Structural Engineering and Mechanics, Vol. 2, No. 3 (1994) 285-294
DOI: http://dx.doi.org/10.12989/sem.1994.2.3.285

Direct Ritz method for random seismic response for non-uniform beams

J.H. Lin†

Research Institute of Engineering Mechanics, Dalian University of Technology, Dalian 116023, P.R. China

F.W. Williams† and P.N. Bennett‡

Division of Structural Engineering, University of Wales College of Cardiff, Cardiff CF21YF, UK

Abstract. Based on a fast and accurate method for the stationary random seismic response analysis for discretized structures (Lin 1992, Lin, et al. 1992), a Ritz method for dealing with such responses of continuous systems is developed. This method is studied quantitatively, using cantilever shear beams for simplicity and clarity. The process can be naturally extended to deal with various boundary conditions as well as non -uniform Bernoulli-Euler beams, or even Timoshenko beams. Algorithms for both proportionally and non-proportionally damped responses are described. For all of such damping cases, it is not necessary to solve for the natural vibrations of the beams. The solution procedure is very simple, and equally efficient for a white or a non-white ground excitation spectrum. Two examples are given where various power spectral density functions, variances, covariances and second spectral moments of displacement, internal force responses, and their derivatives are calculated and analysed. Some Ritz solutions are compared with "exact" CQC solutions.

Key words: random seismic beam Ritz.

1. Introduction

In seismic analysis, many engineering structures can be modeled as non-uniform beams. Several publications(e.g. Crandall and Yildiz 1962, Gasparini, et al. 1981) describe the conventional solution procedure, in which eigenmodes of the structure must be found before the random response is calculated, so that considerable computational effort is generally required. A pseudo-excitation (or fast-CQC) method of structural stationary random seismic response has been described for discretized linear structures (Lin 1992, Lin et al. 1992). This paper shows that when the Ritz method combined with the pseudo-excitation algorithm is applied to random seismic analysis of beams, the procedure is very simple and straightforward and also allows various damping cases to be dealt with easily. Indeed it will be seen that if damping is of the hysteretic, Rayleigh, or even non-proportional types, the random response can be calculated without having to solve an eigenproblem. Two shear beams were analysed

† Professor
‡ Research Student

in detail, which show that satisfactory PSD (power spectral density) values and variances of the beam displacements can be obtained by taking quite few (three or even two) Ritz functions. However, as expected, reliable analysis of the PSD of the internal forces, especially their higher order spectral moment, needs more Ritz functions.

2. Direct Ritz method for PSD analysis

Consider an earthquake excitation acting on a non-uniform shear beam, see Fig.1, for which

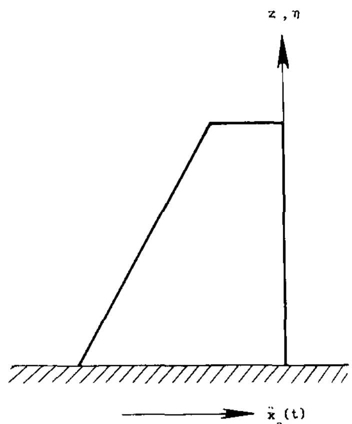

Fig. 1 Non-uniform Shear Beam

the motion equation is (Lin Y.K. 1967, Gasparini 1981):

$$\rho A(z)\ddot{u}+c(z)\,\dot{u}-[G(z)A(z)u']'=-\rho A(z)\ddot{x}_g \tag{1}$$

in which superscripts and represent the partial derivatives with respect to time t and coordinate z respectively; ρ is the density of the beam; $G(z)$ and $A(z)$ are the shear modulus and the sectional area of the beam, which are variable along the coordinate z; $u=u(z,t)$ is the transverse displacement of the beam; $\ddot{x}_g=\ddot{x}_g(t)$ is the ground acceleration for which the PSD $S_a(\omega)$ is given; and the damping distribution $c(z)$ can be either proportional or non-proportional, as discussed later.

