obtained by Sneddon (1995b). According to this reference it can be written:

$$\bar{q}_o(\xi) = \frac{q_o}{D}(\xi^2 + q_o^2)^{-3/2} \tag{46}$$

In similar way, when the loading distribution along a plate is described by a diminishing

Table 1 Solution of the infinite plate on elastic foundation under several types of axisymmetric loadings

Loading type	Loading function	Analytical solution
Arbitrary axisymmetric loading	$q(r)$	$w(r) = H_0^{-1}\left\{\frac{\overline{q}_0(\xi)}{\xi^4+\lambda};r\right\}$ where $\overline{q}_0(\xi) = \frac{1}{D}H_0\{q(r);\xi\}$
Step wised loading	$q(r) = \begin{cases} q_0 & r_i<r<r_0 \\ 0 & r<r_i \text{ and } r>r_0 \end{cases}$	$w(r) = \frac{q_0}{D}H_0^{-1}\left\{\frac{J_1(\xi r_i)-J_1(\xi r_0)}{\xi^4+\lambda};r\right\}$
Concentrated line loading	$q(r) = \begin{cases} P & r=a \\ 0 & r\neq a \end{cases}$	$w(r) = \frac{P}{2D}H_0^{-1}\left\{\frac{\xi(J_2(\xi a)-J_0(\xi a))}{\xi^4+\lambda};r\right\}$
Exponential loading	$q(r) = e^{-q_0 r}$	$w(r) = \frac{q_0}{D}H_0^{-1}\left\{\frac{(\xi^2+q_0^2)^{-3/2}}{\xi^4+\lambda};r\right\}$
Diminishing harmonic loading	$q(r) = \frac{\sin(q_0 r)}{r}$	$w(r) = \frac{1}{D}\int_0^{q_0}\xi\frac{(q_0^2-\xi^2)^{-1/2}}{\xi^4+\lambda}J_0(\xi r)d\xi$

harmonic function of the type

$$q(r) = \frac{\sin(q_o r)}{r} \tag{47}$$

the corresponding function $\overline{q}_o(\xi)$ can be obtained:

$$\overline{q}_o(\xi) = \frac{1}{D}\begin{cases} (q_o^2-\xi^2)^{-1/2} & 0<\xi<q_o \\ 0 & \xi>q_0 \end{cases} \tag{48}$$

Then, the solution of the differential Eq. (4) for the above loading types can be derived by the Eq. (27). Table 1 summarizes the solution of a plate on an elastic foundation under axisymmetric loading for the above described loading types. With the aid of the Table 1, results of typical examples of the above cases are shown in Figs. 3(a)-(d).

3.5 Solution of the boundary-value problem

To solve the boundary-value problem of the simply supported annular plate on elastic foundation under uniform loading $q(r) = q_o$, the boundary conditions given by the Eqs. (7)-(10) must be satisfied. Firstly we calculate the values of four suitable loads P_i (i = 1, …, 4) acting on an infinite circular plate, which produce these boundary conditions (Fig. 4). The loads P_1, P_3 act on the boundaries r_i, r_o of the prospective annular plate respectively, while the loads P_2, P_4 act outside, on the corresponding locations $(r_i - \delta)$ and $(r_o + \delta)$. The distance δ has an arbitrary value lying in the

