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Dynamic interaction analysis of vehicle-bridge system using transfer matrix method

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Abstract. The dynamic interaction of vehicle-bridge is studied by using transfer matrix method in this paper. The vehicle model is simplified as a spring-damping-mass system. By adopting the idea of Newmark-β method, the partial differential equation of structure vibration is transformed into a differential equation irrelevant to time. Then, this differential equation is solved by transfer matrix method. The prospective application of this method in real engineering is finally demonstrated by several examples.

Key words: dynamic interaction of vehicle-bridge; transfer matrix method; Newmark-β method; Euler-Bernoulli beam.

1. Introduction

During the past century, a lot of important works about the dynamic interaction of vehicle-bridge system have been done. The initial work on this subjected was pioneered by Inglis (1934). Then, Hillerborg studied this problem by means of Fourier's transformation method. In recent twenty years, increasingly complex computational models of dynamic interaction of vehicle-bridge were proposed. On the basis of those models, the dynamic responses of vehicle-bridge system are studied and lots of useful conclusions are obtained (Sridharan and Mallik 1979, Wiriyachai *et al*. 1982, Chompooming and Yener 1995, Michaltsos *et al*. 1996, Xia *et al*. 2003). Meantime, it also should be noted that most of those complex models are essentially based on the early theory of Inglis and Hillerborg.

Transfer matrix method is a semi-analytical algorithm, and can be efficiently used for periodic one-dimensional structures. In general, most of large-scale structures are periodicity due to assembly and construction. Once the transfer matrix of a representative element is obtained, the solution of the whole structure can be obtained without requiring great computational effort (Pestel and Leckie 1993). In previous literatures, by adopting transfer matrix method, the stability and free vibration of one-dimensional structures are studied comprehensively (Takahashi 1999, Li *et al*. 2004a,b). Moreover, combining with other methods, transfer matrix method may have more extensively application. Using transfer matrix method in combination with the FORM, the structural stochastic

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dynamic responses are studied by Fang (1995). Wang *et al.* (1999) studied the seismic response of frame-shear wall structures through combining transfer matrix method with Runge-Kutta method. Lee (2000) proposed an algorithm named as spectral transfer matrix method, which is a combination of transfer matrix method and spectral element method, and this method is used in dynamic response analysis in one-dimensional structures.

In this paper, the authors aim at applying transfer matrix method into the dynamic interaction analysis of vehicle-bridge. Transfer matrix method can be conveniently used in the structural dynamic response analysis in frequency domain. However, the works about applying transfer matrix method into the structural dynamic response analysis in time domain is comparably rare. To study the structural dynamic response in time domain, the transfer matrix method is combined with Newmark-β method in this paper. By adopting the idea of Newmark-β method, the partial differential equation of structure vibration is transformed into a differential equation irrelevant to time. Then, this differential equation is solved by transfer matrix method. Through simplifying the vehicle model as a spring-damping-mass system, the dynamic interaction of vehicle-bridge is studied using the algorithm proposed by authors. Example studies demonstrate the prospective application of this method in real engineering.

2. Formulation of the transfer matrix

2.1 The transfer matrix for beam element

For Euler-Bernoulli beam, the vibration equation can be written as (Paz 1991)

$$EI\frac{\partial^4 y}{\partial x^4} + \bar{m}\frac{\partial^2 y}{\partial t^2} = f(x, t) \tag{1}$$

in which y is the vertical deflection of position x at time t, $\bar{m}$ is the mass per unit length, EI is the flexural stiffness, and $f(x, t)$ is the vertical load acting on the beam.

Eq. (1) can be written in the incremental form for time step i as

$$EI\frac{\partial^4 \Delta y_i}{\partial x^4} + \bar{m}\frac{\partial^2 \Delta y_i}{\partial t^2} = \Delta f_i \tag{2}$$

where

$$\Delta y_i = y_{i+1} - y_i \tag{3}$$

$$\Delta f_i = f(x, t_{i+1}) - f(x, t_i) \tag{4}$$

Using the idea of Newmark-β method (Newmark 1959), the incremental acceleration and velocity at time step i can be given as

$$\frac{\partial^2 \Delta y_i}{\partial t^2} = \frac{\Delta y_i}{\beta \Delta t^2} - \frac{\dot{y}_i}{\beta \Delta t} - \frac{\ddot{y}_i}{2\beta} \tag{5}$$

