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The Poisson effect on the curved beam analysis

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Abstract. The bending stress formula that taking into account the transverse deformation is developed for plane-curved, untwisted isotropic beams subjected to loadings that result in deformations in the plane of curvature. In order to account the transverse Poisson contraction effect, a new constitutive relation between force resultants, moment resultants, mid-plane strains and deformed curvatures for a curved plate is derived in a 6×6 matrix form. This constitutive relation will provide the fundamental basis to the analyses of curved structures composing of isotropic or anisotropic materials. Then, the bending stress formula of a curved isotropic beam can be deduced from this newly developed curved plate theory. The stress predictions by the present analysis are compared to those by the analysis that neglected the Poisson contraction effect. The results show that the Poisson effect becomes more significant as the Poisson ratio and the curvature are getting larger.

Key words: curved beam; curved plate; bending stress; Poisson ratio.

1. Introduction

The technical theory of bending generally yields results of good accuracy for straight slender beams. However, there are many applications where the centerline of the beam is curved rather than straight. For example, the airplane fuselage frame, as indicated in Fig. 1, is an instance where stresses and displacements must be determined on the basis of a theory that accounts for the non-straight geometries of the structures.

For a straight beam, the axial strain is distributed in a linear fashion across the cross section under the Navier's hypothesis. Accordingly, the flexure formula of a straight beam is resolved to be $\sigma = -My/I$ by assuming the beam as a one-dimensional elastic isotropic structure. Due to the linear axial strain distribution, the same flexure formula can also be deduced from the two-dimensional plate theory by which the transverse deformation is considered. It means that Poisson's ratio will not affect the bending stress predictions. On the subject of curved beams, the axial strain is distributed in a nonlinear fashion across the cross section due to the geometric curvature and the corresponding bending stress distribution is generally shown in Fig. 2. The bending stress formula of a curved beam has been formulated by assuming the curved beam composing of a one-dimensional elastic isotropic solid behaving according to the stress-strain relation of $\sigma_x = E\varepsilon_x$ and $\tau_{xy} = G\gamma_{xy}$ (Bickford 1998). However, due to the nonlinear axial strain distribution of the curved beam, the different

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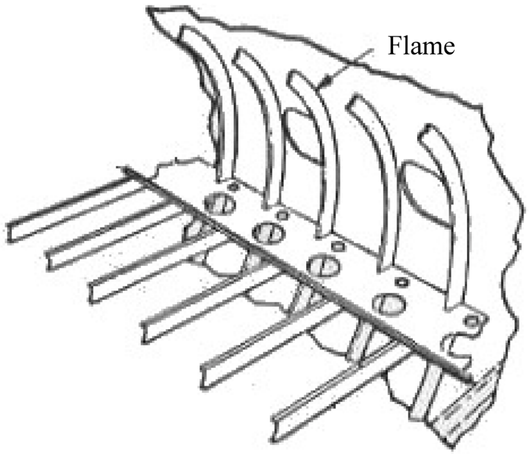


Fig. 1 Airplane fuselage frame

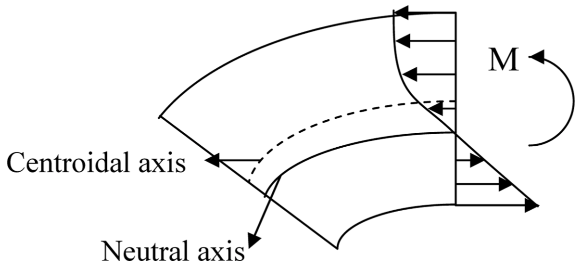


Fig. 2 Bending stress distribution in a curved beam

bending stress formula will be deduced from the two-dimensional curved plate theory by which the transverse Poisson contraction effect is taken into account. This paper will be devoted to develop a new curved plate theory from which the bending stress formula of the curved beam can be deduced to explore the Poisson contraction effect on the stress and strain predictions.

In order to account for the transverse deformation into the curved beam analysis, a constitutive relation for a curved plate needs to be established. Similar to the classic 6×6 ABD matrix constitutive relation of a flat laminated composite plate (Vinson and Chou 1975, Christensen 1979, Vinson and Sierakowski 1987, Herakovich 1998), a new 6×6 constitutive relation between force resultants, moment resultants, mid-plane strains and deformed curvatures is formulated for a curved plate. This new curved plate constitutive relation will provide the fundamental basis to the analyses of curved structures (e.g., curved beams and plates etc.) composing of isotropic or anisotropic materials. By applying this newly derived constitutive relation, the bending stress of a curved isotropic beam subjected to loadings that result in deformations in the plane of curvature can be formulated. Unlike the bending stress formula by one-dimensional approach (Bickford 1998) that is independent to the Poisson's ratio ν, the formula derived by considering the Poisson contraction effect composes of the term of Poisson's ratio ν. It implies that the curved beams composing of different Poisson's ratio will result in different bending stress predictions. The bending stress predictions are investigated as a function of Poisson's ratio ν and the geometric parameter of the thickness-radius (h/R) ratio. The results show that the higher Poisson ratio ν and the larger h/R ratio, the more Poisson contraction effect exhibits on the bending stress predictions.

