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# Storey-based stability analysis of multi-storey unbraced frames

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**Abstract.** This paper presents a practical method to evaluate the effective length factors for columns in multi-storey unbraced frames based on the concept of storey-based elastic buckling by means of decomposing a multi-storey frame into a series of single-storey partially-restrained (PR) frames. The lateral stiffness of the multi-storey unbraced frame is derived and expressed as the product of the lateral stiffness of each storey. Thus, the stability analysis for the multi-storey frame is conducted by investigating the lateral stability of each individual storey, which is facilitated through decomposing the multi-storey frame into a series of single-storey PR frames and applying the storey-based stability analysis proposed by the authors (Xu and Liu 2002) for each single-storey PR frame. Prior to introducing decomposition approaches, the end rotational stiffness of an axially load column is derived and rotational stiffness interaction between the upper and lower columns is investigated. Three decomposition approaches, characterized by means of distributing beam-to-column rotational-restraining stiffness between the upper and lower columns, are proposed. The procedure of calculating storey-based column effective length factors is presented. Numerical examples are then given to illustrate the effectiveness of the proposed procedure.

**Key words:** column effective length; lateral stability; multi-storey unbraced frame; storey-based buckling.

## 1. Introduction

Column and frame stability is of primary importance to the structural design of unbraced multi-storey frames. Although theoretical methods or so-called system buckling methods on elastic buckling of such frames under proportional loading were well established (Majid 1972, Livesley 1975, Chen and Lui 1987), the methods were generally considered impractical because the methods involved solving the minimum positive eigenvalue from either a highly nonlinear or a transcendental equation. In design practice, the effective-length based methods are still the general methods of evaluating the column compressive strength, and the concept of effective-length is considered as an essential part of many analysis procedures and has been recommended in almost all of the current design specifications (AISC 1989, 2001).

Among the various effective-length based methods, the most widely adopted method for designing a frame is the alignment chart method that was originally proposed by Julian and Lawrence (1959).

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However, since this method adopted several simplifications and assumptions that may be not realistic, inaccurate column strengths may result from when these assumptions are not satisfied. Various studies had been carried out, aimed to improve the effectiveness of the alignment chart methods (Bridge and Fraster 1987, Chen and Lui 1985, Duan and Chen 1989, Chen *et al*. 1997).

In evaluating column buckling strength in unbraced frames, the alignment chart method takes into account the rotational restraints by upper and lower assemblages but neglects the interaction of lateral stiffnesses among the columns in the same storey in resisting lateral sway buckling of unbraced frames (possible lateral support provided by other columns in the same storey). Unlike the alignment chart method that considers column buckling as a single subassemblage buckling, the concept of storey-based buckling introduced by Yura (1971) takes into consideration the fact that lateral sway instability of an unbraced frame is a storey phenomenon involving the interaction of lateral sway resistance of each column in the storey and total gravity load in columns in that storey. Based on this concept, different methods were proposed to assess the column and frame buckling loads for unbraced frames (LeMessurier 1977, Lui 1992, ASCE 1997, Aristizabal-Ochoa 1997). The conclusion obtained from different comparative studies (Shanmugam and Chen 1995, Roddis *et al.* 1998) on the alignment chart method, system buckling method, and storey-based buckling method recommended the storey-based buckling method for general use in design practice. The LRFD specification (AISC 2001) addressed the concept of storey-based buckling because the alignment chart method does not consider destabilizing effects due to lean-on columns in a frame. Two methods of determining the storey-based effective length factor, namely, the storey stiffness method (LeMessurier 1977) and storey buckling method (Yura 1971), were presented in the Commentary of the LRFD Specification (AISC 2001).

Considering that the foregoing storey-based buckling methods often require either the first order elastic analysis or the alignment chart while evaluating the storey-based effective length factor, an efficient storey-based buckling method (Xu *et al*. 2001, Xu and Liu 2002) based on the single-storey partially-restrained (PR) frame model was proposed which did not require either conducting frame analysis or using the alignment chart. This paper extends the method into the multi-storey case by decomposing a multiple-storey unbraced frame into to a series of single-storey frames. Different approaches of decomposition which are primarily related to the distribution of end rotational-restraining stiffness of beams to the connected upper and lower columns are investigated. Following the proposed procedure of evaluation column effective length factor for multi-storey unbraced frames, examples were presented to illustrate its efficiency.

## 2. Storey-based stability equation

Considering the elastic buckling of an unbraced frame composed of prismatic members, the concept of storey-based buckling states that lateral sway instability of an unbraced frame is a storey phenomenon involving the interaction of lateral stiffness among columns in the storey. In other words, in resisting the lateral sway instability, stronger columns or columns with larger stiffnesses are able to provide lateral support for weaker columns in the same storey and the weaker columns rely upon such lateral support to maintain the lateral stability. Accordingly, the condition for multi-column storey-based buckling in a lateral sway mode is that the sum of the lateral stiffness of the storey vanishes. For a multi-storey unbraced frame with $m - 1$ bays and $n$ stories as shown in Fig. 1(a), the lateral stiffness of an axially loaded column in such frame can be expressed (Xu and Liu 2002) as

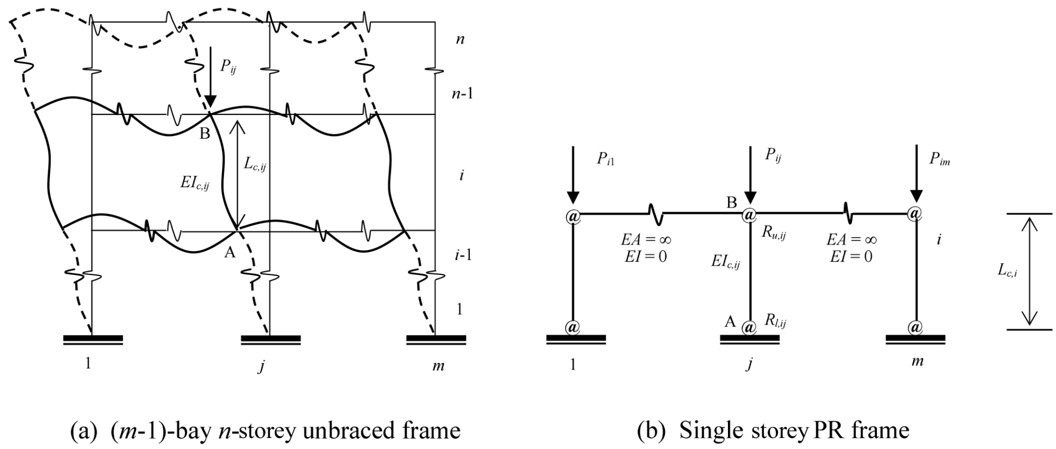


Fig. 1 Analytical model of multistory frames

$$S_{ij} = \beta_{ij}\frac{12EI_{c,ij}}{L_{c,i}^3} \tag{1}$$

where subscripts $i$ and $j$ are storey and column indices; $E$ is Young's modulus; and $I_{c,ij}$ and $L_{c,i}$ are the moment of inertia and length of the column, respectively. The column lateral stiffness modification factor $\beta_{ij}$ is

