Structural Engineering and Mechanics, Vol.2, No.1 (1994) 35-48
DOI: http://dx.doi.org/10.12989/sem.1994.2.1.035

A concrete plasticity model with elliptic failure surface and independent hardening/softening

Hamdan N. Al-Ghamedy†

Dep. of Civil Engineering, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia

Abstract. A plasticity-based concrete model is proposed. The failure surface is elliptic in the σ-τ stress space. Independent hardening as well as softening is assumed in tension, compression, and shear. The nonlinear inelastic action initiates from the origin in the σ-ε (τ-γ) diagram. Several parameters are incorporated to control hardening/softening regions. The model is incorporated into a nonlinear finite element program along with other classical models. Several examples are solved and the results are compared with experimental data and other failure criteria. "Reasonable results" and stable solutions are obtained for different types of reinforced concrete oriented structures.

Key words: concrete model; plasticity; failure surface; independent hardening/softening; ellipse; nonlinear finite element; reinforced concrete.

1. Introduction

Concrete behavior is undoubtedly complex, and its strength and characteristics are functions of many variables. Among many factors, the concrete strength and response behavior depend on the physical and mechanical properties of the matrix-aggregate mesostructure as well as on the nature of loading. It is almost unimaginable to mathematically characterize its behavior under all possible loading conditions in a convenient form. This led to the development of many different models and failure theories each with its own limitations and shortcomings. A model or failure criterion may fit well with some test results for a particular type of concrete and one kind of structure and for a special loading, but not for other conditions. There are several mathematical models which describe the uni-, bi- and tri-axial strength of concrete; however, the description of multi-axial stiffness and ductility under different loading conditions is far less developed. This is the reason for such topic to still be an active area of research in recent years.

In the last three decades, many authors developed several constitutive theories with various assumptions. They include nonlinear isotropic/anisotropic elastic, elasto-plastic with hardening/softening, microcracking theory and endochronic theory models. Parameters ranging from one to five are used in different models under uni- and multi-axial stress states. Since the literature on the subject is so numerous, no attempt is made here to go over these topics or even to discuss some in details. Focus is rather placed on the current model and its unique features. For more information about the published literature, references (Stevens et al. 1991,

† Assistant Professor

Fanella 1990, Levine 1982, Stankowski and Gerstle 1985, Kotsovos and Newman 1977, Scavuzzo et al. 1983, Hsu et al. 1963, Tasuji et al. 1978, Gerstle et al. 1980, Kotsovos and Newman 1978, Willam and Sture 1986, Willam et al. 1984, 1985, 1986, Willam 1984, Hurlbut 1985, Bresler and Pister 1958, Ottosen 1977, Willam and Warnke 1975, van Mier 1984, Chen and Saleeb 1982, Desai and Siriwardane 1984, Nayak and Zienkiewicz 1972, Chen and Chen 1975, Committee on Concrete and Masonry Structures 1982, Elwi and Murray 1979, Bazant and Shieh 1978, Yoder and Iwan 1981, Lade and Nelson 1981, Krajcinovic and Fonseka 1981) can be consulted.

2. Failure criterion

One of the most commonly used failure criteria for brittle materials is Mohr-Coulomb. It was originally developed for soil. When applied to concrete, it is usually necessary to have stress and/or strain cut-offs. This creates problems numerically as well as in the definition of the gradient of the plastic potential in some regions, as tried and reported by several authors (Ghamedy 1986, Day 1985). Because of that, it is tried here to develop a model which avoids such difficulties.

Most, if not all, concrete plasticity models have one or more of the following shortcomings:

1. Discontinuities exist because of the cut-offs and/or cracking, which make it inconvenient to work with.
2. Hardening behavior only is hypothesized, although concrete materials exhibit hardening as well as softening in tension and compression.
3. Initial linear portion on the stress-strain diagram is assumed (which might not be observed in the experiment).
4. "Arbitrary" coupling among different stress components is normally invoked, even though the influence of hardening/softening of a stress component on other components is not fully understood in brittle materials, especially under low stress levels (Chen 1982).

Because of these reasons, a plasticity-based concrete model with a smooth and continuous failure surface and independent hardening/softening loading functions in tension, compression, and shear is proposed. It is meanly intended for "oriented" structures in which the state of stress consists of one normal and one shear stresses as described next.

An ellipse is chosen to represent the failure surface since it can approximate the Mohr's en-

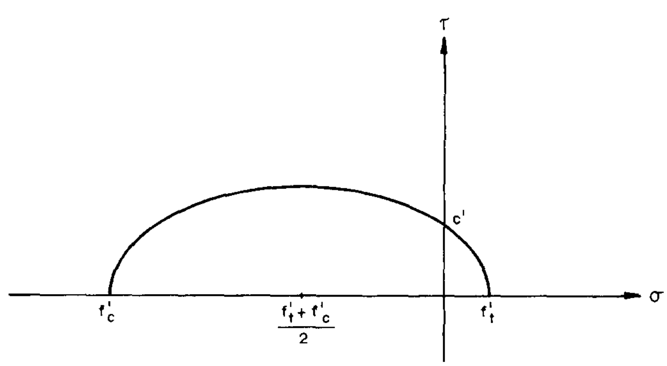


Fig. 1 Elliptic failure surface in σ-τ space

velope with cut-offs. The formulation is written in the σ-τ stress space. The ellipse must meet the following conditions (Fig. 1):

1. The ellipse is symmetric about the σ-axis.
2. The major axis is in the horizontal (σ) direction since $(f_t'-f_c') >> 2c'$.
3. The vertices are at points $(f_t',0)$ and $(f_c',0)$ with $f_t' > 0$ and $f_c' < 0$.
4. The ellipse intersects with $\pm\tau$-axis at point c'.