2. Curved plate theory

Consider a curved plate of thickness h as depicted in Fig. 3. Here, the x-axis is passing everywhere through the centroid of the section and tangent to a circular arc of Radius R, that is, $ds = Rd\theta$, where θ is the angular variable associated with a change in location along the curved section. The z-axis lies along the local direction of the radius R with the y-axis such that a right-handed coordinate system is formed.

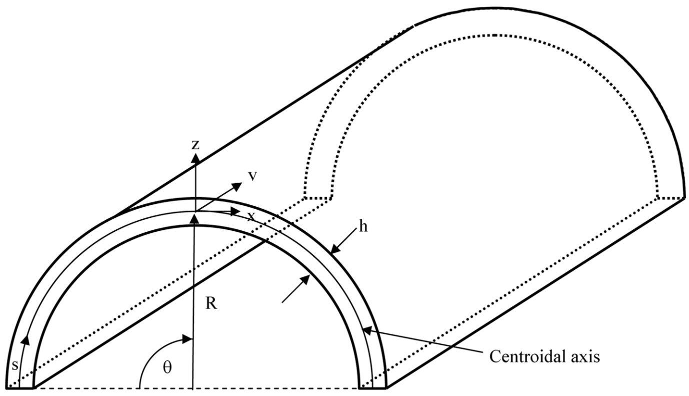


Fig. 3 Geometry for a curved plate

2.1 Strain-displacement relations

Under small deformation, a curved plate is assumed to deform by following the Kirchhoff-Love hypothesis for plates (Vinson and Chou 1975):

(1) A lineal element of the curved plate extending through the plate thickness is normal to the mid-plane (i.e., *xy* plane). Upon the application of loads, the lineal remain straight and normal to the deformed mid-plane.

(2) The lineal element does not change length.

Based upon the foregoing assumptions, the most general forms for the displacements in the x and y directions are

$$u(x, y, z) = u_0(x, y) + z\alpha(x, y) \tag{1}$$

$$v(x, y, z) = v_0(x, y) + z\beta(x, y) \tag{2}$$

where u_0 and v_0 denote the mid-plane displacements in the x and y directions and α and β are notations which will be further defined later. From the assumption (2), requires that $\varepsilon_z = 0$ and in turn means that the lateral deflection w can be expressed as

$$w(x, y, z) = w_0(x, y) = w \tag{3}$$

By specializing the strain-displacement relations in the cylindrical coordination into the present x-y-z coordination, the strain-displacements become

$$\varepsilon_x = \frac{1}{1+\kappa z}\left(\frac{\partial u}{\partial s} + \kappa w\right) = \frac{1}{1+\kappa z}\left(\frac{\partial u_0}{\partial s} + z\frac{\partial \alpha}{\partial s} + \kappa w\right) \tag{4}$$

$$\varepsilon_y = \frac{\partial v_0}{\partial y} + z\frac{\partial \beta}{\partial y} \tag{5}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0 \tag{6}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{1}{1+\kappa z}\frac{\partial v}{\partial s} = \frac{\partial u_0}{\partial y} + z\frac{\partial \alpha}{\partial y} + \frac{1}{1+\kappa z}\left(\frac{\partial v_0}{\partial s} + z\frac{\partial \beta}{\partial s}\right) \tag{7}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} - \frac{\kappa}{1+\kappa z}u + \frac{1}{1+\kappa z}\frac{\partial w}{\partial s} = \frac{\partial u_0}{\partial z} + \alpha - \frac{\kappa}{1+\kappa z}(u_0 + z\alpha) + \frac{1}{1+\kappa z}\frac{\partial w}{\partial s} \tag{8}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{\partial v_0}{\partial z} + \beta + \frac{\partial w}{\partial y} \tag{9}$$

where $\kappa = 1/R$ is the curvature of the curved plate. The assumption (1) requires that the shear strains of $\gamma_{xz} = 0$ and $\gamma_{yz} = 0$ are zero and leads to

$$\alpha = \kappa u_0 - \frac{\partial w}{\partial s} \tag{10}$$

$$\beta = -\frac{\partial w}{\partial y} \tag{11}$$

Substituting Eqs. (10-11) into Eqs. (4-5) and Eq. (7), the remainder planar strains of ε_x, ε_y and γ_{xy} can be expressed as

$$\varepsilon_x = \varepsilon_x^0 + \frac{z}{1+\kappa z}\kappa_x^1 \tag{12}$$

$$\varepsilon_y = \varepsilon_y^0 + z\kappa_y^0 \tag{13}$$

$$\gamma_{xy} = \gamma_{xy}^0 + z\kappa_{xy}^0 + \frac{z}{1+\kappa z}\kappa_{xy}^1 \tag{14}$$

where the mid-plane strains $\{\varepsilon^0\}$ and deformed curvatures $\{\kappa^0\}$ and $\{\kappa^1\}$ are defined as