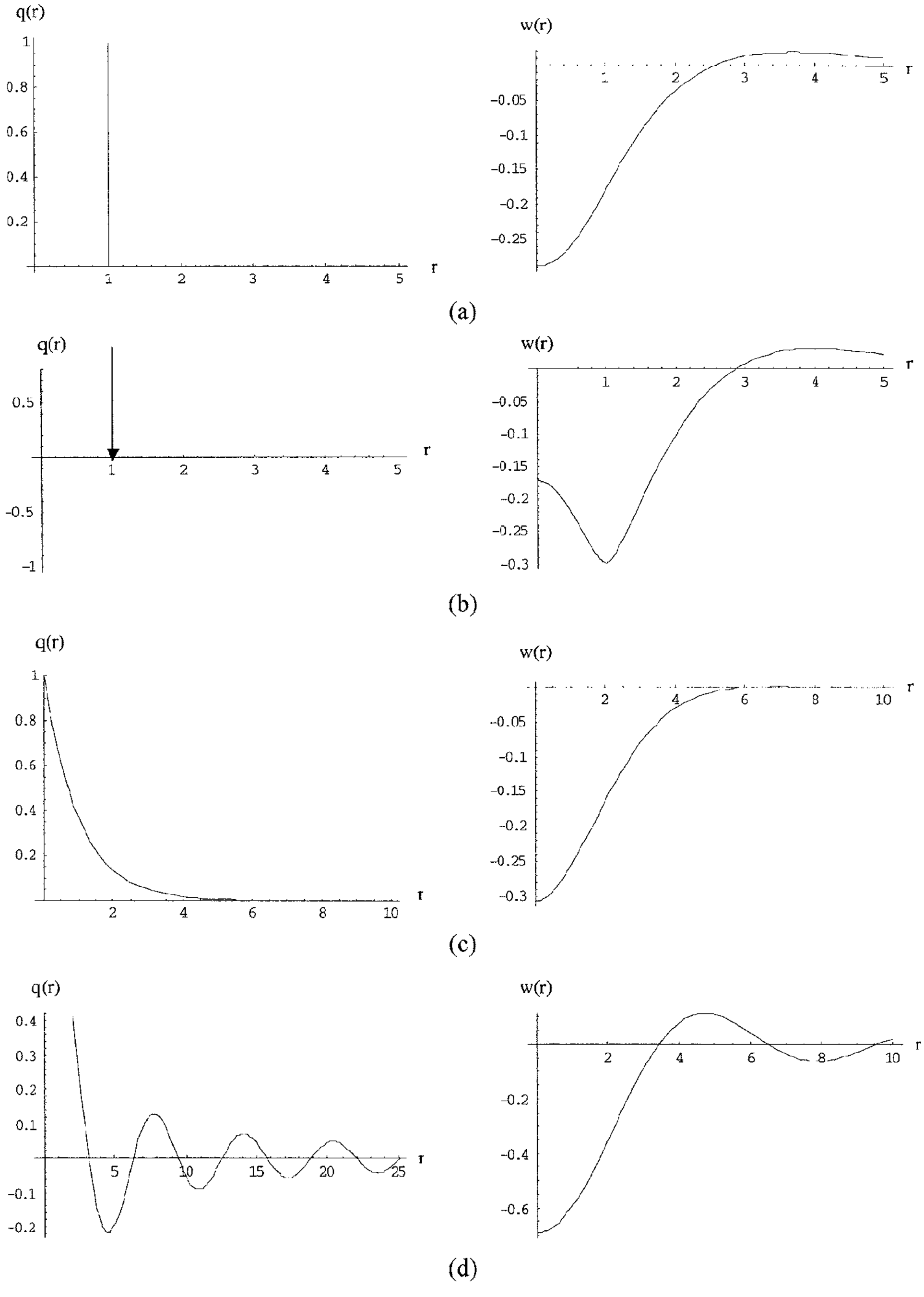


Fig. 4 Solution $w(r)$ of the infinite plate on elastic foundation under the following types of axisymmetric loading $q(r)$: (a) Step wised loading q_o acting in the area $0 < q_o < 1$ (example: $q_o = 1$, $\lambda = 1$, $D = 1$), (b) Concentrated line loading P acting on $r = 1$ (example: $P = 1$, $\lambda = 1$, $D = 1$), (c) Exponential loading $q(r) = e^{-q_o r}$ (example: $q_o = 1$, $\lambda = 1$, $D = 1$), (d) Diminishing harmonic loading $q(r) = \dfrac{\sin(q_o r)}{r}$ (example: $q_o = 1$, $\lambda = 1$, $D = 1$)