$$\frac{\partial \Delta y_i}{\partial t} = \frac{\Delta y_i}{2\beta\Delta t} - \frac{\dot{y}_i}{2\beta} + \left(1 - \frac{1}{4\beta}\right)\Delta t \ddot{y}_i \tag{6}$$

where $\dot{y}_i$ and $\ddot{y}_i$ are the velocity and acceleration at time t_i respectively. Newmark suggested that the value of parameter β is in the range $\frac{1}{6} \le \beta \le \frac{1}{2}$ (Newmark 1959). For $\beta = \frac{1}{4}$ this method is unconditionally stable and it provides the satisfactory accuracy.

Substituting Eq. (5) into Eq. (2) yields

$$EI\frac{d^4 \Delta y_i}{dx^4} + \frac{\bar{m}}{\beta \Delta t^2}\Delta y_i = \Delta f_i + \bar{m}\left(\frac{\dot{y}_i}{\beta\Delta t} + \frac{\ddot{y}_i}{2\beta}\right) \tag{7}$$

Thus, the incremental partial differential equation of beam vibration is transformed into a differential equation. Adopting the transfer matrix, the incremental displacement Δy_i can be solved. Then substituting Δy_i into Eqs. (5) and (6), the incremental acceleration and velocity at time step i are obtained.

For convenience, Eq. (7) is rewritten as

$$\frac{d^4 \Delta y_i}{dx^4} + 4k^4 \Delta y_i = \Delta F_i \tag{8}$$

where

$$k^4 = \frac{\bar{m}}{4\beta\Delta t^2 \cdot EI}$$

$$\Delta F_i = \left[\Delta f_i + \bar{m}\left(\frac{\dot{y}_i}{\beta\Delta t} + \frac{\ddot{y}_i}{2\beta}\right)\right] \Big/ EI \tag{9}$$

The solution of Eq. (8) is given as

$$\Delta y_i = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) + c_4 f_4(x) + \sum_{j=1}^{4} D_j(x) f_j(x) \tag{10}$$

in which

$$\begin{aligned} f_1(x) &= \mathrm{ch}kx \cdot \cos kx, \quad & f_2(x) &= \mathrm{ch}kx \cdot \sin kx \\ f_3(x) &= \mathrm{sh}kx \cdot \cos kx, \quad & f_4(x) &= \mathrm{sh}kx \cdot \sin kx \end{aligned} \tag{11}$$

$c_1 \sim c_4$ is a set of coefficients. $\sum_{j=1}^{4} D_j(x) f_j(x)$ is the special solution of Eq. (8), and by adopting the method of variation of parameters $D_j(x)$ can be determined (Zill and Cullen 2001). The analytical expressions of $D_j(x)$ when ΔF_i is a concentrated load or a linear varying distributing load are given in appendix.

From Eq. (10), and considering $\sum_{j=1}^{4} D_j'(x)f_j(x) = \sum_{j=1}^{4} D_j'(x)f_j'(x) = \sum_{j=1}^{4} D_j'(x)f_j''(x) = 0$, the analytical expressions of incremental rotation $\Delta\theta_i$, incremental moment ΔM_i, and incremental shear force ΔQ_i are obtained

$$\begin{aligned}\Delta\theta_i = \frac{d\Delta y_i}{dx} &= c_1 k[f_3(x) - f_2(x)] + c_2 k[f_4(x) + f_1(x)] + \\ &\quad c_3 k[f_1(x) - f_4(x)] + c_4 k[f_2(x) + f_3(x)] + \sum_{j=1}^{4} D_j(x) f_j'(x)\end{aligned} \tag{12}$$

$$\begin{aligned}\Delta M_i = -EI\frac{d^2\Delta y_i}{dx^2} &= 2c_1 \cdot k^2 \cdot EI \cdot f_4(x) - 2c_2 \cdot k^2 \cdot EI \cdot f_3(x) + \\ &\quad 2c_3 \cdot k^2 \cdot EI \cdot f_2(x) - 2c_4 \cdot k^2 \cdot EI \cdot f_1(x) - EI\sum_{j=1}^{4} D_j(x) f_j''(x)\end{aligned} \tag{13}$$