$$\beta_{ij} = \frac{\phi_{ij}^3}{12}\frac{a_{1,ij}\phi_{ij}\cos\phi_{ij} + a_{2,ij}\sin\phi_{ij}}{18r_{l,ij}r_{u,ij} + (a_{1,ij} - a_{2,ij})\phi_{ij}\sin\phi_{ij} - a_{3,ij}\cos\phi_{ij}} \tag{2}$$

in which

$$a_{1,ij} = 3(r_{l,ij} + r_{u,ij} - 2r_{u,ij}r_{l,ij}) \tag{3a}$$

$$a_{2,ij} = 9r_{l,ij}r_{u,ij} - (1 - r_{l,ij})(1 - r_{u,ij})\phi_{ij}^2 \tag{3b}$$

$$a_{3,ij} = 18r_{l,ij}r_{u,ij} + a_{1,ij}\phi_{ij}^2 \tag{3c}$$

where subscripts $l$ and $u$ denote the lower and upper ends of the column and the corresponding column end-fixity factors are respectively defined as

$$r_{u,ij} = \frac{1}{1 + 3EI_{c,ij}/(R_{u,ij}L_{c,i})} \tag{4a}$$

$$r_{l,ij} = \frac{1}{1 + 3EI_{c,ij}/(R_{l,ij}L_{c,i})} \tag{4b}$$

in which $R_{u,ij}$ and $R_{l,ij}$ are the rotational-restraining stiffnesses which are contributed by the other members connected to the upper and lower ends of column $ij$, respectively.

The parameter $\phi_{ij}$ in Eq. (2) is defined as

$$\phi_{ij} = \sqrt{\frac{P_{ij}L_{c,ij}^2}{EI_{c,ij}}} = \pi\sqrt{\frac{P_{ij}}{P_{e,ij}}} \tag{5}$$

where $P_{e,ij} = \pi^2 EI_{c,ij}/L_{c,i}^2$ is the Euler buckling load of the column; $P_{ij} = \lambda P_{u,ij}$ is the column axial force, in which $P_{u,ij}$ is the applied axial load, and $\lambda$ is a load proportional multiplier. Having the lateral stiffness of the axially loaded column expressed in Eq. (1), the stability equation for single-storey PR frame buckling in a lateral sway mode can be expressed as Xu and Liu (2002)

$$S_i = \sum_{j=1}^{m} S_{ij} = \sum_{j=1}^{m} \beta_{ij}\frac{12EI_{c,ij}}{L_{c,i}^3} = 0 \tag{6}$$

For a multi-storey frame as shown in Fig. 1(a), the relationship between lateral stiffness of the frame, $S$, and that of the storey ($S_i$, $i = 1, 2, 3\ldots n$) is

$$S = \frac{1}{\sum_{i=1}^{n}\frac{1}{S_i}} \qquad (i = 1, 2, 3, \ldots, n) \tag{7}$$

Lateral instability occurs when the lateral stiffness of the frame vanishes, which can be concluded from Eq. (7) to have at least in one storey, say storey $k$ of the frame, such that $S_k = 0$. Pursuant to this condition, the lateral stability equation for unbraced multi-storey frames can be conveniently expressed as

$$\prod_{i=1}^{n} S_i = \prod_{i=1}^{n}\left(\sum_{j=1}^{m} \beta_{ij}\frac{12EI_{c,ij}}{L_{c,i}^3}\right) = 0 \tag{8}$$

Eq. (8) defines that the storey-based buckling of an unbraced multi-storey frame occurs as any one of the stories fails to maintain its lateral stability. Practically, it is desirable to convert the stability analysis of a multi-storey frame into analysis of single-storey frames for the reason of simplicity and this can be achieved through decomposing a multi-storey frame into a series of single-storey frames as shown in Fig. 1(b). In other words, Eq. (8) can be solved by setting the lateral stiffness of each storey $S_i$ to be zero and solving for the corresponding critical load multiplier. The storey that yields the minimum critical load multiplier in lateral sway instability would be the critical storey of the frame, and the critical load or corresponding effective length factor for each individual column in the frame can then be determined by that load multiplier.

However, it should be pointed out that the rotational-restraining stiffnesses $R_{l,ij}$ and $R_{u,ij}$ shown in Fig. 1(b) and employed in Eqs. (4) shall be evaluated differently for the single- and multi-storey cases. In the single-storey frame case, beams that are connected to the upper end of a column are the only members that provide the rotational restraint to the column against column instability. For a column in a multi-storey frame, in addition to the rotational restraints provided by connected beams, the columns located on the upper or lower level may also contribute rotational restraints to the column under consideration. To account for the stiffness interaction between the upper and lower columns that are connected in stability analysis of a multi-storey frame, the end rotational stiffness of an axially load column has to be determined and is discussed in the following section.