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial s} + \kappa w \tag{15}$$

$$\kappa_x^1 = -\left(\frac{\partial^2 w}{\partial s^2} + \kappa^2 w\right) \tag{16}$$

$$\varepsilon_y^0 = \frac{\partial v_0}{\partial y} \tag{17}$$

$$\kappa_y^0 = -\frac{\partial^2 w}{\partial y^2} \tag{18}$$

$$\gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial s} \tag{19}$$

$$\kappa_{xy}^0 = -\left(\frac{\partial^2 w}{\partial s \partial y} - \kappa\frac{\partial u_0}{\partial y}\right) \tag{20}$$

$$\kappa_{xy}^{1} = -\left(\frac{\partial^2 w}{\partial s \partial y} + \kappa\frac{\partial v_0}{\partial s}\right) \tag{21}$$

Combining Eqs. (12-14) and Eqs. (15-21) in matrix form, we obtain

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} 0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix} + \frac{z}{1+\kappa z}\begin{Bmatrix} \kappa_x^1 \\ 0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{22}$$

or more simply

$$\{\varepsilon\} = \{\varepsilon^0\} + z\{\kappa^0\} + \frac{z}{1+\kappa z}\{\kappa^1\} \tag{23}$$

Eq. (23) indicates that the planar strains $\{\varepsilon\}$ at any z-location in the curved plate are in terms of the mid-plane strains $\{\varepsilon^0\}$ and the deformed curvatures $\{\kappa^0\}$ and $\{\kappa^1\}$; it is a fundamental equation of curved plate theory.

2.2 Stresses

The stresses at any z-location can then be determined by substituting strain equation of (22) into the plane stress constitutive equation, and leads to

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [Q]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{24}$$

where $[Q]$ is the stress-strain stiffness matrix of the material which can be either isotropic or anisotropic. Here, we only consider the curved isotropic plate and the stiffness matrix $[Q]$ is, then, given by

$$[Q] = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix} \tag{25}$$

where E, G and ν denote the Young's modulus, the shear modulus and the Poisson's ratio of the material, respectively. Combining Eqs. (22) and (24) gives a general expression for stresses at z-location in the plate:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [Q]\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + [Q]z\begin{Bmatrix} 0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix} + [Q]\frac{z}{1+\kappa z}\begin{Bmatrix} \kappa_x^1 \\ 0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{26}$$

The first term in Eq. (26) corresponds to the stresses associated with the mid-plane strains, and the second and third terms correspond to the stresses associated with bending strains. It is noted that $\{\varepsilon^0\}$, $\{\kappa^0\}$ and $\{\kappa^1\}$, which are only associated with the mid-plane displacements and geometric curvature κ, are independent of z location.

2.3 Force resultants and moment resultants

The force resultants $\{N\}$ refer to the stresses integrated over the thickness of the plate. A similar interpretation can be given to the moments resultants $\{M\}$. Thus, $\{N\}$ and $\{M\}$ in compact forms are, respectively, expressed as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz \tag{27}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z dz \tag{28}$$

Substituting Eq. (26) into Eqs. (27) and (28) gives

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} [Q] \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} dz + \int_{-h/2}^{h/2} [Q] \begin{Bmatrix} 0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix} z dz + \int_{-h/2}^{h/2} [Q] \begin{Bmatrix} \kappa_x^1 \\ 0 \\ \kappa_{xy}^1 \end{Bmatrix} \frac{z}{1+\kappa z} dz \tag{29}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} [Q] \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} z dz + \int_{-h/2}^{h/2} [Q] \begin{Bmatrix} 0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix} z^2 dz + \int_{-h/2}^{h/2} [Q] \begin{Bmatrix} \kappa_x^1 \\ 0 \\ \kappa_{xy}^1 \end{Bmatrix} \frac{z^2}{1+\kappa z} dz \tag{30}$$

Remembering that $\{\varepsilon^0\}$, $\{\kappa^0\}$ and $\{\kappa^1\}$ are independent of z location, the integrals in Eqs. (29-30) are easily carried out. For example,

$$\int_{-h/2}^{h/2} \frac{z}{1+\kappa z} dz = \frac{1}{\kappa}\left(h - \frac{1}{\kappa}\ln\frac{1+\kappa h/2}{1-\kappa h/2}\right) = I_1 \tag{31}$$

$$\int_{-h/2}^{h/2} \frac{z^2}{1+\kappa z} dz = -\frac{1}{\kappa^2}\left(h - \frac{1}{\kappa}\ln\frac{1+\kappa h/2}{1-\kappa h/2}\right) = I_2 = -\frac{I_1}{\kappa} \tag{32}$$

where ln represents natural logarithms.