$$\begin{aligned}\Delta Q_i &= -EI\frac{d^3\Delta y_i}{dx^3} \\ &= 2c_1 \cdot k^3 \cdot EI \cdot [f_2(x) + f_3(x)] - 2c_2 \cdot k^3 \cdot EI \cdot [f_1(x) - f_4(x)] + \\ &\quad 2c_3 \cdot k^3 \cdot EI \cdot [f_4(x) + f_1(x)] - 2c_4 \cdot k^3 \cdot EI \cdot [f_3(x) - f_2(x)] \\ &\quad -EI\sum_{j=1}^{4} D_j(x) f_j'''(x)\end{aligned} \tag{14}$$

Defining the state vector and coefficient vector as

$$\{Z(x)\} = \{\Delta y_i \;\; \Delta\theta_i \;\; \Delta M_i \;\; \Delta Q_i \;\; 1\}^T \tag{15}$$

$$\{C\} = \{c_1 \;\; c_2 \;\; c_3 \;\; c_4 \;\; 1\}^T \tag{16}$$

Then, Eqs. (10), (12), (13), and (14) can be expressed in the matrix form as

$$\{Z(x)\} = [H(x)]\{C\} \tag{17}$$

For a given element, assuming that the coordinates of left and right end are 0 and L, substituting $x = 0$ and $x = L$ into Eq. (17) yields

$$\{Z(0)\} = [H(0)]\{C\} \tag{18}$$

$$\{Z(L)\} = [H(L)]\{C\} \tag{19}$$

From Eqs. (18) and (19), it can be obtained that

$$\{Z(L)\} = [H(L)][H(0)]^{-1}\{Z(0)\} = [T]\{Z(0)\} \tag{20}$$

in which matrix $[T]$ is the transfer matrix, and the analytical expressions of its' non-zero elements are given as

$$T_{11} = f_1(L) \qquad T_{12} = \frac{f_2(L)+f_3(L)}{2k} \qquad T_{13} = \frac{f_4(L)}{2k^2 \cdot EI}$$

$$T_{14} = \frac{f_3(L)-f_2(L)}{4k^3 \cdot EI} \qquad T_{15} = \sum_{j=1}^{4} D_j(L) f_j(L) \qquad T_{21} = k[f_3(L)-f_2(L)]$$

$$T_{22} = f_1(L) \qquad T_{23} = -\frac{f_2(L)+f_3(L)}{2k \cdot EI} \qquad T_{24} = -\frac{f_4(L)}{2k^2 \cdot EI}$$

$$T_{25} = \sum_{j=1}^{4} D_j(L) f_j'(L) \qquad T_{31} = 2k^2 \cdot EI \cdot f_4(L) \qquad T_{32} = k \cdot EI[f_2(L)-f_3(L)]$$

$$T_{33} = f_1(L) \qquad T_{34} = \frac{f_2(L)+f_3(L)}{2k} \qquad T_{35} = -EI\sum_{j=1}^{4} D_j(L) f_j''(L)$$

$$T_{41} = 2k^3 \cdot EI[f_2(L)+f_3(L)] \qquad T_{42} = 2k^2 \cdot EI \cdot f_4(L) \qquad T_{43} = k[f_3(L)-f_2(L)]$$

$$T_{44} = f_1(L) \qquad T_{45} = -EI\sum_{j=1}^{4} D_j(L) f_j'''(L) \qquad T_{55} = 1 \tag{21}$$

2.2 The point transfer matrix for support

For a continuous beam shown in Fig. 1, the relationship between the state vector of the left and right side of the pth support can be written as

$$\begin{Bmatrix} 0 \\ \Delta\theta_p^R \\ \Delta M_p^R \\ \Delta Q_p^R \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \Delta\theta_p^L \\ \Delta M_p^L \\ \Delta Q_p^L + \Delta Q_p' \\ 1 \end{Bmatrix} \tag{22}$$

where $\Delta Q_p'$ is the support reaction of the pth support.