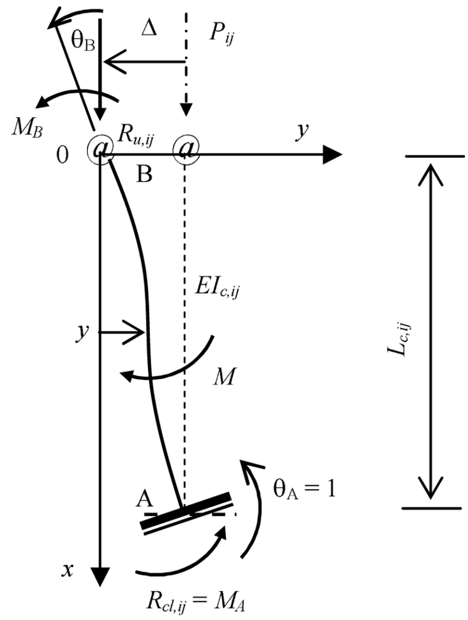


Fig. 2 Axially loaded PR column

## 3. End rotational stiffness of axially loaded column

For an axially loaded and laterally unrestrained column *ij* as shown in Fig. 2, let $R_{u,ij}$ be the external rotational-restraining stiffness induced by the other members connected at the upper end of the column. Noted that $\theta_B$ and $M_B$ are the end rotation and bending moment at the upper end of the column, respectively. The moment at a location $x$ along the column is given by

$$M = M_B + P_{ij}y \tag{9}$$

where, $P_{ij}$ is the column axial force. The equilibrium condition of the column yields

$$EI_{c,ij}\frac{d^2y}{dx^2} + P_{ij}y = -M_B \tag{10}$$

By substituting the following boundary conditions into Eq. (10),

$$y\big|_{x=0} = 0 \tag{11a}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = \theta_B \tag{11b}$$

$$\left.\frac{dy}{dx}\right|_{x=L_{c,ij}} = \theta_A = 1 \tag{11c}$$

$$M_B = R_{u,ij}\theta_B \tag{11d}$$

the rotational stiffness at the lower end of the column, $R_{cl,ij}$, can be solved as

$$R_{cl,ij} = \frac{EI_{c,ij}}{L_{c,ij}} \frac{R_{u,ij}L_{c,ij}/EI_{c,ij} - \phi_{ij}\tan\phi_{ij}}{1 + \dfrac{R_{u,ij}L_{c,ij}}{EI_{c,ij}}\dfrac{\tan\phi_{ij}}{\phi_{ij}}} = \frac{EI_{c,ij}}{L_{c,ij}} \frac{3r_{u,ij} - (1 - r_{u,ij})\phi_{ij}\tan\phi_{ij}}{1 - r_{u,ij} + 3r_{u,ij}\dfrac{\tan\phi_{ij}}{\phi_{ij}}} \tag{12a}$$

where $r_{u,ij}$ is the column end-fixity factor associated with the rotational-restraining stiffness $R_{u,ij}$ and is defined in Eq. (4a), and parameter $\phi_{ij}$ is defined in Eq. (5).

Similarly, the rotational stiffness associated with the upper end of the column, $R_{cu,ij}$, can be expressed as

$$R_{cu,ij} = \frac{EI_{c,ij}}{L_{c,ij}} \frac{R_{l,ij}L_{c,ij}/EI_{c,ij} - \phi_{ij}\tan\phi_{ij}}{1 + \dfrac{R_{l,ij}L_{c,ij}}{EI_{c,ij}}\dfrac{\tan\phi_{ij}}{\phi_{ij}}} = \frac{EI_{c,ij}}{L_{c,ij}} \frac{3r_{l,ij} - (1 - r_{l,ij})\phi_{ij}\tan\phi_{ij}}{1 - r_{l,ij} + 3r_{l,ij}\dfrac{\tan\phi_{ij}}{\phi_{ij}}} \tag{12b}$$

where $r_{l,ij}$ is the end-fixity factor of the column at the lower end and is defined in Eq. (4b).

Fig. 3 illustrates the relationship among the column end rotational stiffness $R_{cl,ij}$, the far end end-fixity factor $r_{u,ij}$ and applied load $P_{ij}$. It is clear that the presence of the axial load leads to the decrease of the column end rotational stiffness. In the case that the column end rotational stiffness $R_{cl,ij}$ turns into zero or a negative value as the result of increasing applied load, the column shown in Fig. 2 becomes laterally unstable. In the context of unbraced frames, if both end rotational stiffnesses of a column become non-positive as the result of increasing applied load, the column turns into a lean-on column which relies upon the external restraints provided by other members of the frame to maintain its stability and sustain the applied load.

In the case that the column is not axially loaded or the effect of the axial load on column end

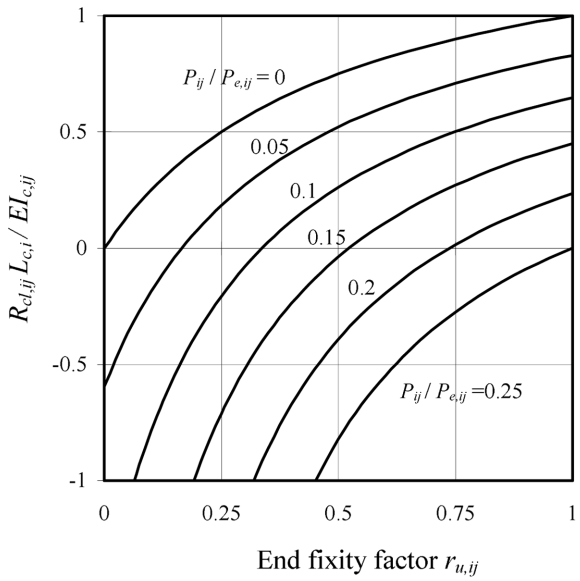


Fig. 3 End rotational stiffness of axially loaded column

rotational stiffness can be neglected, Eqs. (12) are simplified as

$$R_{cl,\,ij} = \frac{EI_{c,\,ij}/L_{c,\,ij}}{1 + EI_{c,\,ij}/R_{u,\,ij}L_{c,\,ij}} = \frac{EI_{c,\,ij}}{L_{c,\,ij}}\frac{3r_{u,\,ij}}{1+2r_{u,\,ij}} \tag{13a}$$

$$R_{cu,\,ij} = \frac{EI_{c,\,ij}/L_{c,\,ij}}{1 + EI_{c,\,ij}/R_{l,\,ij}L_{c,\,ij}} = \frac{EI_{c,\,ij}}{L_{c,\,ij}}\frac{3r_{l,\,ij}}{1+2r_{l,\,ij}} \tag{13b}$$

If the column ends are rigidly connected, Eqs. (13) can be further simplified as

$$R_{cl,\,ij} = R_{cu,\,ij} = \frac{EI_{c,\,ij}}{L_{c,\,ij}} \tag{14}$$

It is noted from Eqs. (12) and (13) that the column end rotational stiffness is a function of the end-fixity factor at the far end of the column. For instance, the rotational stiffness at the lower end of the column, $R_{cl,\,ij}$, is a function of $r_{u,\,ij}$ as that shown in Eq. (12a). As defined in Eq. (4a), $r_{u,\,ij}$ is related to $R_{u,\,ij}$, the rotational-restraining stiffness at the far end, which is contributed by the other members connected to the upper end of column $ij$ including column $(i+1)\,j$. Since the end rotational stiffness of column $(i+1)\,j$ is further involved with that of column $(i+2)\,j$ and so on, therefore, the end rotational stiffness of a column is interrelated with columns at different stories in the same column line. Apparently, considering such stiffness interaction would make the evaluation of column end rotational stiffness, and eventually the frame stability analysis, much more complicated and is not suitable for engineering practice. However, prior to make any simplification on this issue, the following parametric study was carried out to investigate the stiffness interaction.