Where f_t', f_c', and c' are the strength in tension, compression, and shear, respectively. Thus, the elliptic equation becomes

$$\frac{(\sigma-h)^2}{a^2}+\frac{\tau^2}{b^2}=1 \tag{1}$$

where

$$a=(f_t'-f_c')/2 \tag{2}$$

$$h=(f_t'+f_c')/2 \tag{3}$$

At $\sigma = 0$, $\tau = \pm c'$, therefore

$$b^2=\frac{a^2}{a^2-h^2}c'^2 \tag{4}$$

After some mathematical manipulation,

$$a^2-h^2=-f_t'f_c' \tag{5}$$

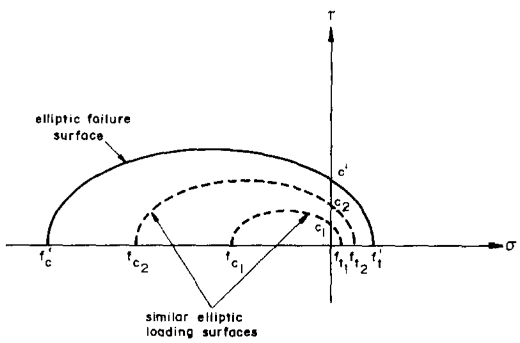
Now after some algebraic operations, the equation for the failure surface can be written as

$$F=(\sigma-h)^2-\frac{f_t'f_c'}{c'^2}\tau^2-a^2=0 \tag{6}$$

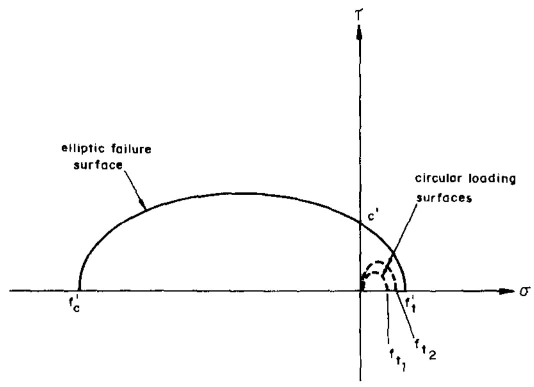
After developing the failure surface which is smooth and continuous, a hardening/softening rule must be specified. Independent hardening and softening in tension, compression, and shear is assumed in this model for the reason stated above. In the past, some researchers suggested different models for concrete displaying independent hardening in tension and compression, e.g. (Chen and Chen 1975, Murray et al. 1979, Epstein and Murray 1978), but the one assumed here is different as outlined below.

Figures 2(a) to 2(f) show some possible shapes for the loading surfaces. All but Fig. 2(f) exhibit some kind of coupling among stress components; therefore, the hardening/softening shown in Fig. 2(f) is chosen since it is completely independent anywhere in the area enclosed by the elliptic surface. Different coupled loading surfaces were first tried in this study, as shown in Figs. 2(a)-2(e), but later they were dropped as they are not based on any theory, in addition to making the formulation more complicated. It should be mentioned that Fig. 2(f) is not based on a theory either. However, the coupling among different stress components under low stress levels is not quite known, as stated earlier, (Chen 1982) , thus the loading surface in Fig. 2(f) is selected since it makes the formulation easier.

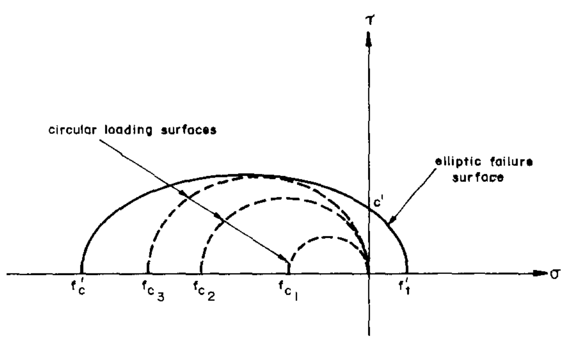
The proposed hardening/softening functions for each component are stated below. They are similar to that suggested by Sargin (1971) for concrete under compression. They are attractive because the shape of the curve, especially the post peak portion, can be modified easily so that it fits the experimental data in compression, tension, as well as shear.



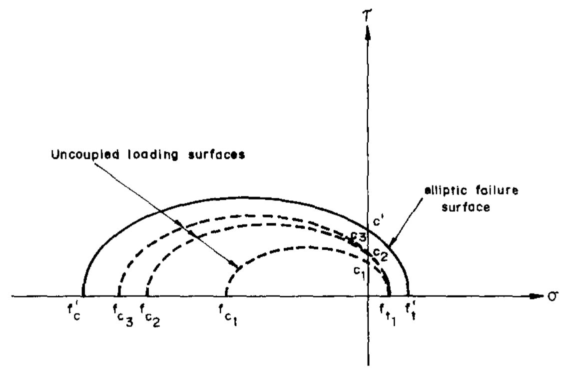
(a) Similar elliptic loading surfaces



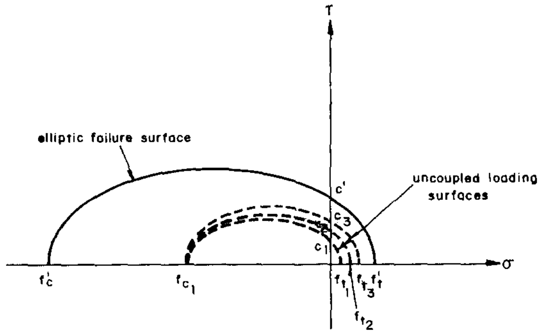
(b) Loading surfaces in uni–axial tension



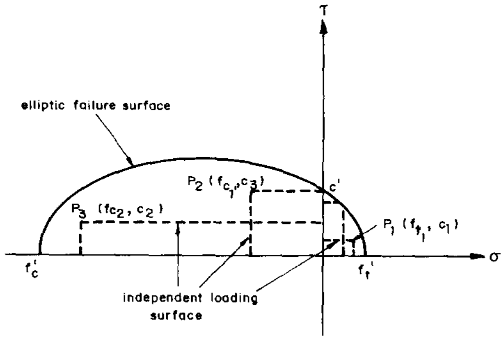
(c) Loading surfaces in uni–axial compression



(d) Uncoupled loading surfaces(compression and shear)



(e) Uncoupled loading surfaces(tension and shear)



(f) Independent rectangular loading surfaces(tension and shear & compression and shear).