2.4 Curved plate constitutive relations

By carrying out the integrations in Eqs. (29) and (30), the fundamental equation of the curved plate theory can be written in the following form:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & 0 & C \\ 0 & D & -C/\kappa \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa^0 \\ \kappa^1 \end{Bmatrix} \tag{33}$$

where $[A]$, $[D]$ and $[C]$ matrices are defined as

$$[A] = h[Q] \tag{34}$$

$$[D] = \frac{h^3}{12}[Q] \tag{35}$$

$$[C] = I_1[Q] \tag{36}$$

Eq. (33) can be written in expanded form as

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & 0 & C_{11} & C_{12} & C_{13} \\ A_{21} & A_{22} & A_{23} & 0 & 0 & 0 & C_{21} & C_{22} & C_{23} \\ A_{31} & A_{32} & A_{33} & 0 & 0 & 0 & C_{31} & C_{32} & C_{33} \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{13} & -C_{11}/\kappa & -C_{12}/\kappa & -C_{13}/\kappa \\ 0 & 0 & 0 & D_{21} & D_{22} & D_{23} & -C_{21}/\kappa & -C_{22}/\kappa & -C_{23}/\kappa \\ 0 & 0 & 0 & D_{31} & D_{32} & D_{33} & -C_{31}/\kappa & -C_{32}/\kappa & -C_{33}/\kappa \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ 0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \\ \kappa_x^1 \\ 0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{37}$$

The form of Eq. (37) is not suitable for matrix operations. Recalling Eqs. (19-21), the deformed curvature κ_{xy}^0 can relate to the mid-plane strain γ_{xy}^0 and the deformed curvature κ_{xy}^1 by the following form

$$\kappa_{xy}^0 = \kappa\gamma_{xy}^0 + \kappa_{xy}^1 \tag{38}$$

Substituting Eq. (38) into Eq. (37), the constitutive relation of Eq. (37) can be expressed as the following 6 × 6 matrix form

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & C_{11} & 0 & C_{13} \\ A_{13} & A_{22} & A_{23} & C_{12} & 0 & C_{23} \\ A_{13} & A_{23} & A_{33} & C_{13} & 0 & C_{33} \\ 0 & 0 & \kappa D_{13} & -C_{11}/\kappa & D_{12} & (-C_{13}/\kappa + D_{13}) \\ 0 & 0 & \kappa D_{23} & -C_{12}/\kappa & D_{22} & (-C_{23}/\kappa + D_{23}) \\ 0 & 0 & \kappa D_{33} & -C_{13}/\kappa & D_{23} & (-C_{33}/\kappa + D_{33}) \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x^1 \\ \kappa_y^0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{39}$$

As can be seen in Eq. (39), the coupling between the stretching and bending responses is existed for a curved plate. This coupling effect is caused by the non-flat geometry of the structure. This stretching-bending coupling effect for a curved plate can not be shown by the flat plate. This new curved plate constitutive relation will provide the fundamental basis to the analyses of curved structures (e.g., curved beams and plates etc.) of isotropic or anisotropic materials. For isotropic materials, Eq. (39) turns into the form of

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{hE}{1-\nu^2} & \frac{h\nu E}{1-\nu^2} & 0 & \frac{I_1 E}{1-\nu^2} & 0 & 0 \\ \frac{h\nu E}{1-\nu^2} & \frac{hE}{1-\nu^2} & 0 & \frac{I_1 \nu E}{1-\nu^2} & 0 & 0 \\ 0 & 0 & hG & 0 & 0 & I_1 G \\ 0 & 0 & 0 & \frac{I_2 E}{1-\nu^2} & \frac{h^3 \nu E}{12(1-\nu^2)} & 0 \\ 0 & 0 & 0 & \frac{I_2 \nu E}{1-\nu^2} & \frac{h^3 E}{12(1-\nu^2)} & 0 \\ 0 & 0 & \frac{\kappa h^3 G}{12} & 0 & 0 & \left(I_2 + \frac{h^3}{12}\right)G \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x^1 \\ \kappa_y^0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{40}$$