Shown in Fig. 4 are simplified models for a series of symmetric one-bay unbraced frames with different number of stories, in which all columns and beams have identical flexural stiffnesses $EI_c/L_c$

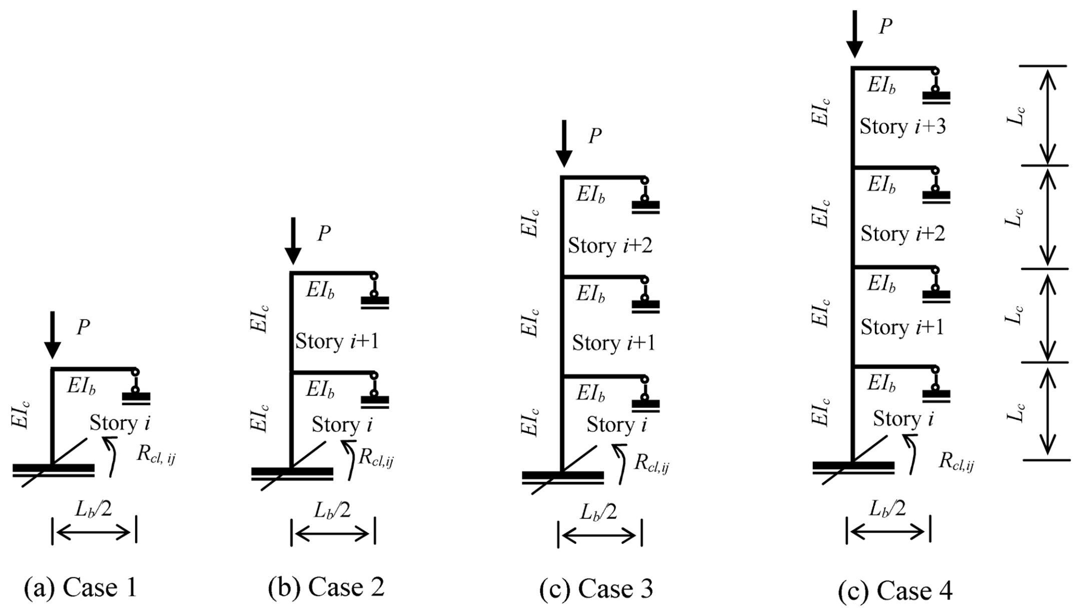


Fig. 4 Simplified models of one-bay multistory frames

Table 1 Comparison of column end rotational stiffnesses

| $P/P_e$ | Case 1 | Case 2 | | Case 3 | | Case 4 | |
|---|---|---|---|---|---|---|---|
| | $R_{cl}L_c/E_{Ic}$ | $R_{cl}L_c/E_{Ic}$ | Error (%) | $R_{cl}L_c/E_{Ic}$ | Error (%) | $R_{cl}L_c/E_{Ic}$ | Error (%) |
| 0.00 | 0.8571 | 0.8727 | –1.82 | 0.8730 | –1.86 | 0.8730 | –1.86 |
| 0.05 | 0.6567 | 0.6719 | –2.31 | 0.6723 | –2.38 | 0.6723 | –2.38 |
| 0.10 | 0.4356 | 0.4486 | –2.98 | 0.4490 | –3.08 | 0.4490 | –3.08 |
| 0.15 | 0.1889 | 0.1963 | –3.92 | 0.1966 | –4.07 | 0.1966 | –4.07 |
| 0.20 | –0.0903 | –0.0951 | –5.23 | –0.0953 | –5.54 | –0.0954 | –5.65 |
| 0.25 | –0.4112 | –0.4415 | –7.37 | –0.4439 | –7.95 | –0.4441 | –8.00 |

and $2EI_b/L_b$, respectively. The lower end rotational stiffness of column $ij$ is denoted as $R_{cl,\,ij}$. Table 1 illustrates the ratios of $R_{cl}L_c/EI_c$ computed according to Eq. (12a) with consideration for the foregoing described stiffness interaction among columns in different stories with respect to the variation of the applied load for the four cases shown in Fig. 4. In Case 1, the column stiffness interaction is not considered, while in Cases 2, 3, and 4, the interaction is considered for the effects of one column above, two columns above, and three columns above, respectively. The differences of $R_{cl,\,ij}$ of Case 1 to that of Cases 2 to 4 are shown as errors in Table 1. The negative values of the errors denote that $R_{cl,\,ij}$ associated with Case 1 is less than that of the other cases, which indicates that columns located above and beyond the adjacent stories contribute positively to the column end rotational stiffness, $R_{cl,\,ij}$. Comparing the errors associated with Cases 2 to 4 for each given applied load magnitude, it is found that there is almost no difference among the three cases, which suggests that the effects of columns beyond those in the adjacent stories on $R_{cl,\,ij}$ may be negligible. Furthermore, the trivial values of the errors between Cases 1 and 2 suggest that the stiffness interaction between columns in adjacent stories is insignificant. Therefore, for the reason of practicality, the rotational stiffness interaction among columns in the same column line and different stories is neglected in this study for calculating the column end rotational stiffness.

## 4. Decomposition of multi-storey frames

In the case of single storey frames, the beam-to-column rotational restraint is directly applied to the upper ends of connected columns. In a multi-storey frame case, floor beams provide rotational restraints for both the lower and upper columns at a joint. To decompose a multi-storey frame into a series of single-storey PR frames and apply the storey-based buckling method proposed by Xu and Liu (2002), one of the challenges is how to distribute the beam-to-column rotational-restraining stiffness between the lower and upper columns with consideration of the effects of axial load on column end rotational stiffness. In this section, the beam-to-column rotational-restraining stiffness is discussed at first. Then, different approaches of decomposing a multi-storey frame into a series of single-storey PR frames are proposed.