Fig. 2 Different options for the loading surfaces

$$\sigma_t = \frac{\dfrac{E\epsilon_t}{f'_t} + (D_t - 1)\left(\dfrac{\epsilon_t}{\epsilon'_t}\right)^2}{1 + \dfrac{E\epsilon_t}{f'_t} - 2\dfrac{\epsilon_t}{\epsilon'_t} + D_t\left(\dfrac{\epsilon_t}{\epsilon'_t}\right)^2} f'_t \tag{7}$$

$$\sigma_c = \frac{\dfrac{E\epsilon_c}{f'_c} + (D_c - 1)\left(\dfrac{\epsilon_c}{\epsilon'_c}\right)^2}{1 + \dfrac{E\epsilon_c}{f'_c} - 2\dfrac{\epsilon_c}{\epsilon'_c} + D_c\left(\dfrac{\epsilon_c}{\epsilon'_c}\right)^2} f'_c \tag{8}$$

$$\tau = \frac{\dfrac{G\gamma}{c'} + (D_s - 1)\left(\dfrac{\gamma}{\gamma'}\right)^2}{1 + \dfrac{G\gamma}{c'} - 2\dfrac{\gamma}{\gamma'} + D_s\left(\dfrac{\gamma}{\gamma'}\right)^2} c' \tag{9}$$

The parameters are illustrated in Fig. 3. Note that D is a parameter which mainly controls the softening regimes as shown in Fig. 4. Thus, it can be changed depending on the state of stress as well as the confinement the concrete is under.

The gradient of the plastic potential Q or the flow vector (assuming associative plasticity) for the failure surface is given by

$$\frac{\partial F}{\partial \sigma} = \frac{\partial Q}{\partial \sigma} = \left\{ \begin{array}{c} 2(\sigma - h) \\ 0 \\ -\dfrac{2f'_t f'_c}{c'^2}\tau \end{array} \right\} \tag{10}$$

By differentiation, the tangential modulus in tension can be shown to be

$$E_t^t = \frac{[1+(A-2)B+DB^2]\,[A+2(D-1)B] - [AB+(D-1)B^2]\,[A-2-2DB]}{[1+(A-2)B+DB^2]\,\epsilon'_t} f'_t \tag{11}$$

where

$$A = \frac{E\epsilon'_t}{f'_t} \tag{12}$$

and

$$B = \frac{\epsilon_t}{\epsilon'_t} \tag{13}$$

Similar expressions are obtained for the tangential moduli in compression and shear.

This concrete plasticity model presented above is formulated and incorporated into a nonlinear finite element computer program. An incremental-iteration solution algorithm, in which the applied load/prescribed displacement is divided into small increments within each iterations are performed until both equilibrium and constitutive relations are satisfied, is written and utilized in the current research. The initial, forward-Euler, as well as tangential stiffness methods are all supported by then present nonlinear solution technique. Either load or displacement norm may be selected for convergence check. Load or displacement control strategy can be chosen.

In the analysis, layered degenerated isoparametric curved beam elements with nonhomo-

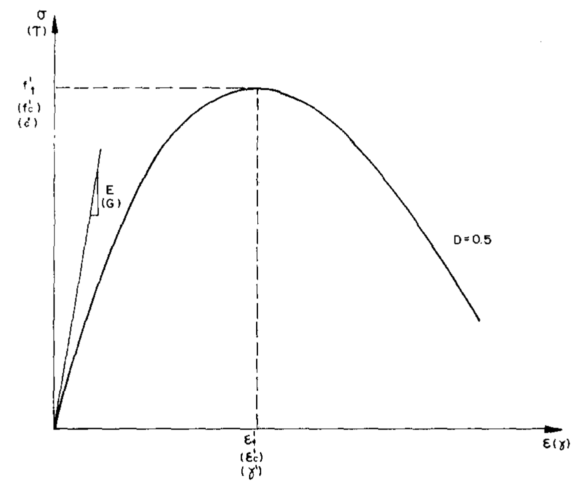


Fig. 3 Hardening/softening in tension, compression and shear

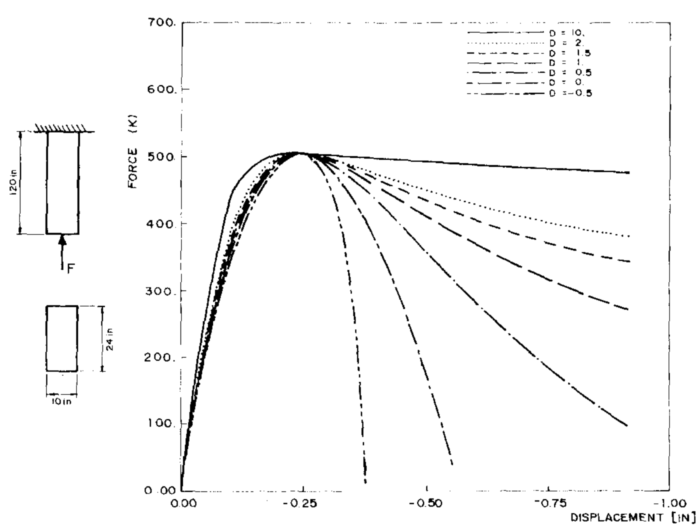


Fig. 4 Influence of the softening parameter D

geneous cross-sections are utilized (Al-Ghamedy and Willam 1990). They have three degrees of freedom per node, two displacements and a rotation. Axial, bending and shear deformations are all accounted for. Such elements have proven to be very efficient and general, especially in the elasto-plastic analysis of reinforced concrete.

3. Application examples

Several examples of reinforced concrete (RC) structures are solved and the results are compared with experimental data and/or other solutions using different failure criteria. Maximum normal stress (Rankine) and Tresca theories are incorporated for reinforcing steel (S), while for concrete (C), maximum normal stress and "modified" Mohr-Coulomb criteria are added. Tension and compression cut-offs are made in Mohr-Coulomb, and limited ductility in the compressive strain is invoked; in addition linear hardening only is assumed with no softening allowed in compression (Ghamedy 1986). For more details, that reference can be referred to.