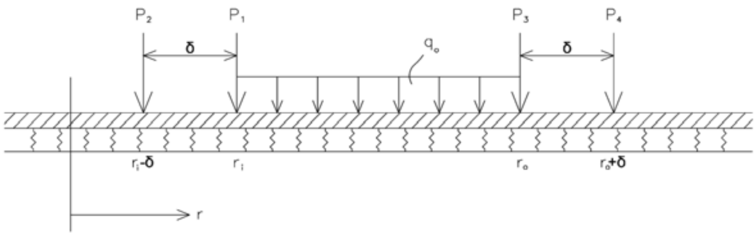


Fig. 3 Action of suitable loads P_i (i = 1, …, 4) on an infinite plate on elastic foundation in order to satisfy the boundary conditions of the corresponding annular plate

area $0 < \delta < r_i$ which can be taken $\delta = r_i/2$. However, the value of δ is not important (Oztorun 2002). Taking into account Eqs. (7)-(10) it can be written:

$$w^{q_0}(r_i) + w^{P_1}(r_i) + w^{P_2}(r_i) + w^{P_3}(r_i) + w^{P_4}(r_i) = 0 \tag{49}$$

$$w^{q_0}(r_o) + w^{P_1}(r_o) + w^{P_2}(r_o) + w^{P_3}(r_o) + w^{P_4}(r_o) = 0 \tag{50}$$

$$M_{rr}^{q_0}(r_i) + M_{rr}^{P_1}(r_i) + M_{rr}^{P_2}(r_i) + M_{rr}^{P_3}(r_i) + M_{rr}^{P_4}(r_i) = 0 \tag{51}$$

$$M_{rr}^{q_0}(r_o) + M_{rr}^{P_1}(r_o) + M_{rr}^{P_2}(r_o) + M_{rr}^{P_3}(r_o) + M_{rr}^{P_4}(r_o) = 0 \tag{52}$$

The algebraic linear system given by Eqs. (49)-(52) leads to the suitable values P_i ($i = 1, \ldots, 4$) which produce the boundary conditions of the real simply supported annular plate on elastic foundation considered to be embedded within the corresponding infinite plate. Similar procedure is followed in the method of boundary elements (e.g. Pavlou 2002). Then, the deflection $w(r)$ and the bending moment $M_{rr}(r)$ at any point of this annular plate can be calculated:

$$w(r) = w^{q_0}(r) + w^{P_1}(r) + w^{P_2}(r) + w^{P_3}(r) + w^{P_4}(r) \tag{53}$$

$$M(r) = M_{rr}^{q_0}(r) + M_{rr}^{P_1}(r) + M_{rr}^{P_2}(r) + M_{rr}^{P_3}(r) + M_{rr}^{P_4}(r) \tag{54}$$

4. Verification of the method in a representative example

With the aid of Eqs. (31), (34), (42), (43) the linear algebraic system of Eqs. (49)-(52) takes the form:

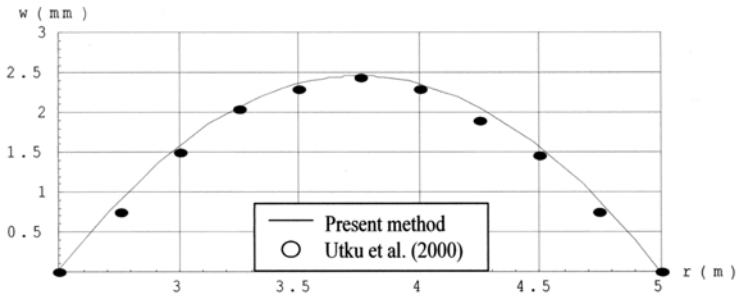
$$\sum_{j=1}^{4} I_{ij} P_j = b_i q_0 \qquad (i = 1, \ldots, 4) \tag{55}$$

where the integrals I_{ij} are given in appendix I. Then, Eqs. (53), (54) can be written:

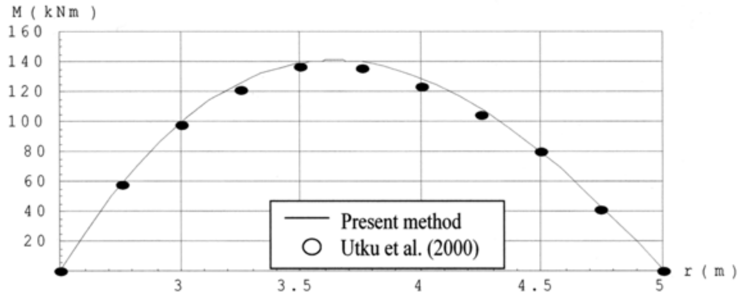
$$D \cdot w(r) = [G_1 \ \ G_2 \ \ G_3 \ \ G_4 \ \ G_5][q_0 \ \ P_1 \ \ P_2 \ \ P_3 \ \ P_4]^T \tag{56}$$

$$M_{rr}(r) = [T_1 \ \ T_2 \ \ T_3 \ \ T_4 \ \ T_5][q_0 \ \ P_1 \ \ P_2 \ \ P_3 \ \ P_4]^T \tag{57}$$

where the functions G_i, T_i (i = 1, …, 5) are given in appendix II.



(a)



(b)

Fig. 5 (a) Comparison of the deflection distribution of annular plate on elastic foundation with results taken by Utku *et al.* (2000), (b) Comparison of the bending moment distribution of annular plate on elastic foundation with taken by Utku *et al.* (2000)

To verify the above procedure, data of a representative problem obtained in the literature (Utku *et al.* 2000) are used. The geometrical parameters of the annular plate under consideration have the values $r_i = 2.5$ m, $r_o = 5.0$ m, $t = 0.25$ m. This plate rests on a subgrade with $k_s = 10000$ kN/m^3, while the uniform distribured load has the value $q_0 = 20$ kN/m^2. The material properties of the plate are $E = 2.7 \times 10^7$ kN/m^2 and $\nu = 0.2$. For the parameter δ, the value $\delta = 1$ is used. The coefficients I_{ij} of the algebraic system (55) are obtained by integrations carried out by the commercial program Mathematica (Wolfram Research Europe Ltd. 2000). Taking into account the described procedure, the deflection and bending moment distribution along the above annular plate are obtained by Eqs. (56), (57). Fig. 5 shows the comparison of the derived results with results obtained by Utku *et al.* (2000), indicating excellent agreement.

To investigate the influence of the parameter δ on the results, the deflection in the middle $r = (2.5 + 5.0)/2 = 3.75$ m of the annular plate is calculated for several values of δ. The obtained error (%) derived by the comparison of the proposed method's results with the results of Utku *et al.* (2000) is presented in Fig. 6. This figure indicates that the error is very small for the values of $\delta > 0$. When $\delta \to 0$ the location of the load P_2 converges to the location of the load P_1 and the location of the load P_4 converges to the location of the load P_3. In this case, the required number of four independed loads (suitable to produce the four boundary conditions) is reduced to the number of two independed loads, increasing thus the error rapidly. From mathematical point of view, when δ tends to zero, the members of the pairs (I_{i1}, I_{i2}), (I_{i1}, I_{i2}) of the coefficients of the linear system as well as the members of the pairs (G_2, G_3), (G_4, G_5), (T_2, T_3), (T_4, T_5) of Eqs. (52), (53) tend to be identical, leading thus to numerical discrepancies.