### *4.1 Beam-to-column restraining stiffness*

Shown in Fig. 5 is the deformed profile of column subassemblage in an unbraced frame in the

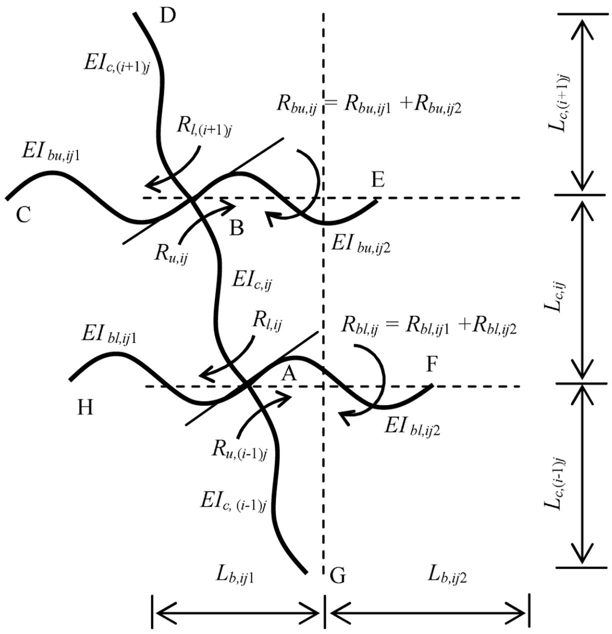


Fig. 5 Beam-to-column rotational restraining stiffness

lateral sway buckling mode. The beam-to-column rotational-restraining stiffnesses at the upper and lower joints of column $ij$ are denoted as $R_{bu,ij}$ and $R_{bl,ij}$ and can be expressed as

$$R_{bu,ij} = \sum_{k=1}^{2} R_{bu,ijk} \tag{15a}$$

$$R_{bl,ij} = \sum_{k=1}^{2} R_{bl,ijk} \tag{15b}$$

where subscript $k$ ($k = 1, 2$) denotes the beam on each side of column $ij$. $R_{bu,ijk}$ and $R_{bl,ijk}$ are end rotational stiffnesses of the beams that are connected to the upper and lower ends of column $ij$, respectively, and they are Xu and Liu (2002)

$$R_{bu,ijk} = \frac{6r_{k,1}}{4 - r_{k,1}r_{k,2}} \frac{EI_{bu,ijk}}{L_{bu,ijk}}(2 + \nu r_{k,2}) \tag{16a}$$

$$R_{bl,ijk} = \frac{6r_{k,1}}{4 - r_{k,1}r_{k,2}} \frac{EI_{bl,ijk}}{L_{bl,ijk}}(2 + \nu r_{k,2}) \tag{16b}$$

in which $r_{k,1}$ and $r_{k,2}$ are end-fixity factors associated with the near and far ends of beam $k$, respectively. Parameter $\nu$ accounts for the deformed shape of the beam in frame buckling. For unbraced frames buckling in lateral sway mode, $\nu$ is taken as one.

Let $R_{l,ij}$ and $R_{u,ij}$ be the end restraining stiffnesses of the lower and upper ends of column $ij$, respectively. Based on the principle that the distribution of beam-to-column restraining stiffness shall be proportional to the column end rotational stiffness at each joint, the end rotational-restraining stiffnesses of the upper and lower ends of column $ij$ can be expressed as,

$$R_{u,\,ij} = \mu_{u,\,ij} R_{bu,\,ij} \tag{17a}$$

$$R_{l,\,ij} = \mu_{l,\,ij} R_{bl,\,ij} \tag{17b}$$

where the stiffness distribution factors $\mu_{u,\,ij}$ and $\mu_{l,\,ij}$ are

$$\mu_{u,\,ij} = \frac{R_{cu,\,ij}}{R_{cl,\,(i+1)j} + R_{cu,\,ij}} \tag{18a}$$

$$\mu_{l,\,ij} = \frac{R_{cl,\,ij}}{R_{cl,\,ij} + R_{cu,\,(i-1)j}} \tag{18b}$$

in which $R_{cu,\,(i-1)j}$ and $R_{cu,\,ij}$ are the rotational stiffnesses of the upper end for columns $(i-1)\,j$ and $ij$; $R_{cl,\,ij}$ and $R_{cl,\,(i+1)j}$ are the rotational stiffnesses of the lower end for columns $ij$ and $(i+1)\,j$, respectively.

### 4.2 Distribution beam-to-column rotational-restraining stiffness

As stated previously, to decompose a multi-storey frame into a series of single-storey PR frames, one of the challenges is how to distribute the beam-to-column rotational-restraining stiffness between the lower and upper columns at a joint so that the column end restraining stiffnesses and corresponding end-fixity factors can be evaluated in accordance with Eqs. (17) and (4), respectively. It is noted from Eqs. (2) and (3) that the column end-fixity factors are essential for computing the column lateral stiffness modification factor which characterizes the column stability in the lateral sway mode.

It is clear from Eqs. (17) and (18) that the determination of the end restraining stiffness of a column requires the evaluation of column end rotational stiffness which can be computed in accordance with any one of Eqs. (12) to (14) depending on the desired accuracy of the results. Therefore, there are three approaches for computing the distribution factor of beam-to-column rotational-restraining stiffness, in which the column end rotational stiffness are evaluated corresponding to Eqs. (12), (13) and (14), respectively. For the column subassemblage shown in Fig. 6, the relationship between the stiffness distribution factors for columns that are joined together satisfies

$$\mu_{l,\,(i+1)j} = 1 - \mu_{u,\,ij} \tag{19}$$

thus, only the stiffness distribution factor associated with upper end of column $ij$ needs to be evaluated.