3.1. Shear beams

Four shallow and deep RC beams, two under-reinforced (UR) and two over-reinforced (OR), are analyzed. They are depicted in Fig. 5. The load-displacement curves are drawn in Figs. 6-9. It can be seen that shallow beams are not very sensitive to the failure criteria, however, OR section is relatively more sensitive than UR section. This is expected as concrete plays more role in OR structures. For deep beams, however, the behavior is distinctly different, depending on the failure theory chosen. As expected Rankine theory gives greater value for the ultimate load, while the proposed criterion gives the lowest value; it is more stable and smoother than "modified" Mohr-Coulomb. Since no experimental data is available, no comparison can be made. This is done in all other examples.

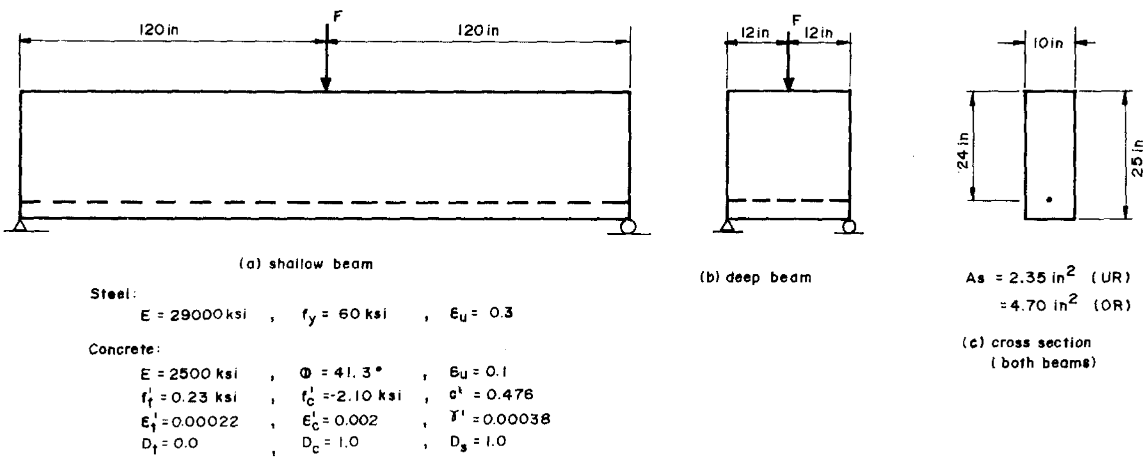


Fig. 5 Shallow and deep beams

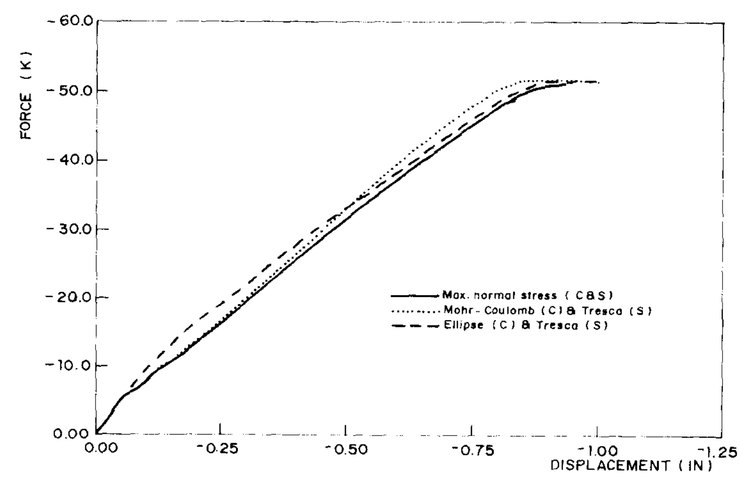


Fig. 6 Load-displacement curve for UR shallow beam

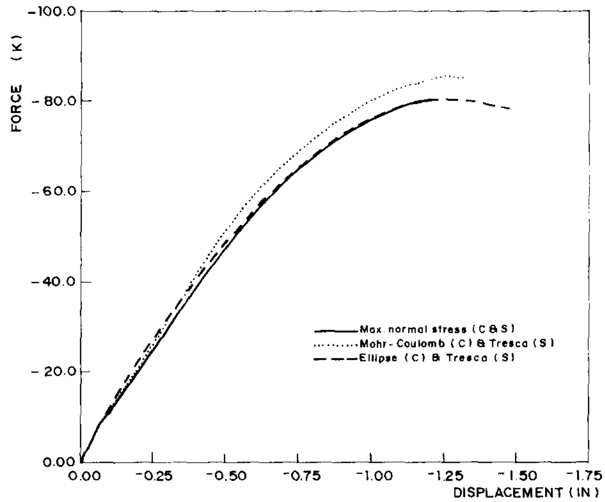


Fig. 7 Load-displacement curve for OR shallow beam

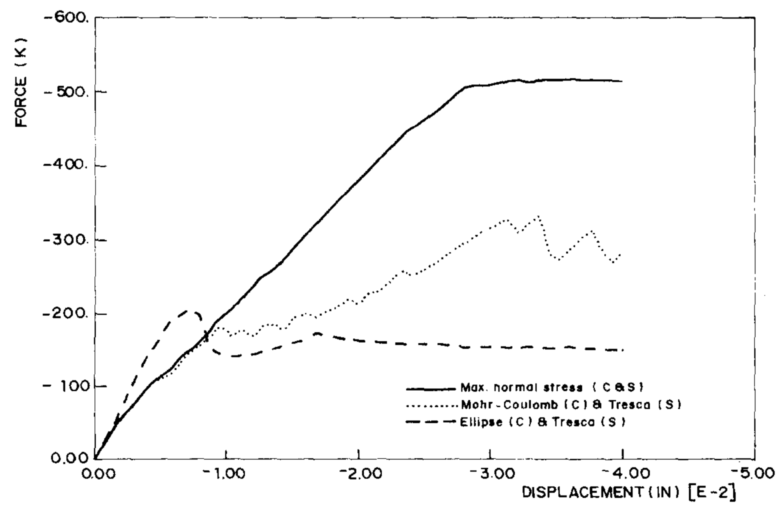


Fig. 8 Load-displacement curve for UR deep beam

3.2. Dutch and German Beams

Two laboratory-tested RC beams are analyzed, and the results are compared with experimental data. A Dutch beam (Walvaren 1982, Blaauwendraad et al. 1983) and a beam tested at Karlsruhe, Germany (Franz and Brenker 1967, Stankowski 1980) are shown in Figs. 10 and 11, respectively. Parameters not given are assumed just like the first example above.