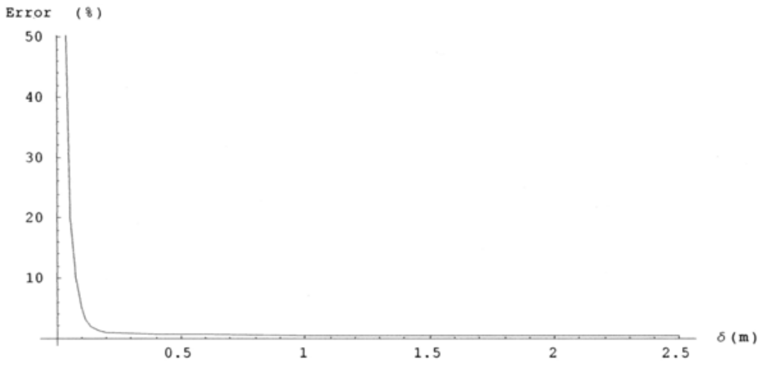


Fig. 6 Sensitivity of the obtained results by the parameter δ

5. Conclusions

A new analytical method based on Hankel integral transform and Bessel functions' properties was derived to solve the problem of annular plate on elastic foundation. For this purpose, the fundamental solutions of the infinite plate on elastic foundation under the action of several loading types were obtained. Advantages of the proposed solution are: (a) provides a unified expression of any axisymmetric loading type $q(r)$ (see Eq. (27)), and (b) although it needs numerical integration tools for the integral calculations, the solution has a closed analytical form (see Table 1). However, the method has two main limitations: (a) the loading functions $q(r)$ must be Hankel integral transformable, and (b) in some cases, it is difficult to calculate the integrals of the type $\int_0^{\infty} \xi \overline{q}(\xi) J_0(\xi r) d\xi$. In most cases this task requires numerical procedures.

In order to solve a typical boundary-value problem, the values of four suitable loads acting on the boundaries and outside the boundaries of a prospective annular plate, embedded within the infinite plate, were determined to produce the real boundary conditions. The sensitivity of the solution by the location of above suitable loads was investigated. The derived results compared with existing numerical results obtained by the literature, were in excellent agreement.

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Appendix I

$$I_{11} = \frac{1}{2D}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}[J_2(\xi r_i) - J_0(\xi r_i)]J_0(\xi r_i)d\xi$$

$$I_{12} = \frac{1}{2D}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}[J_2(\xi(r_i-\delta)) - J_0(\xi(r_i-\delta))]J_0(\xi r_i)d\xi$$

$$I_{13} = \frac{1}{2D}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}[J_2(\xi r_0) - J_0(\xi r_0)]J_0(\xi r_i)d\xi$$

$$I_{14} = \frac{1}{2D}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}[J_2(\xi(r_0+\delta)) - J_0(\xi(r_0+\delta))]J_0(\xi r_i)d\xi$$

$$I_{21} = \frac{1}{2D}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}[J_2(\xi r_i) - J_0(\xi r_i)]J_0(\xi r_0)d\xi$$

$$I_{22} = \frac{1}{2D}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}[J_2(\xi(r_i-\delta)) - J_0(\xi(r_i-\delta))]J_0(\xi r_0)d\xi$$

$$I_{23} = \frac{1}{2D}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}[J_2(\xi r_0) - J_0(\xi r_0)]J_0(\xi r_0)d\xi$$

$$I_{24} = \frac{1}{2D}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}[J_2(\xi(r_0+\delta)) - J_0(\xi(r_0+\delta))]J_0(\xi r_0)d\xi$$

$$I_{31} = \frac{1}{\pi}\left[(\nu+1)\frac{1}{r_i}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}\frac{J_2(\xi r_i) - J_0(\xi r_i)}{4r_i}J_1(\xi r_i)d\xi - \int_0^\infty \frac{\xi^3}{\xi^4+\lambda}\frac{J_2(\xi r_i) - J_0(\xi r_i)}{4r_i}J_2(\xi r_i)d\xi\right]$$