The first approach of computing the distribution factor, $\mu_{u,\,ij}$, is referred to as the geometrical stiffness distribution (GSD) approach in which the effect of column axial force is accounted for, and the column end rotational stiffnesses are calculated based on Eqs. (12). For column $ij$, the column end rotational stiffness $R_{cu,\,ij}$, can be directly obtained from Eq. (12b) in terms of end fixity factor $r_{l,\,ij}$. Applying Eq. (12a) for column $(i+1)\,j$ to compute column end rotational stiffness $R_{cl,\,(i+1)j}$ yields

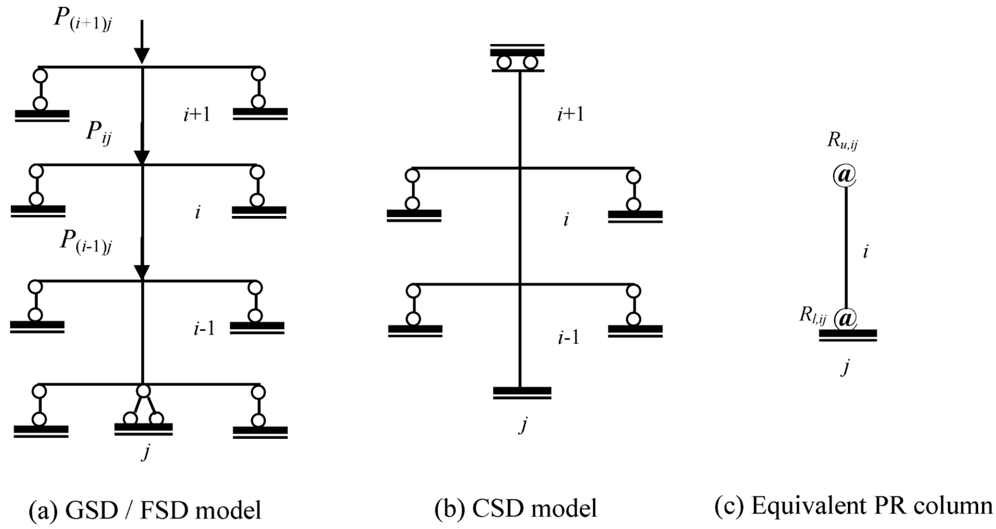


Fig. 6 Models of decomposition

$$R_{cl,(i+1)j} = \frac{R_{u,(i+1)j} - EI_{c,(i+1)j}\phi_{(i+1)j}\tan\phi_{(i+1)j}/L_{c,(i+1)j}}{1 + \dfrac{R_{u,(i+1)j}L_{c,(i+1)j}}{EI_{c,(i+1)j}}\dfrac{\tan\phi_{(i+1)j}}{\phi_{(i+1)j}}} \tag{20}$$

Noted that $R_{u,(i+1)j}$ in Eq. (20) can be taken as $R_{u,(i+1)j} = R_{bu,(i+1)j}$ based on the foregoing discussion which concludes that the rotational stiffness interaction between upper and lower columns in the same column line can be neglected in calculating the column end rotational stiffness. The model of the decomposition is shown in Fig. 6(a). By substituting Eqs. (12b) and (20) into Eq. (18a), the stiffness distribution factor associated with the upper end of column *ij* can be obtained as

$$\mu_{u,ij} = \frac{\dfrac{EI_{c,ij}}{L_{c,ij}}\dfrac{3r_{l,ij} - (1-r_{l,ij})\phi_{ij}\tan\phi_{ij}}{1 - r_{l,ij} + 3r_{l,ij}\dfrac{\tan\phi_{ij}}{\phi_{ij}}}}{\dfrac{EI_{c,ij}}{L_{c,ij}}\dfrac{3r_{l,ij} - (1-r_{l,ij})\phi_{ij}\tan\phi_{ij}}{1 - r_{l,ij} + 3r_{l,ij}\dfrac{\tan\phi_{ij}}{\phi_{ij}}} + \dfrac{R_{bu,(i+1)j} - EI_{c,(i+1)j}\phi_{(i+1)j}\tan\phi_{(i+1)j}/L_{c,(i+1)j}}{1 + \dfrac{R_{bu,(i+1)j}L_{c,(i+1)j}}{EI_{c,(i+1)j}}\dfrac{\tan\phi_{(i+1)j}}{\phi_{(i+1)j}}}} \tag{21}$$

where $r_{l,ij}$ is defined in Eq. (4b), and $\phi_{ij}$ and $\phi_{i(j+1)}$ are defined in accordance with Eq. (5).

Similarly, while Eqs. (13) are employed to evaluate $\mu_{u,ij}$ based on Eq. (18a), the corresponding stiffness distribution factor can be expressed as

$$\mu_{u,ij} = \frac{\dfrac{EI_{c,ij}}{L_{c,ij}}\dfrac{3r_{l,ij}}{1+2r_{l,ij}}}{\dfrac{EI_{c,ij}}{L_{c,ij}}\dfrac{3r_{l,ij}}{1+2r_{l,ij}} + \dfrac{EI_{c,(i+1)j}}{L_{c,(i+1)j}}\dfrac{1}{1 + EI_{c,(i+1)j}/R_{u,(i+1)j}L_{c,(i+1)j}}} \tag{22}$$

where $R_{u,(i+1)j} = R_{bu,(i+1)j}$ as discussed in Eq. (21) and $r_{l,ij}$ is defined in Eq. (4b). The approach based on Eq. (22) to compute the stiffness distribution factor is referred to as the frame-based stiffness distribution (FSD) approach, and the corresponding model of decomposition is the same as that shown in Fig. 6(a) except the effects of axial loads are neglected.

If Eqs. (14) are employed to evaluate $\mu_{u,ij}$ based on Eq. (18a), this yields

$$\mu_{u,\,ij} = \frac{EI_{c,\,ij}/L_{c,\,ij}}{EI_{c,\,ij}/L_{c,\,ij} + EI_{c,\,(i+1)j}/L_{c,\,(i+1)j}} \tag{23}$$

The approach based on Eq. (23) to evaluate the stiffness distribution factor is referred to as the column-based stiffness distribution (CSD) approach. The model of decomposition associated with the CSD approach is similar to that of the FSD approach except that the rotational stiffness of beams at the far end of the column in adjacent stories is taken as infinite as that shown in Fig. 6(b). It is shown in the Appendix that the stability equation associated with alignment chart can be obtained from that of the CSD model in the case of single column subassemblage.

It is noted that the evaluation of the stiffness distribution factor, $\mu_{u,\,ij}$, in accordance with either the GSD or FSD approach, requires the end-fixity factor at the far end of the column $r_{l,\,ij}$ to be known, as indicated in Eqs. (21) and (22). In such case, the decomposition process can be conveniently initiated from the first storey because the end-fixity factors associated with column bases are known and continued toward to the upper stories. For the case of using the CSD approach, the decomposition process can be initiated from any storey. Having the distribution factor $\mu_{u,\,ij}$ be evaluated, the corresponding distribution factor for the lower end of column $(i+1)j$, $\mu_{l,\,(i+1)j}$, can obtained from Eq. (19). Consequently, the corresponding column end restraining stiffnesses $R_{u,\,ij}$ and $R_{l,\,(i+1)j}$ shown in Fig. (6c) can be obtained from Eqs. (17). Having column end restraining stiffnesses for all columns being computed, the multi-storey frame can now be represented a series of single-storey PR frames as shown in Fig. 1(b).