The load-displacement curves for the two beams are drawn in Figs. 12 and 13 using several criteria along with the experimental curves. It can be seen that all failure theories give acceptable results, however, the criterion with elliptic surface is the best, especially when the softening parameter in tension (D_t) is set equal to 0.5.

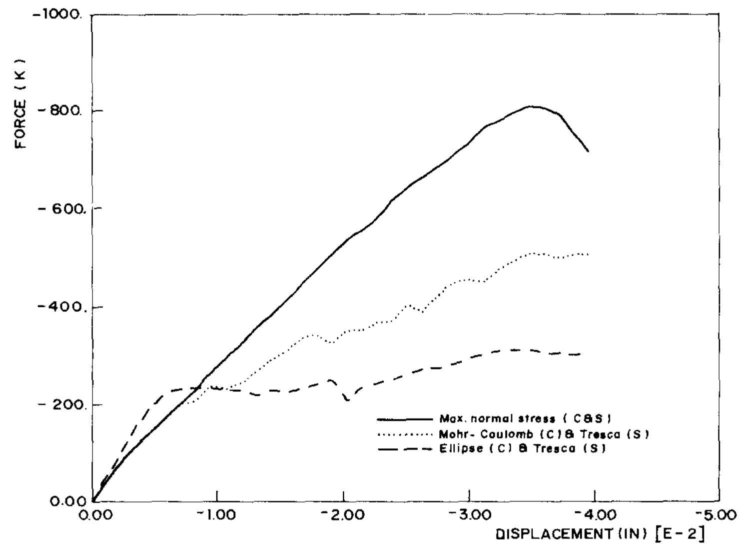


Fig. 9 Load-displacement curve for OR deep beam

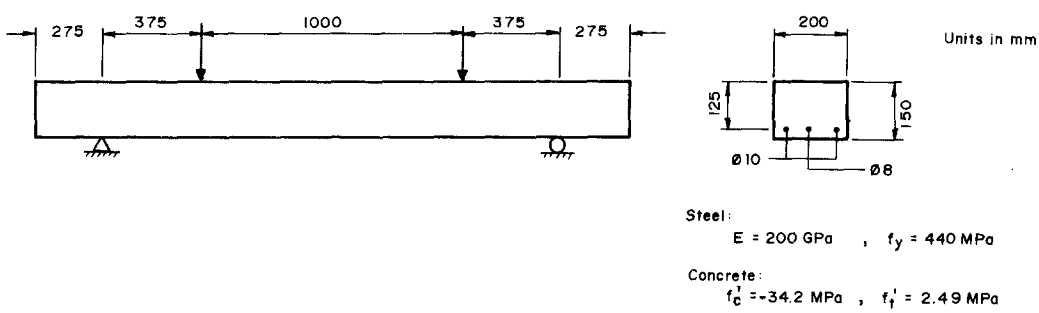


Fig. 10 Dutch beam

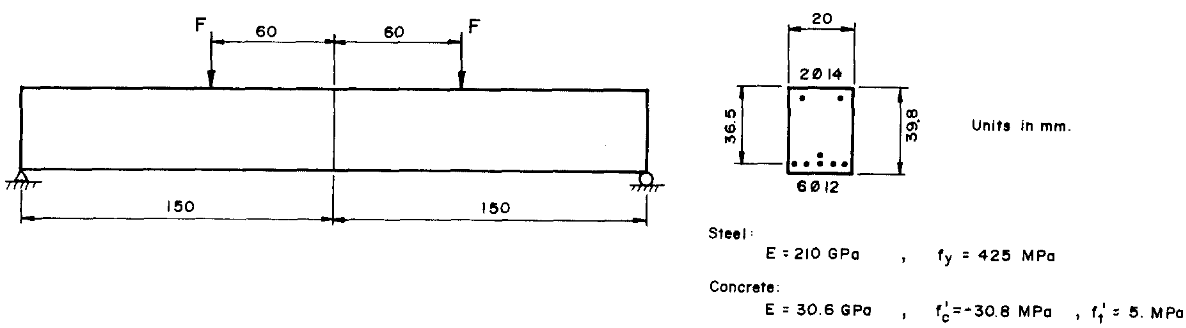


Fig. 11 German beam

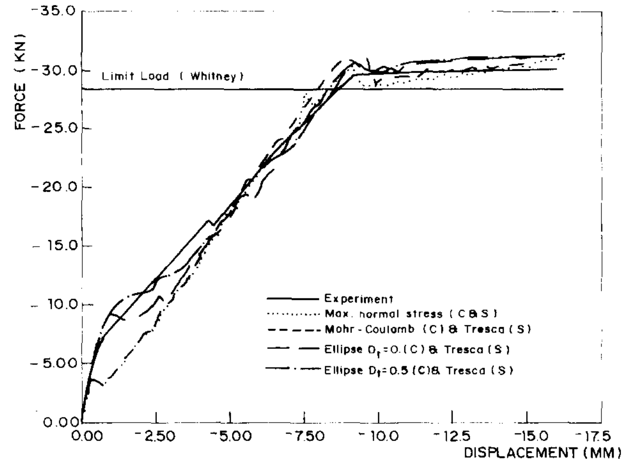


Fig. 12 Load-displacement curve for Dutch beam

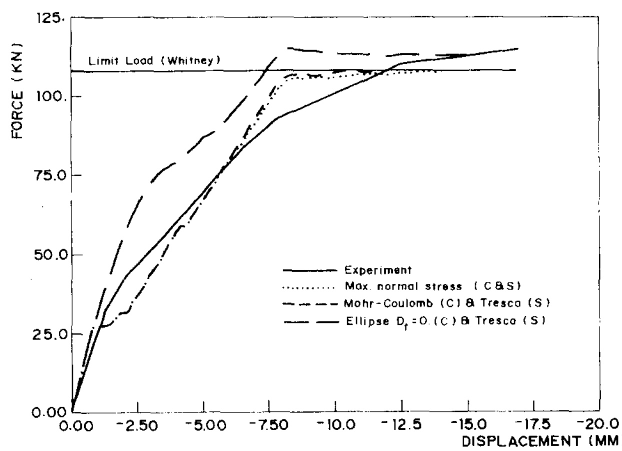


Fig. 13 Load-displacement curve for German beam

3.3. Circular rings

The RC circular conduits shown in Figs. 14 and 15 were tested at the U.S. Army Engineer Waterways Experiment Station (Chiarito and Wright 1984, Wright and Chiarito 1984). The load-displacement responses for the thin and thick rings are illustrated in Figs. 16 and 17.

Compared with the thin ring test value of 7.3 k for the crown load at ultimate, the maximum normal stress criterion gives a value of 6.4 k, while the proposed model predicts a value of 7.07 k for the collapse load. In another analysis (Gerstle 1985), worse results were obtained assuming different failure criteria and state of stress. For the thick ring, the ultimate load values are 22.15 k, 21.10 k, 23.44, and 16.24 in the experiment, elliptic surface, maximum normal stress, and Mohr-Coulomb, respectively. The correct failure mechanism is captured in both rings; hinges are formed at quarter points. Parameters not determined by the test are assumed as in previous examples. The experimental load-displacement curves could not be obtained by the author.