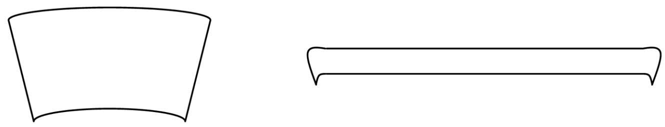
It is found in Eq. (40) that the normal responses are unrelated to the shear responses. Thus, Eq. (40) can be separated into two matrix equations as

$$\begin{Bmatrix} N_x \\ N_y \\ M_x \\ M_y \end{Bmatrix} = \begin{bmatrix} \frac{hE}{1-\nu^2} & \frac{h\nu E}{1-\nu^2} & \frac{I_1 E}{1-\nu^2} & 0 \\ \frac{h\nu E}{1-\nu^2} & \frac{hE}{1-\nu^2} & \frac{\nu I_1 E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{I_2 E}{1-\nu^2} & \frac{h^3 \nu E}{12(1-\nu^2)} \\ 0 & 0 & \frac{I_2 \nu E}{1-\nu^2} & \frac{h^3 E}{12(1-\nu^2)} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \kappa_x^1 \\ \kappa_y^0 \end{Bmatrix} \tag{41}$$

and

$$\begin{Bmatrix} N_{xy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} hG & I_1 G \\ \frac{\kappa h^3 G}{12} & \left(I_2 + \frac{h^3}{12}\right)G \end{bmatrix} \begin{Bmatrix} \gamma_{xy}^0 \\ \kappa_{xy}^1 \end{Bmatrix} \tag{42}$$

It is noted that the curved plate constitutive equation of Eq. (39) for anisotropic materials exhibits the coupling between the normal and shear responses and cannot be separated to two matrices as the forms of Eqs. (41-42).



(a) Beam cross-section (b) Plate cross-section

Fig. 4 Illustration of anticlastic effect in curved beam and plate showing the distortion of the cross-section

3. Curved beam analysis

As the aspect ratio of the cross-section, that is, the ratio of the cross-section width to height of a curved plate is reduced to a certain value such that the axial strain gives rise to a free transverse distortion of the cross-section because of the Poisson contraction effect, as shown in Fig. 4(a). Then, the configurations of curved plates can be referred to as the curved beams. On the other hand, a curved plate does not show the "anticlastic" effect except the outer edges of the cross-section, as illustrated in Fig. 4(b).

3.1 Bending stress

Consider a curved isotropic beam with rectangular cross-section subjected to the loadings that result in deformations in the plane of curvature. The situation of a curved beam can be described mathematically by setting the transverse moment resultant $M_y = 0$ and the transverse force resultant $N_y = 0$. Then, the curved plate constitutive relation of Eq. (41) with the inverse becomes

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \kappa_x^1 \\ \kappa_y^0 \end{Bmatrix} = \begin{bmatrix} \dfrac{hE}{1-\nu^2} & \dfrac{h\nu E}{1-\nu^2} & \dfrac{I_1 E}{1-\nu^2} & 0 \\ \dfrac{h\nu E}{1-\nu^2} & \dfrac{hE}{1-\nu^2} & \dfrac{\nu I_1 E}{1-\nu^2} & 0 \\ 0 & 0 & \dfrac{I_2 E}{1-\nu^2} & \dfrac{h^3 \nu E}{12(1-\nu^2)} \\ 0 & 0 & \dfrac{I_2 \nu E}{1-\nu^2} & \dfrac{h^3 E}{12(1-\nu^2)} \end{bmatrix} \begin{Bmatrix} N_x \\ 0 \\ M_x \\ 0 \end{Bmatrix} \tag{43}$$

It should be noted that the stress resultant N_x and moment resultant M_x are on a unit width basis, and must be multiplied by the width of the beam to get the axial force and moment used in the beam theory. For the axial force denoted as N (=bN_x) and the beam moment denoted as M (=bM_x), the axial and transverse strains can be obtained from Eq. (43):

$$\varepsilon_x = \varepsilon_x^0 + \frac{z}{1+\kappa z}\kappa_x^1 = \frac{N+\kappa M}{AE} + \frac{z}{1+\kappa z}\frac{M}{bI_2 E} \tag{44}$$

$$\varepsilon_y = \varepsilon_y^0 + z\kappa_y^0 = -\frac{\nu N}{AE} - z\frac{\nu M}{IE} \tag{45}$$

where A (=bh) and I (=$bh^3/12$) are, respectively, the area and the moment of the inertial of the cross section. It is clearly shown in Eq. (44) that the axial strain, ε_x, is distributed in a nonlinear fashion across the cross section. The bending and transverse stresses are obtained from the stress-strain relation given by Eq. (25):