$$I_{32} = \frac{1}{\pi}\left[(\nu+1)\frac{1}{r_i}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}\frac{J_2(\xi(r_i-\delta))-J_0(\xi(r_i-\delta))}{4(r_i-\delta)}J_1(\xi r_i)d\xi - \int_0^\infty \frac{\xi^3}{\xi^4+\lambda}\frac{J_2(\xi(r_i-\delta))-J_0(\xi(r_i-\delta))}{4(r_i-\delta)}J_2(\xi r_i)d\xi\right]$$

$$I_{33} = \frac{1}{\pi}\left[(\nu+1)\frac{1}{r_i}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}\frac{J_2(\xi r_0)-J_0(\xi r_0)}{4r_0}J_1(\xi r_i)d\xi - \int_0^\infty \frac{\xi^3}{\xi^4+\lambda}\frac{J_2(\xi r_0)-J_0(\xi r_0)}{4r_0}J_2(\xi r_i)d\xi\right]$$

$$I_{34} = \frac{1}{\pi}\left[(\nu+1)\frac{1}{r_i}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}\frac{J_2(\xi(r_0+\delta))-J_0(\xi(r_0+\delta))}{4(r_0+\delta)}J_1(\xi r_i)d\xi - \int_0^\infty \frac{\xi^3}{\xi^4+\lambda}\frac{J_2(\xi(r_0+\delta))-J_0(\xi(r_0+\delta))}{4(r_0+\delta)}J_2(\xi r_i)d\xi\right]$$

$$I_{41} = \frac{1}{\pi}\left[(\nu+1)\frac{1}{r_0}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}\frac{J_2(\xi r_i)-J_0(\xi r_i)}{4r_i}J_1(\xi r_o)d\xi - \int_0^\infty \frac{\xi^3}{\xi^4+\lambda}\frac{J_2(\xi r_i)-J_0(\xi r_i)}{4r_i}J_2(\xi r_0)d\xi\right]$$

$$I_{42} = \frac{1}{\pi}\left[(\nu+1)\frac{1}{r_0}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}\frac{J_2(\xi(r_i-\delta))-J_0(\xi(r_i-\delta))}{4(r_i-\delta)}J_1(\xi r_0)d\xi - \int_0^\infty \frac{\xi^3}{\xi^4+\lambda}\frac{J_2(\xi(r_i-\delta))-J_0(\xi(r_i-\delta))}{4(r_i-\delta)}J_2(\xi r_0)d\xi\right]$$

$$I_{43} = \frac{1}{\pi}\left[(\nu+1)\frac{1}{r_o}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}\frac{J_2(\xi r_0)-J_0(\xi r_0)}{4r_0}J_1(\xi r_0)d\xi - \int_0^\infty \frac{\xi^3}{\xi^4+\lambda}\frac{J_2(\xi r_0)-J_0(\xi r_0)}{4r_0}J_2(\xi r_0)d\xi\right]$$

$$I_{44} = \frac{1}{\pi}\left[(\nu+1)\frac{1}{r_0}\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}\frac{J_2(\xi(r_o+\delta))-J_0(\xi(r_0+\delta))}{4(r_0+\delta)}J_1(\xi r_0)d\xi - \int_0^\infty \frac{\xi^3}{\xi^4+\lambda}\frac{J_2(\xi(r_0+\delta))-J_0(\xi(r_0+\delta))}{4(r_0+\delta)}J_2(\xi r_0)d\xi\right]$$

$$b_1 = -\frac{1}{D}\int_0^\infty \frac{\xi}{\xi^4+\lambda}[J_1(\xi r_i)-J_1(\xi r_0)]J_0(\xi r_i)d\xi$$

$$b_2 = -\frac{1}{D}\int_0^\infty \frac{\xi}{\xi^4+\lambda}[J_1(\xi r_i)-J_1(\xi r_0)]J_0(\xi r_0)d\xi$$