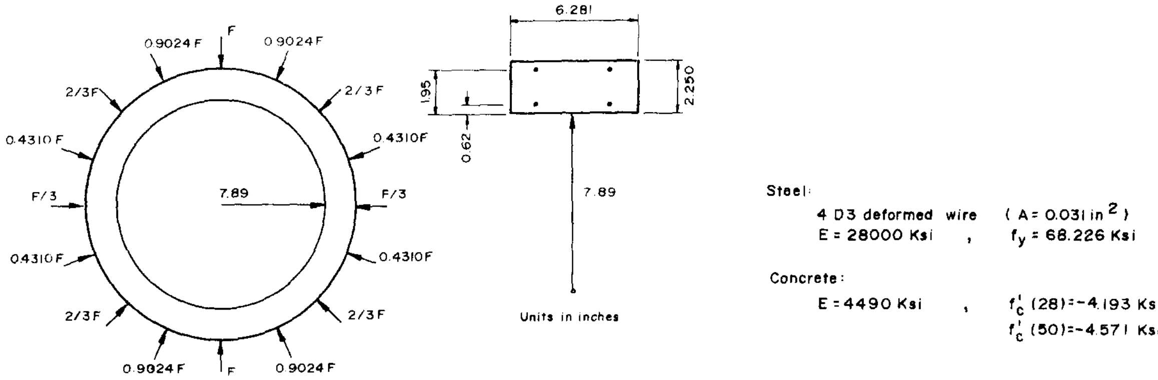


Fig. 14 Thin ring

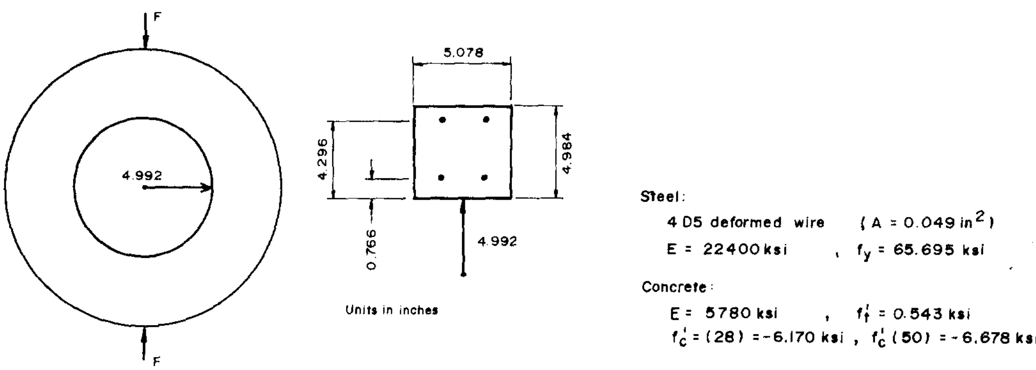


Fig. 15 Thick ring

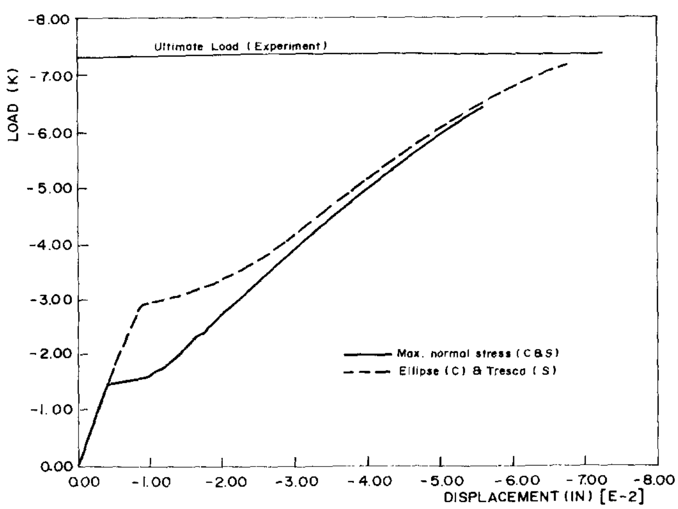


Fig. 16 Load displacement curve for thin ring

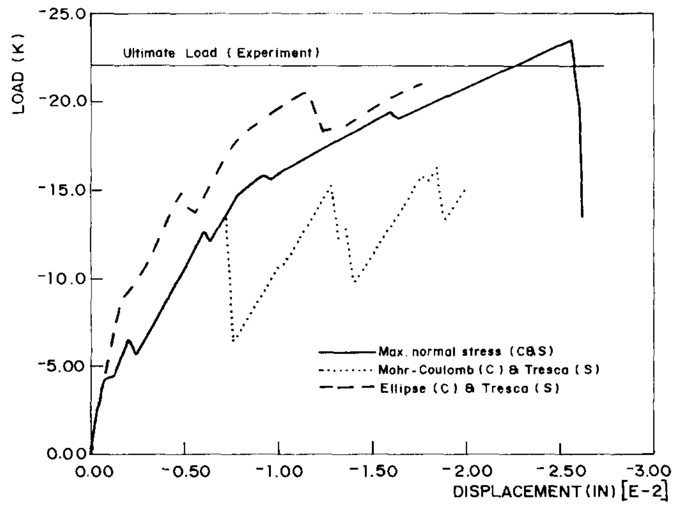


Fig. 17 Load displacement curve for thick ring

4. Conclusions

The proposed concrete model is shown to be good, reasonable, and (to some extent) practical. The results for different reinforced concrete structures compare well with experimental observations and are better than the predictions of other old models. The algorithm is numerically stable and the convergence rate is good. Several parameters are included in the model so that they can be modified, depending on the concrete on hand and the structure considered. One shortcoming of the model is the decrease in shear strength under high compressive stress due to the elliptic shape of the failure surface, however, this is in a small region only. A "skewed" ellipse may be used to remedy this problem, but, the formulation gets more complicated, and the model may become impractical to use. Nevertheless, it is the intention of the author to try this in the future and to compare the results with the present model.

Acknowledgements

The author extends his appreciation to Prof. Kaspar J. Willam of University of Colorado, Boulder, USA, for the discussions and ideas exchanged with him. The support of King Fahd University of Petroleum & Minerals is also acknowledged.

References

Bazant, Z.P. and Shieh, C.L. (1978), "Endochronic models for nonlinear triaxial behavior of concrete", *Nuclear Engineering and Design*, **47**, 305-315.

Blaauwendraad, J., Berg, F.J. and Merks, P.J. (1983), "Bench mark problems", *Rijkwaterstaat Structural Research*, Delft University.

Bresler, B. and Pister, K.S. (1958), "Strength of concrete under combined stresses", *J. ACI*, **55**, 321-345.

Chen, W.F. and Saleeb, A.F. (1982), "Constitutive equations for engineering materials", **1**, *Elasticity and*

Modeling, John Wiley and Sons, New York.

Chen, A.C.T. and Chen, W.F. (1975), "Constitutive relations for concrete", *J. Eng. Mech. Div., ASCE,* **101** (EM4), 465-481.

Chen, W.F. (1982), *Plasticity in Reinforced Concrete*, McGraw-Hill, New York.