The state vector of the left side of the pth support can be expressed in terms of the state vector of the first support as

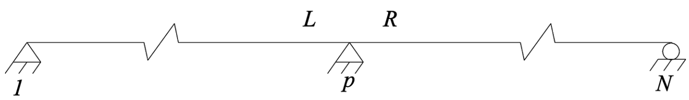

Fig. 1 Continuous beam

$$\begin{Bmatrix} 0 \\ \Delta\theta_p^L \\ \Delta M_p^L \\ \Delta Q_p^L \\ 1 \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} \\ T_{21} & T_{22} & T_{23} & T_{24} & T_{25} \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ \Delta\theta_p^1 \\ 0 \\ \Delta Q_p^1 \\ 1 \end{Bmatrix} = \begin{bmatrix} T_{12} & T_{14} & T_{15} \\ T_{22} & T_{24} & T_{25} \\ T_{32} & T_{34} & T_{35} \\ T_{42} & T_{44} & T_{45} \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \Delta\theta_p^1 \\ \Delta Q_p^1 \\ 1 \end{Bmatrix} \tag{23}$$

From Eqs. (22) and (23), the point transfer matrix of the pth support is obtained

$$\begin{Bmatrix} 0 \\ \Delta\theta_p^R \\ \Delta M_p^R \\ \Delta Q_p^R \\ 1 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ T_{22} - T_{12}T_{24}/T_{14} & 0 & T_{25} - T_{15}T_{24}/T_{14} \\ T_{32} - T_{12}T_{34}/T_{14} & 0 & T_{35} - T_{15}T_{34}/T_{14} \\ T_{42} - T_{12}T_{44}/T_{14} & 1 & T_{45} - T_{15}T_{44}/T_{14} \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \Delta\theta_p^1 \\ \Delta Q_p' \\ 1 \end{Bmatrix} \tag{24}$$

3. Dynamic interaction of vehicle-bridge

As shown in Fig. 2, the vehicle model is simplified as a spring-damping-mass system. Using D'Alembert's principle, the motion equations of the beam, the bogie, and the vehicle body are obtained

$$EI\frac{\partial^4 y}{\partial x^4} + \bar{m}\frac{\partial^2 y}{\partial t^2} = \left[(m_b + m_t)g - m_b\frac{d^2 y_b}{dt^2} - m_t\frac{d^2 y_t}{dt^2}\right] \cdot \delta(x - vt) \tag{25}$$

$$m_b\frac{d^2 y_b}{dt^2} + m_t\frac{d^2 y_t}{dt^2} + c_a\frac{dy_t}{dt} + K_a y_t = K_a y(vt) + c_a\frac{dy(vt)}{dt} \tag{26}$$

$$m_b\frac{d^2 y_b}{dt^2} + c_b\left(\frac{dy_b}{dt} - \frac{dy_t}{dt}\right) + K_b(y_b - y_t) = 0 \tag{27}$$

where y, y_b, and y_t are the displacement of the beam, the bogie, and the vehicle body respectively. m_b and m_t are the mass of the bogie and the vehicle body. v is the velocity of the vehicle.

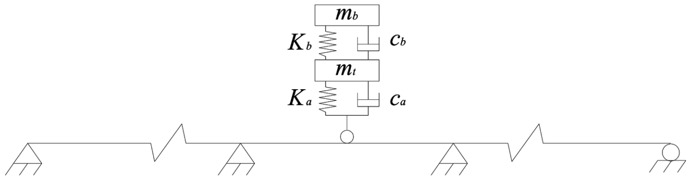

Fig. 2 The interaction model of vehicle-bridge system

Following the same procedure adopted in Eqs. (2)-(7), Eqs. (25)-(27) can be expressed as

$$EI\frac{d^4\Delta y_i}{dx^4} + \frac{\overline{m}}{\beta\Delta t^2}\Delta y_i = [(m_b + m_t)g - m_b\ddot{y}_{bi} - m_t\ddot{y}_{ti}] \cdot \delta(x - vt_{t+1}) - [(m_b + m_t)g - m_b\ddot{y}_{bi} - m_t\ddot{y}_{ti}] \cdot \delta(x - vt_i) + \overline{m}\left(\frac{\dot{y}_i}{\beta\Delta t} + \frac{\ddot{y}_i}{2\beta}\right) \tag{28}$$