The direct Ritz method is based on the fast CQC, or pseudo excitation method (Lin 1992, Lin, et al. 1992) which, in the case of stationary single excitation, can be described as follows:

A linear system is subjected to a stationary random excitation whose PSD $S_{xx}(\omega)$ has been specified. If $y(t)$ is an arbitrary response due to the pseudo harmonic excitation $x(t)=\sqrt{S_{xx}(\omega)}e^{i\omega t}$, then $y^*y=|y(t)|^2$ must be the PSD of the corresponding response due to the excitation $S_{xx}(\omega)$. And if $z(t)$ is another arbitrary response due to this pseudo excitation $x(t)$, then y^*z and z^*y must be, respectively, the cross-PSD's $S_{yz}(\omega)$ and $S_{zy}(\omega)$. According to this pseudo excitation principle, a pseudo sinusoidal ground acceleration can be established in virtue of the given ground acceleration PSD $S_a(\omega)$ as

$$\ddot{x}_g(t)=\sqrt{S_a(\omega)}e^{i\omega t} \tag{2}$$

Substituting Eq. (2) into Eq. (1) produces the motion equation

$$\rho A\ddot{u}+c\dot{u}-[GAu']'=-\rho A\sqrt{S_a(\omega)}e^{i\omega t} \tag{3}$$

This harmonic motion equation can be easily solved by means of the Ritz method(e.g., Warburton 1976), as follows. Let

$$u(z,t) = \sum_{j=1}^{q} y_j(t)\phi_j\ (z) = [\phi]\ \{y\} \tag{4}$$

where: ϕ_j $(j=1,....,q)$ are the experience-based Ritz functions. Substituting Eq. (4) into Eq. (3), and pre-multiplied by $[\Phi]^T$, gives

$$[M]\ \{\ddot{y}\} + [C]\ \{\dot{y}\} - [R]\ \{y\} = -\{\xi\}\sqrt{S_a(\omega)}e^{i\omega t} \tag{5}$$

in which

$$[M] = \int_o^L \rho A(z)\ [\phi]^T\ [\phi]\ dz \tag{6}$$

$$[C] = \int_o^L c(z)\ [\phi]^T\ [\phi]\ dz \tag{7}$$

$$\begin{aligned}[R] &= \int_o^L [\phi]^T\ (G(z)\ A(z)\ [\phi]')'\ dz \\ &= ([\phi]^T\ G(z)\ A(z)\ [\phi]')\Big|_o^L - \int_0^L G(z)\ A(z)\ [\phi]'^T\ [\phi]'\ dz\end{aligned} \tag{8}$$

$$\{\xi\} = \int_o^L \rho A(z)\ [\phi]^T\ dz \tag{9}$$

Provided the shear beam has simple boundary conditions, i.e. either the boundary displacement $u=0$ or the boundary shear force $Q=GAu'=0$, then the first item on the RHS of Eq. (8) must vanish so that only the second term, denoted as $-[K]$, remains, where

$$[K] = \int_0^L G(z)\ A(z)\ [\phi]'^T\ [\phi]'\ dz \tag{10}$$

Hence, Eq. (5) becomes

$$[M]\ \{\ddot{y}\} + [C]\ \{\dot{y}\} + [K]\ \{y\} = \{p\}e^{i\omega t} \tag{11}$$

$$\{p\} = -\{\xi\}\sqrt{S_a(\omega)} \tag{12}$$

where $[M]$, $[C]$ and $[K]$ are in general non-diagonal because there are no orthogonality relations between the experience-based Ritz functions. Therefore Eq. (11) cannot be decoupled directly. Alternative solutions to Eq. (11) are given below for different damping properties.

2.1. Case I. Proportional damping with given damping ratios

This is the commonest case for engineering applications. The solution vector $\{y\}$ of Eq. (11) should be decomposed according to the $q\times q$ eigenvector matrix $[W]$ which satisfies the following eigenproblem

$$[K]\ [W] = [M]\ [W]\ [\Lambda] \tag{13}$$

Denoting

$$\{y\}=\sum_{j=1}^{q} v_j\{w_j\}=[W]\{V\} \tag{14}$$

then substituting Eq. (14) into Eq. (11) and pre-multiplying by $[W]^T$ gives the following motion equation

$$\ddot{v}_j+2\zeta_j\omega_j\dot{v}_j+\omega_j^2 v_j=p_j^*/m_j^*=-\gamma_j\sqrt{S_a}e^{i\omega t} \tag{15}$$

in which v_j and m_j^* are the generalized displacement and generalized mass associated with the j-th mode respectively, ζ_j is the given j-th modal damping ratio, and

$$\gamma_j=\{w_j\}^T\{\xi\}/m_j^*=\{w_j\}^T\{\xi\}/\{w_j\}^T[M]\{w_j\} \tag{16}$$