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) = \frac{N}{A} + \frac{M}{1-\nu^2}\left(\frac{\kappa}{A} - z\frac{\nu^2}{I} + \frac{z}{1+\kappa z}\frac{1}{bI_2}\right) \tag{46}$$

$$\sigma_y = \frac{E}{1-\nu^2}(\nu\varepsilon_x + \varepsilon_y) = \frac{\nu M}{1-\nu^2}\left(\frac{\kappa}{A} - z\frac{1}{I} + \frac{z}{1+\kappa z}\frac{1}{bI_2}\right) \tag{47}$$

The curved beam analysis developed by Bickford (1998), in which the beam was assumed composing of a one-dimensional elastic isotropic solid behaving according to $\sigma_x = E\varepsilon_x$ and $\tau_{xy} = G\gamma_{xy}$, gave the following bending stress formula:

$$\sigma_x = \frac{N}{A} + M\left(\frac{\kappa}{A} + \frac{z}{1+\kappa z}\frac{1}{bI_2}\right) \tag{48}$$

Unlike the bending stress formula of Eq. (46), which is the function of the Poisson's ratio ν, Eq. (48) is independent of the Poisson's ratio ν.

4. Results and discussions

The influences of the Poisson effect on the stress predictions of a curved beam will be illuminated in the following example. Consider a curved beam of thickness h and width b with a radius R subjected to the loading P, as shown in Fig. 5. This is a statically determined problem in which the axial force $N = P\sin\theta = bN_x$ and the bending moment $M = PR\sin\theta = bM_x$ can be resolved from force equilibrium relations.

For specific examples of $h = R$ and $h = 1.5R$, the through-thickness bending stress distributions σ_x at the location of $\theta = \pi/2$ are, respectively, plotted in Figs. 6 and 7 for different Poisson's ratio values of $\nu = 0$, 0.25 and 0.5. For the case of Poisson's ratio $\nu = 0$, the bending stress equation of Eq. (46) reduces to Eq. (48) of Bickford (1998). As illustrated in Figs. 6 and 7, the maximum

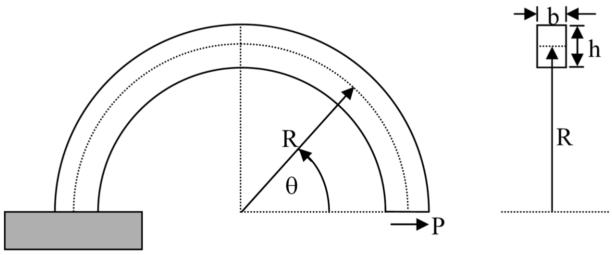


Fig. 5 Bending of a curved beam with rectangular cross section

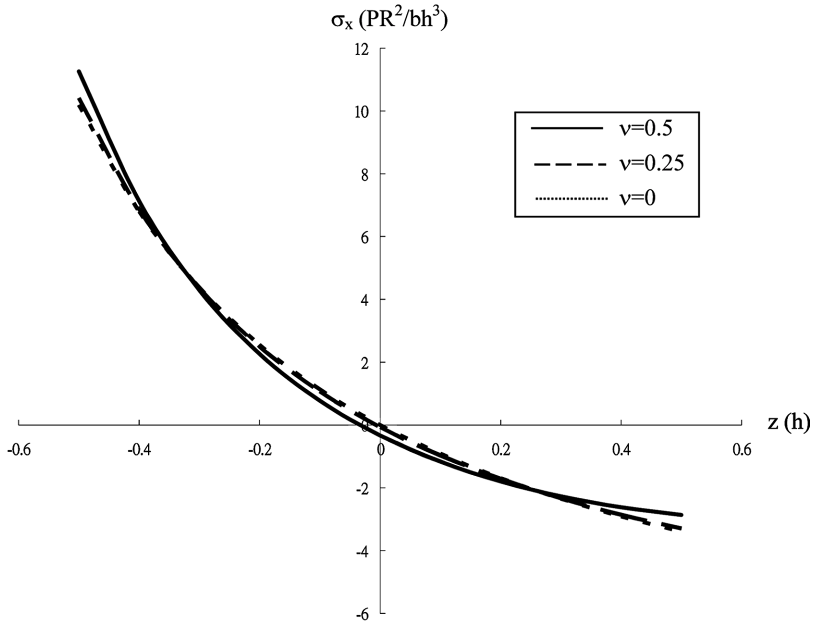


Fig. 6 Bending stress distributions σ_x for various value of Poisson's ratio ν at the case of $h = R$