$$\frac{m_b}{\beta\Delta t^2}\Delta y_{bi} + \left(\frac{m_t}{\beta\Delta t^2} + \frac{c_a}{2\beta\Delta t} + K_a\right)\Delta y_{ti} = K_a\Delta y_i(vt_{i+1}) + c_a\Delta\dot{y}_i(vt_{i+1}) + m_b\left(\frac{\dot{y}_{bi}}{\beta\Delta t} + \frac{\ddot{y}_{bi}}{2\beta}\right) + m_t\left(\frac{\dot{y}_{ti}}{\beta\Delta t} + \frac{\ddot{y}_{ti}}{2\beta}\right) + c_a\frac{\dot{y}_{ti}}{2\beta} - c_a\left(1 - \frac{1}{4\beta}\right)\Delta t\ddot{y}_{ti} \tag{29}$$

$$\left(\frac{m_b}{\beta\Delta t^2} + \frac{c_b}{2\beta\Delta t} + K_b\right)\Delta y_{bi} - \left(\frac{c_b}{2\beta\Delta t} + K_b\right)\Delta y_{ti} = m_b\left(\frac{\dot{y}_{bi}}{\beta\Delta t} + \frac{\ddot{y}_{bi}}{2\beta}\right) + c_b\left[\frac{\dot{y}_{bi}}{2\beta} - \left(1 - \frac{1}{4\beta}\right)\Delta t\ddot{y}_{bi}\right] - c_b\left[\frac{\dot{y}_{ti}}{2\beta} - \left(1 - \frac{1}{4\beta}\right)\Delta t\ddot{y}_{ti}\right] \tag{30}$$

Using the transfer matrix method developed above, Eq. (28) can be solved. Then, from Eqs. (29) and (30), the incremental displacement of bogie and vehicle body are obtained. Substituting incremental displacement into Eqs. (5) and (6), the incremental acceleration and velocity can be finally determined.

4. Example studies

4.1 Dynamic interaction of vehicle and simply supported beam

A simply supported beam is taken as numerical example. Its' span length is 16 m, the flexural stiffness is 2.05×10^{10} N · m^2, and the mass per unit length is 9.36×10^3 kN/m. The parameters of