Chiarito, V.P. and Wright, R.S. (1984), *Ultimate Behavior of Reinforced Concrete Circular Conduits*, Struct. Mech. Div., Structures Laboratory, USAE Waterways Experiment Station, Vicksburg, Mississippi.

Committee on Concrete and Masonry Structures, *State of the Art Report on Finite Element Analysis of Reinforced Concrete, Chapter 2-Constitutive Relations and Failure Theories*, Task Committee on Finite Element Analysis of Reinforced Concrete Structures (Arthur H. Nilson, Chairman), ASCE 1982.

Day, S.E. (1985), "A softening plasticity model for concrete", *M.S. Thesis*, CEAE Department, University of Colorado, Boulder.

Desai, C.S. and Siriwardane, H.J. (1984), *Constitutive Laws for Engineering Materials with Emphasis on Geologic Materials*, Prentice-Hall Inc., Englewood Cliffs, NJ.

Elwi, A.A. and Murray, D.W. (1979), "A 3D hypoelastic concrete constitutive relationship", *J. Eng. Mech. Div., ASCE,* **105** (EM4), 623-641.

Epstein, M. and Murray, D.W. (1978), "A biaxial law for concrete incorporated in BOSORS Code", *Computers & Structures,* **9** (1), 57-63.

Fanella, D.A. (1990), "Fracture and failure of concrete in uniaxial and biaxial loading", *J. Eng. Mech., ASCE,* **116** (11), 2341-2362.

Franz, G. and Brenker, H. (1967), "Verformungsuersche an stahlbetonbalken mit hochfestem bewehrungsstahl", *Heft,* **188**, Von Wilhelm Ernst & Sohn, Deutscher Ausschuss fur Stahlbeton, Berlin.

Gerstle, K.H., Aschi, H., Bellotti, R., Bertacchi, P., Kotsovos, M.D., Ko, H.Y., Linse, D., Newman, J.B., Russi, P., Schickert, G., Taylor, M.A., Traina, L.A., Winkler, H. and Zimmerman, R.M. (1980), "Behavior of concrete under multiaxial stress states", *J. Eng. Mech. Div., ASCE,* **106**, 1383-1403.

Gerstle, K.H. (1985), "Analytical study of WES ring tests", *Structural Research Series 8504,* CEAE Department, University of Colorado, Boulder.

Ghamedy, H.N. (1986), "Nonlinear finite element analysis of inelastic structures with applications to reinforced concrete", *Ph.D. Dissertation*, CEAE Department, University of Colorado, Boulder.

Al-Ghamedy, H.N. and Willam, K.J. (1990), "Layered nonhomogeneous curved beam elements for inelastic analysis", *Computers & Structures,* **37** (4), 521-534.

Hurlbut, B.S. (1985), "Experimental and computational investigation of strain softening in concrete", *M.S. Thesis*, CEAE Department, University of Colorado.