The solution to Eq. (15) is

$$v_j=-\gamma_j H_j\sqrt{S_a}e^{i\omega t} \tag{17}$$

$$H_j(i\omega)=(\omega_j^2-\omega^2+i2\zeta_j\omega_j\omega)^{-1} \tag{18}$$

The displacement and shear force can be expressed as

$$u(z,t)=[\phi][W]\{V\} \tag{19}$$

$$Q(z,t)=G(z)A(z)[\phi]'[W]\{V\} \tag{20}$$

Thus, using the principle described above, the PSD functions of u and Q can be expressed as

$$S_{uu}(z,\omega)=|u(z,t)|^2,\quad S_{QQ}(z,\omega)=|Q(z,t)|^2 \tag{21}$$

The dimensionless forms of S_{uu}, S_{QQ} and ω are respectively

$$S_U=S_{uu}C_s^4/(L^4S_a),\quad S_Q=S_{QQ}/(\rho^2L^2A^2S_a),\quad \omega_o=\omega L/C_s \tag{22}$$

where $C_s=\sqrt{G/\rho}$. In Eqs. (21), the variable t will naturally disappear when taking the two norms. When the PSD of the responses (displacements, internal forces, etc.), denoted as S_{res}, have been worked out, their spectral moments can then be calculated easily. Among them, the most useful are the zeroth moments (i.e. the variance), and the second moments:

$$\begin{aligned}\lambda_{0,res}&=\sigma_{res}^2=2\int_o^\infty S_{res}\,d\omega\\ \lambda_{2,res}&=2\int_o^\infty \omega^2S_{res}\,d\omega\end{aligned} \tag{23}$$

2.2. Case II. Rayleigh, hysteretic and non-proportional damping

Eq. (11) is not necessarily decouplable as above. For such problems, the stable state solution {y} of Eq. (11) can be decomposed into a real part $\{y_r\}$ and an imaginary part $\{y_i\}$:

$$\begin{aligned}\{y\}&=\{y_r\}+i\{y_i\}\equiv(\{a\}+i\{b\})e^{i\omega t}\\ \{\ddot{y}\}&=-\omega^2(\{a\}+i\{b\})e^{i\omega t}\end{aligned} \tag{24}$$

where $\{a\}$ and $\{b\}$ are two unknown real vectors. Substituting Eqs. (24) into Eq. (11) and se parating real and imaginary parts gives the following simultaneous eqs.

$$[E]\{a\}+[D]\{b\}=\{p\}$$
$$-[D]\{a\}+[E]\{b\}=\{0\} \tag{25}$$

Herein the expressions for $[E]$ and $[D]$ are listed in Table 1 for some typical damping cases.

Table 1 Expressions of [C], [E] and [D] for Various Damping Cases

Damping Property	Damping Matrices [C]	[E]	[D]
Rayleigh Damping	$\alpha\,[M]+\beta\,[K]$	$[K]-\omega^2[M]$	$-\omega\,(\alpha\,[M]+\beta\,[K])$
Hysteretic Damping	$i\varepsilon\,[K]$	$[K]-\omega^2[M]$	$-\varepsilon\,[K]$
Non-proportional	$[C]$	$[K]-\omega^2[M]$	$-\omega\,[C]$

Vectors $\{a\}$ and $\{b\}$, or $\{y_r\}$ and $\{y_i\}$, are very easy to solve for. Thus $u(z, t)$ can be readily obtained in virtue of Eq. (4), and then the shear force can be worked out by the following equation:

$$Q(z, t)=G(z)\ A(z)[\Phi]'\{y\} \tag{26}$$

Finally, all kinds of PSD functions and their moments can be calculated by means of Eqs. (21)−(23).