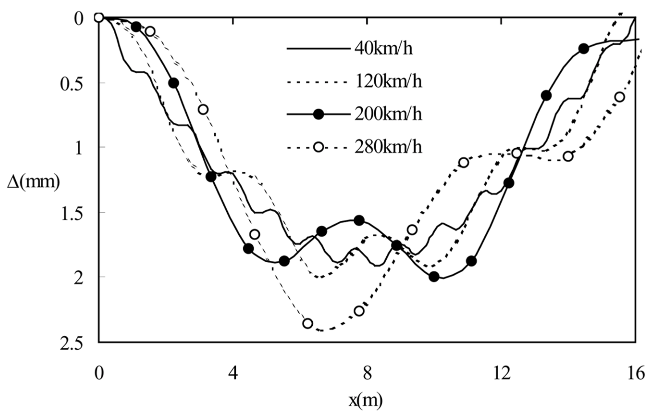

Fig. 3 Response of mid-span deflection to the vehicle position

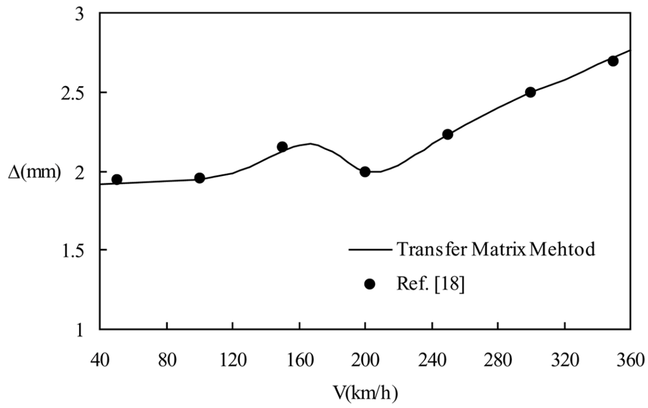

Fig. 4 Response of maximum mid-span deflection to the vehicle velocity

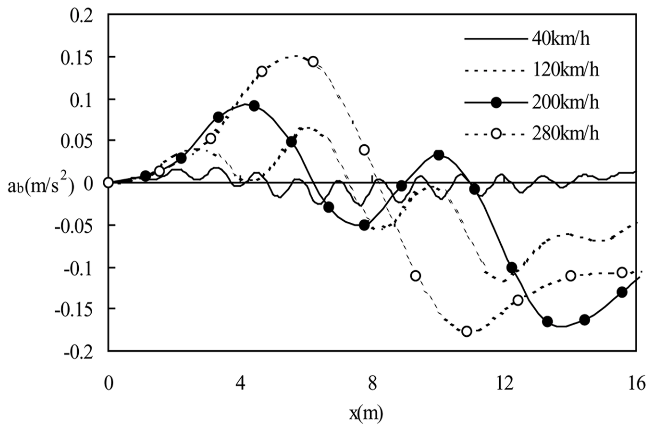

Fig. 5 Acceleration history of vehicle body

vehicle model are taken as: m_t = 8660 kg, m_b = 38500 kg, $K_a = 8.56 \times 10^6$ N/m, $K_b = 5.07 \times 10^6$ N/m, $c_a = 1.96 \times 10^5$ kg/s, and $c_b = 3.82 \times 10^5$ kg/s. The response of the mid-span deflection to the vehicle position as the vehicle moves with various velocities along the beam is given in Fig. 3. The relationship between the maximum mid-span dynamic deflection and the vehicle velocity is plotted in Fig. 4. To investigate the dynamic response of vehicle, the acceleration history of vehicle body is shown in Fig. 5.

From Fig. 3, it can be found that the fluctuation of dynamic deflection response of the beam decreases gradually with the vehicle speed, and the maximum deflection occurs when the vehicle is located adjacent to mid-span. The problem is also analyzed by Ruge-Kutta method in Xiao and Shen (2004), and the result comparison shown in Fig. 4 demonstrates the accuracy of the method proposed by authors. Moreover, from Fig. 4, it can be concluded that the maximum deflection of beam doesn't increase linearly with the vehicle speed, and the maximum deflection increases quickly when the vehicle speed exceeds 200 km/h. As shown in Fig. 5, with the increment of vehicle speed, although the fluctuation of vehicle body acceleration decreases, the maximum vehicle body acceleration increases enormously.