Hsu, T.T.C., Slate, F.O., Sturman, G.M. and Winter, G. (1963), "Microcracking of plain concrete and the shape of.the stress-strain curve", *J. ACI,* **60** (2), 209-224.

Kotsovos, M.D. and Newman, J.B., (1977), "Behavior of concrete under multiaxial stress", *J. ACI,* **74** (9), 443-446.

Kotsovos, M.D. and Newman, J.B. (1978), "Generalized stress-strain relations for concrete", *J. Eng. Mech. Div., ASCE,* **104** (EM4), 843-856.

Krajcinovic, D. and Fonseka, G.V. (1981), "The continuous damage theory of brittle materials, Part 1: General theory", *ASME Trans., Journal of Applied Mechanics,* **4** (4), 809-815.

Lade, P.V. and Nelson, R.B. (1981), "Incrementalization procedure for elasto-plastic constitutive model with mutiple, simultaneous yield surfaces", In: *Implementation of Computer Procedures and Stress-Strain Laws in Geotechnical Engineering,* **II**, (Desai, C.S., Saxena, S.K., eds.), Acorn Press.

Levine, H.S. (1982), "A two-surface plastic and microcracking model for plain concrete", *Nonlinear Numerical Analysis of Reinforced Concrete, Winter Annual Meeting, ASME,* Phoenix, Az, Nov. 14-19, 1982, 27-47.

Murray, D.W., Chituyanondh, L., Rijub-Agha, K.Y. and Wong, C. (1979), "A concrete plasticity theory for biaxial stress analysis", *J. Eng. Mech. Div., ASCE,* **105** (GM6), 989-1106.

Ottosen, N.S. (1977), "A failure criterion for concrete", *J. Eng. Mech. Div., ASCE,* **103** (EM4), 527-535.

Nayak, G.C. and Zienkiewicz, O.C. (1972), "Elasto-plastic stress analysis generalization of various constitutive relations including strain softening", *Int. Journal for Numerical Methods in Engineering,* **5**, 113-

135.

Sargin, M. (1971), "Stress-strain relationships for concrete and the analysis of structural concrete sections", *Study No. 4, Solid Mech. Div.*, University of Waterloo, Canada.

Scavuzzo, R., Stankowski, T., Gerstle, K.H. and Ko, H.Y. (1983), "Stress-strain curves for concrete under multiaxial load histories", *Report*, University of Colorado, Boulder.

Stankowski, T. and Gerstle, K.H. (1985), "Simple formulation of concrete behavior under multiaxial load histories", *J. ACI,* **82** (2), 213-221.

Stankowski, T. (1980), *Grenzlastanalyse eines stahlbetonbalkens mit der F.E. methode*, Diplom-Arbeit, Inst. Fuer Massivbau, Universitaet Stuttgart, Germany.

Stevens, N.J. Uzumeri, S.M., Collins, M. P. and Will, G.T. (1991). "Constitutive model for reinforced concrete finite element analysis", *ACI Structural Journal,* **88** (1), 49-59.

Tasuji, M.E., Slate, F.O. and Nilson, A.H. (1978), "Stress-strain response and fracture of concrete in biaxial loading", *J. ACI,* **75** (7), 306-312.

van Mier, J.G.M. (1984), "Strain-softening of concrete under multiaxial loading conditions", *Ph.D. Dissertation,* University of Technology, Eindhoven.

Walraven, J.C. (1982), "The influence of depth on the shear strength of lightweight concrete beams without shear reinforcement", Stevin Laboratory, *Report No. 5-78-4*, Delft University of Technology, Delft, The Netherlands.

Willam, K.J. and Sture, S. (1986), "Finite elements and localized failure: Final technical report", *Structural Research Series 8601*, CEAE Department, University of Colorado, Boulder.

Willam, K.J., Bicanic, N. and Sture, S. (1984), "Constitutive and computational aspects of strain-softening and localization in solids", *Symp. on Constitutive Equations, Macro and Computational Aspects, ASME /WAM 84 Meeting,* New Orleans, ASME (Willam, K., ed.), **G00274**, New York, 233-252.

Willam, K.J., Hurlbut, B. and Sture, S. (1985), "Experimental, constitutive and computational aspects of concrete failure", *Proc. U.S.-Japan Joint Seminar on Finite Element Analysis of Reinforced Concrete Structures*, **1**, University of Tokyo, May 21-24, 1985, 149-171.

Willam, K.J. (1984), "Experimental and computational aspect of concrete fracture", *Int. Conf. Computer-Aided Analysis and Design of Concrete Structures*, Split, Yugoslavia, September 17-21, 1984, 33-69 (invited paper).

Willam, K., Bicanic, N., Pramono, E. and Sture, S. (1986), "Composite fracture model for strain-softening computation of concrete failure", *Proc. Int. Conf. Fracture Mechanics of Concrete-Fundamentals and Applications*, EPF Lausanne, Switzerland, October 1-3, 1985, Elsevier Publ., Amsterdam.

Willam, K.J. and Warnke, E.P. (1975), "Constitutive models for the triaxial behavior of concrete", Int. Assoc. Bridge Struct. Eng. Sem. Concr. Struct. Subjected Triaxial Stresses, Bergamo, Italy, 1974, *Int. Assoc. Bridge Struct. Eng. Proc.,* **19**, 1-30.

Willam, K.J., Pramono, E. and Sture, S. (1985), "Stability and uniqueness of strain-softening computations", Europe-US Symposium on Finite Element Methods for Nonlinear Problems, NIT Trondheim, August 12-16, 1985, *Structural Research Series 8503*, University of Colorado, Boulder, 1-24.

Wright, R.S. and Chiarito, V.P. (1984), *The Effect of Confining Pressures on the Behavior of Reinforced Concrete Circular Conduits*, Struct. Mech. Div., Structures Laboratory, USAE Waterways Experiment Station, Vicksburg, Mississippi.

Yoder, P.J. and Iwan, W.D. (1981), "On the formulation of strain-space plasticity with multiple loading surfaces", *ASME Trans., Journal of Applied Mechanics,* **48** (4), 773-778.