3. Examples

3.1. Example 1.

A uniform shear cantilever beam subjected to a stationary Gaussian random ground motion with a given acceleration PSD $S_a(\omega)=S_o$ (white spectrum) is taken as an example. Four sets of Ritz functions are used as listed in Table 2 in which Case A takes 10 exact

Table 2 Ritz Functions Used in Example 1

Case	Ritz Functions($\eta=Z/L$)	Remarks
A	$\sum_{j=1}^{10} a_j Sin(j-0.5)\pi\eta$	CQC(Exact)
E		SRSS(Approximate)
B	$a_1\eta+a_2\eta^2$	Ritz I
C	$a_1\eta+a_2\eta^2+a_3\eta^3$	Ritz II
D	$a_1(2\eta-\eta^2)+a_2(3\eta^2-2\eta^3)$	Ritz III

eigenmodes as the Ritz functions and in fact produces the exact CQC solution, while case E produces the corresponding SRSS solution by neglecting the cross-correlation items between the participant modes (Clough and Penzien 1975, Lin 1992). The responses associated with case E, as a widely used approximation, have been worked out for comparison by means of the traditional method (Clough and Penzien 1975) instead of the present Ritz method. Cases B and C take the simplest power functions as the Ritz bases. The deflection curves of the beam subjected to uniformly or triangularly distributed loadings are taken as the Ritz functions for Case D. The PSD functions of the beam-top displacement $U(L)$ and the beam

-base shear force $Q(0)$, as well as their variances and second spectral moments, are found and compared for the case of proportional damping with given damping ratios.

The PSD curves of the top displacement $U(L)$ for damping ratios $\zeta=0.05$ and 0.20 are shown in Figs. 2-3. It is seen that even the simplest Ritz set, Case B, gives excellent PSD curves. The PSD curve for the SRSS approximation with $\zeta=0.20$ is herein the poorest, but is still quite acceptable.

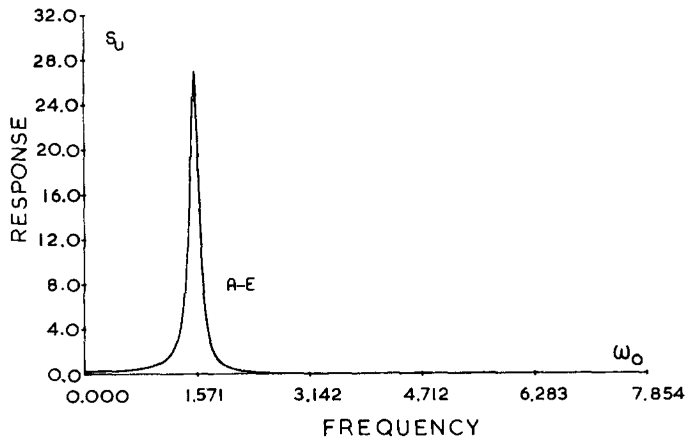

Fig. 2 PSD of Beam-top Displacement($\zeta=0.05$)

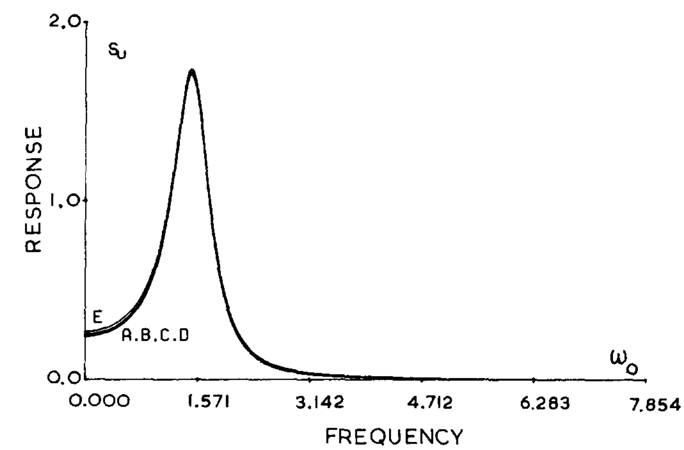

Fig. 3 PSD of Beam-top Displacement($\zeta=0.20$)

Figs. 4 and 5 show that better and/or more Ritz functions are required for the computations of good shear-force PSD curves.

Table 3 lists the variances of U(L). Clearly the three Ritz cases all give satisfactory results, such that even the simplest Ritz basis(Case B) results in a maximum error of only 3.7% (when $\zeta=0.01$), with the errors for cases C and D always being lower than 1%. However for the variances of $Q(0)$ the errors are much bigger, about 20~30% for case B and 6~13% for cases C and D, see Table 4.