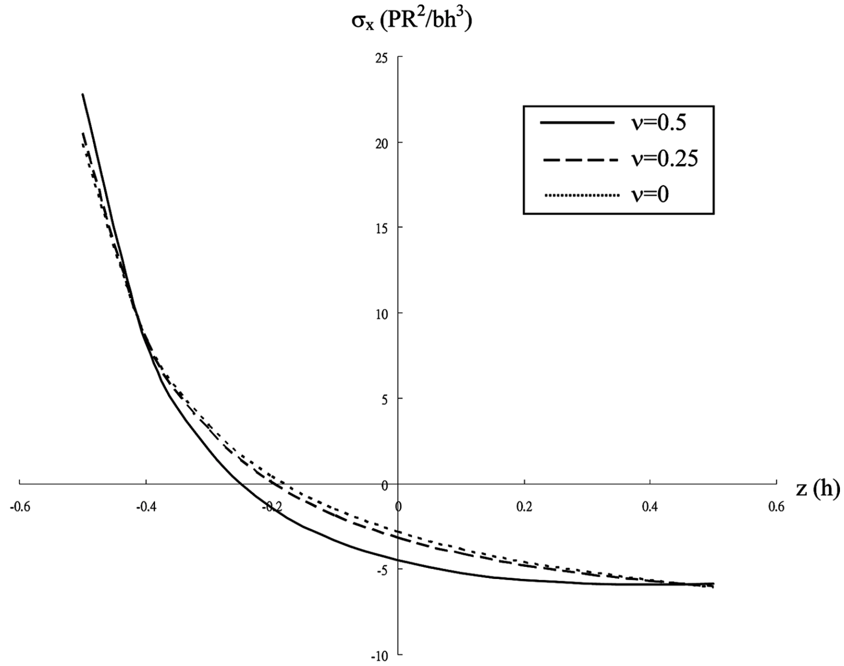


Fig. 7 Bending stress distributions σ_x for various value of Poisson's ratio ν at the case of $h = 1.5R$

bending stresses occur at the inner surface. At the presence of non-zero Poisson ratio, the bending stresses will shift to higher tensile stress at the inner radius and shift to smaller compressive stress at

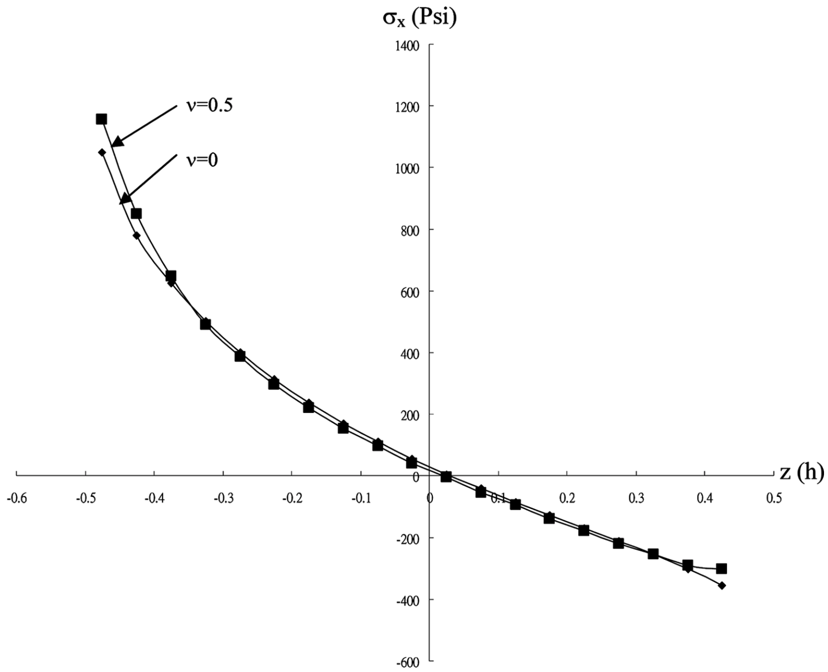


Fig. 8 Bending stress distributions σ_x for $\nu = 0$ and 0.5 at $h = R$ by finite element analysis

the outer radius, as shown in Figs. 6 and 7. The amount of shifting will be more significant as the Poisson ratio ν is getting larger. The influence of the h/R ratio on the bending stress predictions can be illustrated by the comparison between Figs. 6 and 7. It is indicated that the Poisson effect becomes more noticeable as the h/R ratio is getting larger.

The Poisson effect on the bending stress predictions is investigated by the finite element analysis (FEA). For the curved AISI 4340 steel beam of $h = R = 20$ in and $b = 10$ in with the loading $P = 23100$ lb, the through-thickness bending stress distribution σ_x at the location of $\theta = \pi/2$ for different Poisson's ratio values of $\nu = 0$ and 0.5 is plotted in Fig. 8, where the maximum tensile and compressive stresses occur at the inner and outer surfaces, respectively. As analogous to the theoretical predictions as shown in Figs. 6 and 7, the FEA results also indicates that the bending stress of $\nu = 0.5$ shifts to higher tensile stress at the inner radius and shifts to smaller compressive stress at the outer radius.