Among the three proposed approaches, the GSD approach is the one in which the effects of the axial force on column end rotational stiffness are accounted for. However, in a frame buckling analysis, the critical axial force of each column at the buckling state is unknown in advance, and as the axial force and column end rotational stiffness are interrelated, numerical iterations are required to obtain the results. As the iterative process may be quite cumbersome from the viewpoint of practice, it is recommended to initiate the process of evaluation of the stiffness distribution factors with either the FSD or CSD approach and compute column the critical axial force $P_{ij}$ in accordance with the procedure of evaluating the column effective length factor, described in the next section. After that, recalculate the stiffness distribution factor based on the obtained column critical axial force $P_{ij}$ and Eq. (21) and decompose the frame accordingly. In such a way, more accurate effective length factors of columns can be obtained with only two iterations as shown in the demonstrated numerical examples.

## 5. Evaluation of column effective length factors

Having a multi-storey frame decomposed into a series of single-storey PR frames as discussed in the previous section, the critical load multiplier associated with each storey in lateral sway buckling can be obtained from Eq. (6) for the corresponding single-storey PR frame. However, the transcendental relationship between $\beta_{ij}$ and $\phi_{ij}$ expressed in Eq. (2) is complicated and inconvenient for solving the critical loads of each storey. For engineering practice, Eq. (2) can be simplified and approximated by means of the first-order of Taylor series expansion as Xu and Liu (2002)

$$\beta_{ij} = \beta_{0,\,ij} - \beta_{1,\,ij}\phi_{ij}^2 \tag{24}$$

where the values of $\beta_{0,\,ij}$ and $\beta_{1,\,ij}$ can be computed from the following expressions

$$\beta_{0,\,ij} = \frac{r_{l,\,ij} + r_{u,\,ij} + r_{u,\,ij}r_{l,\,ij}}{4 - r_{l,\,ij}r_{u,\,ij}} \tag{25a}$$

$$\beta_{1,\,ij} = \frac{8(5 + r_{u,\,ij}^2) - (34 - r_{u,\,ij})r_{u,\,ij}r_{l,\,ij} + (8 + r_{u,\,ij} + 3r_{u,\,ij}^2)r_{l,\,ij}^2}{30(4 - r_{l,\,ij}r_{u,\,ij})^2} \tag{25b}$$

By substituting Eq. (24) into Eq. (1), the lateral stiffness of column $ij$ can be written as

$$S_{ij} = 12\left(\frac{EI_{c,\,ij}}{L_{c,\,ij}^3}\beta_{0,\,ij} - \frac{P_{u,\,ij}}{L_{c,\,ij}}\beta_{1,\,ij}\lambda_i\right) \tag{26}$$

in which $L_{c,\,ij}$ and $P_{u,\,ij}$ are the length and applied axial load of column $j$ in the $i$th storey, respectively. $\lambda_i$ is the proportional load multiplier associated with the $i$th storey of the frame. The previous study by Xu and Liu (2002) on single-storey PR frames demonstrated that Eq. (24) provided an adequate approximation for Eq. (2) for evaluating column lateral stiffness. Substituting Eq. (26) into Eq. (6), the stability equation for storey $i$ buckling in a lateral sway mode can be expressed as

$$S_i = \sum_{j=1}^{m} 12\left(\frac{EI_{c,\,ij}}{L_{c,\,ij}^3}\beta_{0,\,ij} - \frac{P_{u,\,ij}}{L_{c,\,ij}}\beta_{1,\,ij}\lambda_i\right) = 0 \tag{27}$$

from which the critical load multiplier can be solved as

$$\lambda_{icr} = \sum_{j=1}^{m}\frac{EI_{c,\,ij}\beta_{0,\,ij}}{L_{c,\,ij}^3}\Bigg/\sum_{j=1}^{m}\frac{P_{u,\,ij}\beta_{1,\,ij}}{L_{c,\,ij}} \tag{28}$$

and the critical axial force of the column is

$$P_{ij} = \lambda_{icr}P_{u,\,ij} \qquad (j = 1, 2, 3, \ldots, m) \tag{29}$$

Finally, the storey-based effective length factor of the column can be evaluated as

$$K_{ij} = \frac{\pi}{L_{c,\,ij}}\sqrt{\frac{EI_{c,\,ij}}{\lambda_{icr}P_{u,\,ij}}} \qquad (j = 1, 2, 3, \ldots, m) \tag{30}$$

The proposed procedure for evaluation of the storey-based effective length factor for columns in a multi-storey unbraced frame with $m - 1$ bays and $n$ stories can be summarized as follows:

(1) Compute the rotational stiffness of each beam and beam-to-column restraining stiffness of each joint according to Eqs. (16) and (15), respectively. Set storey index $i = 1$ ($i = 1$ for the first storey);

(2) Evaluate upper-end stiffness distribution factors $\mu_{u,\,ij}$ from Eqs. (21) to (23) based on selected decomposing approach (GSD, FSD and CSD) for all of the columns ($j$ = 1, 2, 3…$m$) in the storey; If $i \neq n$, then evaluate lower-end stiffness distribution factors for upper columns $\mu_{l,\,i(j+1)}$ from Eq. (19);

(3) Calculate the beam-to-column rotational-restraining stiffnesses $R_{u,\,ij}$ and $R_{l,\,ij}$ based on Eqs. (17) and compute corresponding end-fixity factors $r_{u,\,ij}$ and $r_{l,\,ij}$ from Eqs. (4) for all of the columns in the $i$th storey;

(4) Compute the column lateral stiffness modification coefficients $\beta_{0,\,ij}$ and $\beta_{1,\,ij}$ ($j$ = 1, 2, 3…$m$) from Eqs. (25);

(5) Solve the critical load multiplier associated with the $i$th $\lambda_{icr}$ from Eq. (28) and compute the corresponding storey-based effective length factors $K_{ij}$ ($j$ = 1, 2, 3…$m$) from Eq. (30) for columns in the storey;

(6) If $i \neq n$, then set storey index $i = i + 1$ and go to step 2, otherwise the procedure is terminated.