The second moments of $U(L)$ and $Q(0)$ are listed in Tables 5 and 6, respectively. Table 5 shows that the Ritz approximations of the second moments of $U(L)$ are reasonably good

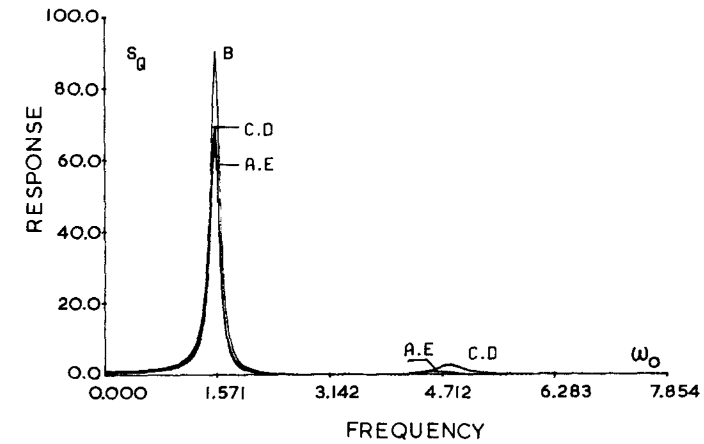

Fig. 4 PSD of Beam-base Shear Force(ζ=0.05)

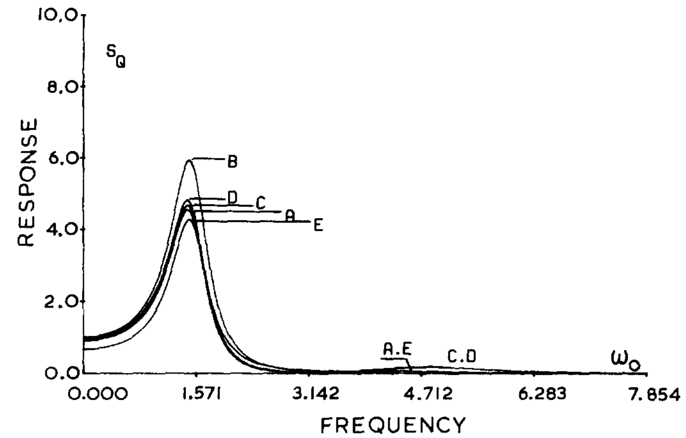

Fig. 5 PSD of Beam-base Shear Force(ζ=0.20)

with errors within about 5%, except for the heavily damped analyses (ζ =0.20), for which the errors are within 8%. This reasonably good accuracy is a consequence of the closeness of the Ritz and the exact displacement PSD values shown in Figs. 2 and 3. Unfortunately, the

Table 3 Variances of Displacement $U(L)$(to be multiplied by$(L/C_s)^3 S_a$)

case / ζ	A CQC	B Ritz(1)	C Ritz(2)	D Ritz(3)	E SRSS
0.01	71.922	69.271	72.041	72.548	71.923
0.05	13.180	13.414	13.204	13.296	13.191
0.10	6.570	6.691	6.582	6.630	6.591
0.20	3.253	3.319	3.256	3.284	3.291

Table 4 Variances of Shear Force $Q(0)$(to be multiplied by $\rho^2 A^2 L C_s S_a$)

ζ \ case	A CQC	B Ritz(1)	C Ritz(2)	D Ritz(3)	E SRSS
0.01	184.78	232.52	204.11	207.87	184.73
0.05	34.23	45.08	37.85	38.59	34.00
0.10	17.40	22.58	19.05	19.49	16.99
0.20	9.14	11.38	9.77	10.11	8.48

Table 5 Second Moment of $U(L)$ (to be multiplied by LS_a/C_s)

ζ \ case	A CQC	B Ritz(1)	C Ritz(2)	D Ritz(3)	E SRSS
0.01	184.60	181.86	190.70	184.40	184.50
0.05	33.77	32.21	35.15	33.78	33.77
0.10	16.63	17.49	17.48	16.74	16.76
0.20	7.99	8.52	8.55	8.12	8.25