The Poisson effect on the stress predictions can be exhibited more profound in the through-thickness transverse stress distributions σ_y, which are plotted in Fig. 9 for the case of $h = 1.5R$ at the location of $\theta = \pi/2$ for different Poisson's ratio values of $\nu = 0$, 0.25 and 0.5. In the case of Poisson ratio $\nu = 0$, the transverse stress equals to zero through the thickness. On the other hand, at the presence of non-zero Poisson ratio ν, the transverse stress is distributed in a nonlinear fashion which exhibits tensile stress at both end and compressive stress in between. The amount of deviation from the case of $\nu = 0$ will be more significant as the Poisson ratio ν is getting larger.

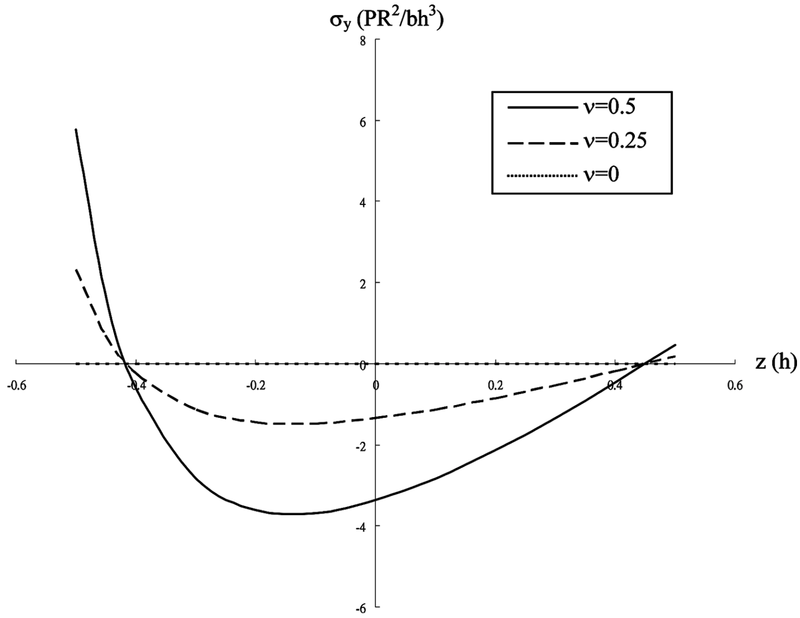


Fig. 9 Transverse stress distributions σ_y for various value of Poisson's ratio ν at the case of h = 1.5R

5. Conclusions

1. The equations that describe the linear elastic response of a curved plate subjected to in-plane loads and bending moments has been developed in this paper. Similar to the classic 6 × 6 ABD matrix constitutive relation of a laminated composite plate, a new 6 × 6 constitutive relation between force resultants, moment resultants, mid-plane strains and deformed curvatures for a curved plate has been formulated. This new curved plate constitutive relation will provide the fundamental basis to the analyses of curved structures composing of isotropic or anisotropic materials.
2. The bending stress formula of Eq. (46) for a curved beam has been deduced from the newly derived constitutive relation of a curved plate. The bending stress predictions by Eq. (46) are compared to those by Eq. (48), which was derived by one-dimension approach and, therefore, neglected the Poisson effect. The results show that the Poisson effect becomes more significant as the Poisson ratio ν and the *h/R* ratio are getting larger.
3. The Poisson effect on the stress predictions can be exhibited more profound in the through-thickness transverse stress distributions σ_y, which are distributed in a nonlinear fashion and exhibit tensile stress at both end and compressive stress in between at the presence of non-zero Poisson ratio. The transverse stress σ_y is caused by the nonlinear distribution of the bending stress through the thickness.

References

Bickford, W.B. (1998), *Advance Mechanics of Materials*, 1st ed. Addison Wesley Longman, Inc., Menlo Park.

Christensen, R.M. (1979), *Mechanics of Composite Materials*, New York: John Wiley & Sons, Inc.
Herakovich, C.T. (1998), *Mechanics of Fibrous Composites*, New York: John Wiley & Sons, Inc.
Vinson, J.R. and Sierakowski, R.L. (1987), *The Behavior of Structures Composed of Composite Materials*, Kluwer Adademic Publishers.
Vinson, J.R. and Chou, T.W. (1975), *Composite Materials and Their Use in Structure*, Applied Science Publishers, London.