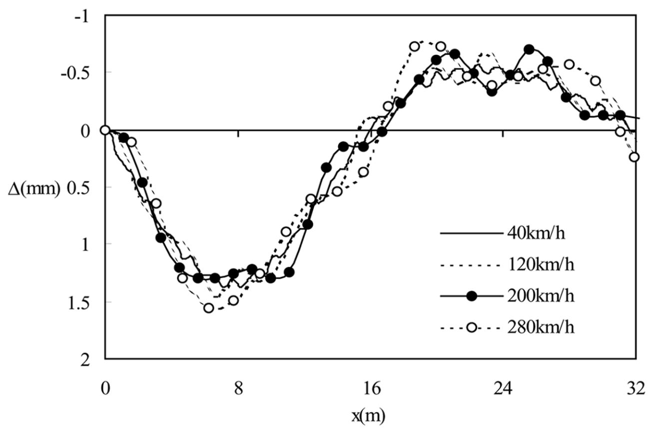

Fig. 6 Response of mid-span deflection to the vehicle position

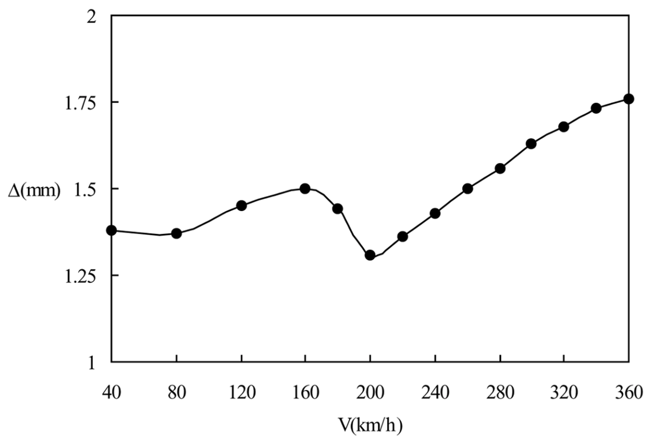

Fig. 7 Response of maximum mid-span deflection to the vehicle velocity

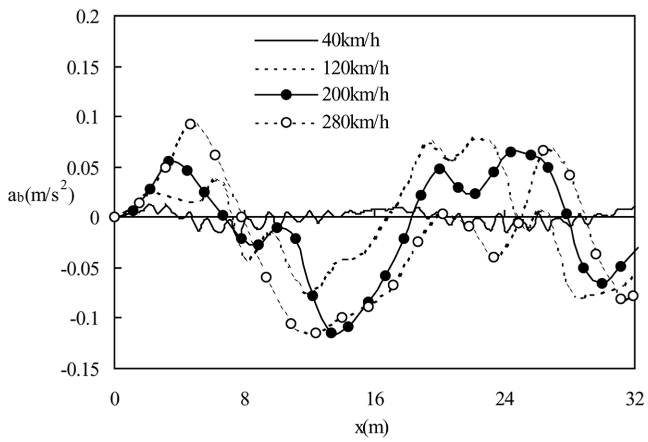

Fig. 8 Acceleration history of vehicle body

4.2 Dynamic interaction of vehicle and continuous beam

To study this problem more deeply, the dynamic interaction of vehicle and a two-span continuous beam is studied. The length of each span is 16 m. The other parameters used in calculation are the same as those in example 4.1. The dynamic response of the beam and the acceleration history of vehicle body are also plotted in Figs. 6-8.

It can be found from Figs. 6-8 that due to improvement of structure stiffness, the dynamic responses of beam and vehicle body are smaller than those of simply supported beam.

5. Conclusions and acknowledgements

From the numerical results presented above, it can be concluded that the method used in this paper is accuracy, and able to deal with the dynamic interaction of vehicle-bridge effectively. Moreover, although a comparably simply vehicle model is adopted, it should be noted that the method can be extended to spatial dynamic interaction of vehicle-bridge where the complicated vehicle model is usually used. In the following studies, the authors will deal with this topic more extensively. The work of this paper is financially supported by Science and Technology Foundation of Southwest Jiaotong University (2003A14).

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Appendix

When $\Delta F_i = P \cdot \delta(x - x_0)$

$$D_1(x) = \frac{f_3(x_0) - f_2(x_0)}{4k^3} \qquad D_2(x) = \frac{f_1(x_0) + f_4(x_0)}{4k^3}$$

$$D_3(x) = \frac{f_4(x_0) - f_1(x_0)}{4k^3} \qquad D_4(x) = \frac{-f_2(x_0) - f_3(x_0)}{4k^3}$$

When $\Delta F_i = P_0 + T \cdot x$

$$\begin{aligned} D_1(x) = \frac{\exp(kx)}{16k^5}[&2Tkx\cos(kx) + 2Tkx\cos(kx)\exp(-2kx) - T\cos(kx) \\ &+ T\cos(kx)\exp(-2kx) - T\sin(kx) - T\sin(kx)\exp(-2kx) \\ &+ 2P_0k\cos(kx) + 2P_0k\cos(kx)\exp(-2kx) - 4P_0\exp(-2kx)] \end{aligned}$$

$$\begin{aligned} D_2(x) = \frac{\exp(kx)}{16k^5}[&2Tkx\sin(kx) - T\sin(kx) + T\sin(kx)\exp(-2kx) \\ &+ T\cos(kx) + T\cos(kx)\exp(-2kx) - 2T\exp(-2kx) \\ &+ 2P_0k\sin(kx)\exp(-2kx)] \end{aligned}$$

$$\begin{aligned} D_3(x) = \frac{\exp(kx)}{16k^5}[&-2Tkx\cos(kx) + 2Tkx\cos(kx)\exp(-2kx) + T\cos(kx) \\ &+ T\cos(kx)\exp(-2kx) + T\sin(kx) - 2P_0k\cos(kx) \\ &+ 2P_0k\cos(kx)\exp(-2kx) - 2T\exp(-kx)] \end{aligned}$$

$$\begin{aligned} D_4(x) = \frac{\exp(kx)}{16k^5}[&-2Tkx\sin(kx) - 2P_0k\sin(kx) + 2Tkx\sin(kx)\exp(-2kx) \\ &+ T\sin(kx) + T\sin(kx)\exp(-2kx) + T\cos(kx)\exp(-2kx) \\ &+ \exp(-2kx)] \end{aligned}$$