$$b_3 = -\left[(\nu+1)\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}\frac{J_1(\xi r_i)-J_1(\xi r_0)}{r_i}J_1(\xi r_i)d\xi - \int_0^\infty \frac{\xi^3}{\xi^4+\lambda}[J_1(\xi r_i)-J_1(\xi r_0)]J_2(\xi r_i)d\xi\right]$$

$$b_4 = -\left[(\nu+1)\int_0^\infty \frac{\xi^2}{\xi^4+\lambda}\frac{J_1(\xi r_i)-J_1(\xi r_0)}{r_0}J_1(\xi r_o)d\xi - \int_0^\infty \frac{\xi^3}{\xi^4+\lambda}[J_1(\xi r_i)-J_1(\xi r_0)]J_2(\xi r_0)d\xi\right]$$

Appendix II

$$G_1 = H_0^{-1}\left\{\frac{J_1(\xi r_i)-J_1(\xi r_0)}{\xi^4+\lambda};r\right\}$$

$$G_2 = \frac{1}{2}H_0^{-1}\left\{\xi\,\frac{J_2(\xi r_i)-J_0(\xi r_i)}{\xi^4+\lambda};r\right\}$$

$$G_3 = \frac{1}{2}H_0^{-1}\left\{\xi\,\frac{J_2(\xi(r_i-\delta))-J_0(\xi(r_i-\delta))}{\xi^4+\lambda};r\right\}$$

$$G_4 = \frac{1}{2}H_0^{-1}\left\{\xi\,\frac{J_2(\xi r_0)-J_0(\xi r_0)}{\xi^4+\lambda};r\right\}$$

$$G_5 = \frac{1}{2}H_0^{-1}\left\{\xi\,\frac{J_2(\xi(r_0+\delta))-J_0(\xi(r_0+\delta))}{\xi^4+\lambda};r\right\}$$

$$T_1 = (\nu+1)\frac{1}{r}H_1^{-1}\{\xi f_1(\xi);r\} - H_2^{-1}\{\xi^2 f_1(\xi);r\}$$

$$T_2 = \frac{1}{\pi}\left((\nu+1)\frac{1}{r}H_1^{-1}\{\xi f_{21}(\xi);r\} - H_2^{-1}\{\xi^2 f_{21}(\xi);r\}\right)$$

$$T_3 = \frac{1}{\pi}\left((\nu+1)\frac{1}{r}H_1^{-1}\{\xi f_{22}(\xi);r\} - H_2^{-1}\{\xi^2 f_{22}(\xi);r\}\right)$$

$$T_4 = \frac{1}{\pi}\left((\nu+1)\frac{1}{r}H_1^{-1}\{\xi f_{23}(\xi);r\} - H_2^{-1}\{\xi^2 f_{23}(\xi);r\}\right)$$

$$T_5 = \frac{1}{\pi}\left((\nu+1)\frac{1}{r}H_1^{-1}\{\xi f_{24}(\xi);r\} - H_2^{-1}\{\xi^2 f_{24}(\xi);r\}\right)$$

$$f_{21} = \frac{\xi}{\xi^4+\lambda}\left[\frac{J_2(\xi r_i)-J_0(\xi r_i)}{4r_i}\right]$$

$$f_{22} = \frac{\xi}{\xi^4+\lambda}\left[\frac{J_2(\xi(r_i-\delta))-J_0(\xi(r_i-\delta))}{4(r_i-\delta)}\right]$$

$$f_{23} = \frac{\xi}{\xi^4+\lambda}\left[\frac{J_2(\xi r_o)-J_0(\xi r_o)}{4r_o}\right]$$

$$f_{24} = \frac{\xi}{\xi^4+\lambda}\left[\frac{J_2(\xi(r_0+\delta))-J_0(\xi(r_0+\delta))}{4(r_0+\delta)}\right]$$