Table 6 Second Moment of $Q(0)$ (to be multiplied by $\rho^2 A^2 L^{-1} C_s^3 S_a$)

ζ \ case	A CQC	B Ritz(1)	C Ritz(2)	D Ritz(3)	E SRSS
0.01	724.55	643.94	1071.11	959.32	748.05
0.05	126.61	124.04	205.63	181.49	139.94
0.10	64.05	61.48	101.56	89.97	68.50
0.20	36.83	30.66	49.04	45.55	32.94

Ritz approximations of the second spectral moments of the shear force $Q(0)$ are quite disappointing, see Table 6; and so a Ritz basis better than Cases C and D is required for such higher order spectral moments of internal force(s).

3.2. Example 2.

The soil dam shown in Fig.1 is regarded as a shear beam. Its height is $L=20$m, and its sectional area per unit width varies linearly along the vertical direction, i.e. $A(\eta)=A_o(1-b\eta)$, in which $\eta=z/L$ and $b=0.5$ for this example. The other parameters of the dam are shear modulus$=8\times10^4$kN/m^2, density$=2.0$ t/m^3 and $A_o=15$m^2. The PSD of the ground acceleration follows the widely used Kanai-Tajimi filtered-white-noise spectrum formula:

$$S_{\ddot{x}_g}(\omega) = \frac{1+4\zeta_g^2(\frac{\omega}{\omega_g})^2}{[1-(\frac{\omega}{\omega_g})^2]^2+4\zeta_g^2(\frac{\omega}{\omega_g})^2} S_o \tag{27}$$

in which $\omega_g=15.6\text{sec}^{-1}$ and $\zeta_g=0.6$. A q-dimensional Ritz set is defined as $R(q)=[\eta, \eta^2, \eta^3, \ldots, \eta^q]$.

In Table 7, the variances and second spectral moments of the top displacement and base shear force are listed in the first four columns for $q=2, 3, 4, 10$. The integration interval was $\omega \in [0, 8]$ sec^{-1}, and the step length used was $\delta\omega=0.01$ sec^{-1}. Hysteretic damping was assumed with $\varepsilon=0.1$.

The solutions for $q=10$ are regarded as the exact ones. Obviously, the two-dimensional Ritz set is good enough for σ_u^2, the variance of displacement $U(L)$, whereas the three-dimensional Ritz set is required for satisfactory results for the second displacement spectral moments $\lambda_{2,U}$ or for the shear force variances σ_Q^2. However, $q=4$ is necessary if the second spectral moment of the shear force, $\lambda_{2,Q}$, is of interest.

In general, the computation of covariances is not required so often as that of variances. When they are of interest, e.g. that of the ground acceleration and beam-top velocity, $C_{\ddot{X}g,\dot{U}}$, or that of the ground acceleration and the first derivative of the beam-base force $C_{\ddot{X}g,\dot{Q}}$, the basic principles described above Eq. (2) enable them to be computed from the pseudo excitation $\ddot{X}_g(t)$ and the pseudo responses $\dot{U}(t)$ and $\dot{Q}(t)$ as follows

$$\dot{U}=i\omega\, U, \qquad \dot{Q}=i\omega\, Q \tag{28}$$

$$S_{\ddot{X}g,\dot{U}}=\ddot{X}_g^*\dot{U}, \qquad S_{\ddot{X}g,\dot{Q}}=\ddot{X}_g^*\dot{Q} \tag{29}$$

$$C_{\ddot{X}g,\dot{U}}=\int_{-\infty}^{\infty} S_{\ddot{X}g,\dot{U}}(\omega)d\omega, \qquad C_{\ddot{X}g,\dot{Q}}=\int_{-\infty}^{\infty} S_{\ddot{X}g,\dot{Q}}(\omega)d\omega \tag{30}$$

The processes are as simple as those needed to compute the variances. The imaginary parts of Eqs. (30) automatically vanish (Lin Y.K.1967). Some numerical results for different Ritz dimensions were computed simultaneously with those associated with the auto-PSD moments, see the fifth and sixth columns of Table 7. Finally,the seventh column of Table 7 lists the total processing time for each Ritz set on a IBM/AT personal computer and shows that this method is very efficient indeed.

Table 7 Variation of Spectral Moments with q

Ritz Dimen	σ_U^2	$\lambda_{2,U}$	σ_Q^2	$\lambda_{2,Q}$	$C_{\ddot{X}g,\dot{U}}$	$C_{\ddot{X}g,\dot{Q}}$	IBM/AT Computer Time
q=2	14.81	46.39	40.69	124.23	−4.110	−10.31	11.0
q=3	14.39	44.25	27.92	91.13	−4.102	−11.46	15.4
q=4	14.39	44.17	27.41	81.91	−4.221	−10.31	19.7
q=10	14.34	44.09	27.10	81.23	−4.285	−10.09	73.0
units	$S_o(L/C_s)^3$	$S_o(L/C_s)$	$S_o\rho^2A_o^2LC_s$	$S_o\rho^2A_o^2C_s^3/L$	S_o	$S_o\rho A_oC_s^2/L$	sec

4. Conclusions

It can be seen from the above results that when the direct Ritz analysis is used:

(1) It is comparatively easy to achieve a satisfactory precision for those calculations associated with displacement, i.e. the PSD, the variances, the covariances, and even, to a lower precision, the second moments. However, more and/or better Ritz functions are needed when the PSD of internal forces and their spectral moments are required to ensure their precision.

(2) Special care should be taken when using the Ritz analysis to find the higher order

spectral moments.

For general non-uniform beams, application of the conventional CQC method, or even of the SRSS methods, to random seismic response analysis is rather tedious, whereas the present Ritz method is much less tedious. In fact, if eigenmodes are taken as the Ritz functions, the present direct Ritz method will give various PSD and their moments which are exactly identical to those given by conventional CQC method (Clough and Penzien 1975).In addition, for various damping cases including those listed in Table 1, the direct Ritz method is also a very convenient tool.

Selection of Ritz functions is a common concern when using any Ritz method. In the method presented, if the analysis is being performed by hand with the help of a calculator, the use of as few Ritz functions as possible may be preferred but the quality of such functions should be good, e. g. those listed as case D in Table 2. However, when computers are being used power functions, possibly with higher order than for case C of Table 2, are suggested because such Ritz functions are very easy to deal with on computers and the increase in computational effort is not severe, e. g. see the final column of Table 7.

It is seen from example 2 that all formulae in this paper apply not only to white noise excitation, but also to filtered white noise or any other specified excitation. As a computer -oriented method, all computations should be carried out over discrete ω points. Different excitation spectra only result in different amplitudes of the pseudo harmonic excitations expressed by Eq. (2). Therefore, if the frequency intervals remain the same, the computational effort required by any sort of excitation spectrum is almost exactly equal to that for the case of white noise excitation. However, if a non-white excitation varies rapidly with ω, a smaller frequency interval will have to be taken and so the computational effort will increase accordingly.

Acknowledgements

This work has been supported jointly by the Natural Science Foundation of China and the British Royal Society. The School of Engineering, University of Wales College of Cardiff also provided much help in the provision of good computer facilities as well as working conditions. Mr. S.M. Hoh gave much help and the third author was supported by a studentship from the British Science and Engineering Research Council.

References

Clough, R.W. and Penzien, J.(1975), *Dynamics of Structures*, McGraw-Hill Inc., New York.

Crandall, S.H. and Yildiz, A. (1962), "Random vibration of beams", *Jnl. Applied Mechanics*, **29**(3), 267-274.

Gasparini, D.A., Debchaudhury and Gazetas, G.(1981), "Random vibration of cantilever shear beams", *Jnl.Earthquake. Eng.Struct. Dyn.*, **9**(6), 599-612.

Lin, Jiahao(1992), "A fast CQC algorithm of PSD matrices for random seismic responses", *Computers & Structures*, **44**(3), 683-687.

Lin, J.H., Williams, F.W. and Zhang, W.S.(1992), "A new approach to multiphase excitation stochastic seismic response", *Proc. Int. Conf. EPMESC-IV*, Dalian, July, 1992, Dalian University of Technology Press, Dalian, China, 515-524.

Lin, Y.K.(1967), *Probabilistic Theory of Structural Dynamics*, McGraw-Hill, New York.

Warburton, G.B.(1976), *The Dynamic Behaviour of Structures,* Pergamon Press, Oxford.