It is noted that the above procedure is primarily developed for cases when either the FSD or CSD approach is selected for the decomposition process. If the GSD approach is selected, then the iterative process to account for the effect of axial force in columns would have to take place. As discussed in the previous section, a process that involves only two iterations is recommended by minor modification on the foregoing procedure for the first iteration as follows:

1. In Step 2, evaluate upper-end stiffness distribution factors $\mu_{u,\,ij}$ based on Eq. (22) or (23) instead of Eq. (21);
2. In Step 5, instead of computing column effective length factors $K_{ij}$ ($j$ = 1, 2, 3… $m$), calculate column critical axial force $P_{ij}$ ($j$ = 1, 2, 3…$m$) in accordance with Eq. (29) so that $P_{ij}$ can be used for evaluating upper-end stiffness distribution factors $\mu_{u,\,ij}$ in accordance with Eq. (21) in the second iteration.

It is noted that the effective length factor obtained from Eq. (30) for columns in the $i$-th storey is referred to as the storey-based effective length factor because it is evaluated based on the critical load multiplier, $\lambda_{icr}$, which is associated with lateral instability of the same storey. Since the critical load multiplier corresponding to the lateral instability of a particular storey may not be the most critical one for the multi-storey frame, therefore, the column effective length factor associated with the most critical load multiplier of the multi-storey frame as defined in the system buckling method (Majid 1972, Livesley 1975, Chen and Lui 1987) can be obtained as

$$K_{ij} = \frac{\pi}{L_{c,\,ij}}\sqrt{\frac{EI_{c,\,ij}}{\lambda_{cr}P_{u,\,ij}}} \qquad (j = 1, 2, 3, \ldots, m) \tag{31}$$

where $\lambda_{cr} = \min\{\lambda_{1cr}, \lambda_{2cr}, \lambda_{3cr} \ldots \lambda_{ncr}\}$ is the critical load multiplier associated with the multi-storey frame.

## 6. Numerical examples

The proposed storey-based stability analysis procedure for unbraced multi-storey frames is illustrated in the following two steel frame examples. The frames are comprised of steel beams and columns with W-shape sections, where Young's modulus of the steel $E$ = 200 GPa. In the first example, the column-based stiffness distribution (CSD) approach is adopted for a one-bay three-storey frame, and the detailed calculations of effective length factors for columns of each storey are presented. Then, the application of the geometrical stiffness distribution (GSD) approach is demonstrated through the calculations for columns in the first storey of the frame. While in the second example, the frame-based stiffness distribution (FSD) approach is employed to analyze a two-bay two-storey frame. For both examples, comparisons are made among the results obtained from the proposed analysis procedure and from other methods to show the validity of the proposed analysis.

### 6.1 Example 1: One-bay three-storey frame

A one-bay three-storey frame is shown in Fig. 7 (Shanmugam *et al*. 1995), in which the moment of inertia associated with W-shape sections are: W8 × 35, $I_x$ = 5.286 × $10^{-5}$ m$^4$ (127 in$^4$); W8 × 48, $I_x$ = 7.659 × $10^{-5}$ m$^4$ (184 in$^4$); W14 × 30, $I_x$ = 12.112 × $10^{-5}$ m$^4$ (291 in$^4$); W21 × 44, $I_x$ = 35.088 × $10^{-5}$ m$^4$ (843 in$^4$).

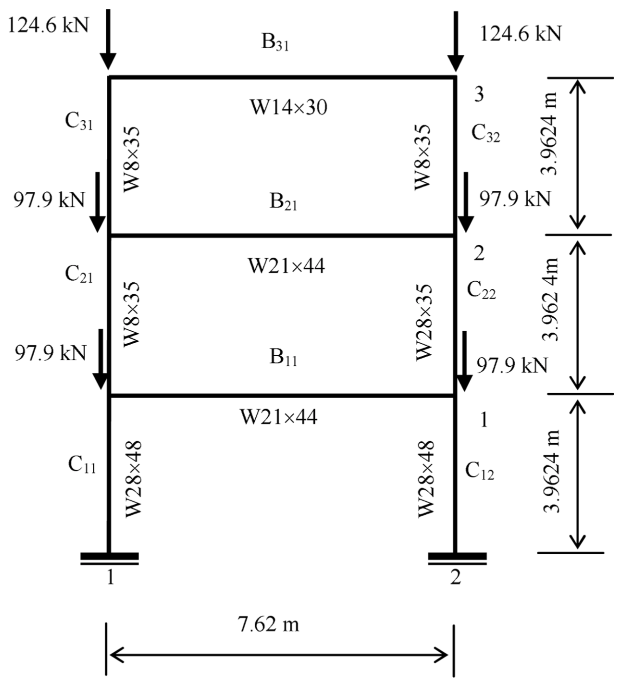


Fig. 7 Example 1: One-bay three-storey frame

Table 2 Example 1: Column effective length factors

| Column | System buckling | Alignment chart | LeMessurier | Lui | CSD Eq. (30) | FSD Eq. (30) | GSD Eq. (30) | CSD Eq. (31) | FSD Eq. (31) | GSD Eq. (31) |
|---|---|---|---|---|---|---|---|---|---|---|
| Lower | 1.14 | 1.11 | 1.12 | 1.14 | 1.11 | 1.11 | 1.13 | 1.21 | 1.21 | 1.15 |
| Middle | 1.14 | 1.21 | 1.21 | 1.21 | 1.21 | 1.21 | 1.15 | 1.21 | 1.21 | 1.15 |
| Upper | 1.52 | 1.23 | 1.23 | 1.30 | 1.23 | 1.23 | 1.43 | 1.61 | 1.61 | 1.53 |

*Column-based Stiffness Distribution (CSD) approach*

The detailed calculations of critical load multipliers of each storey and column effective length factors based on the CSD approach are demonstrated in the following. The resulting column effective length factors are also summarized in Table 2.

**Storey 1:**

(1) The rotational stiffness of beam $B_{11}$ which is connected to columns $C_{11}$ and $C_{12}$ can be evaluated in accordance with Eqs. (16) as

$$R_{bu,\,11} = 6\frac{EI_{b,\,11}}{L_{b,\,11}} = 6\frac{2\times10^{8}\times35.088\times10^{-5}}{7.62} = 55256.69 \text{ kN-m/rad}$$

(2) The distribution factors associated with the upper ends of columns $C_{11}$ and $C_{12}$ can be obtained from Eq. (23) as

$$\mu_{u,\,11} = \mu_{u,\,12} = \frac{EI_{c,\,11}/L_{c,\,11}}{EI_{c,\,11}/L_{c,\,11} + EI_{c,\,11}/L_{c,\,11}}$$

$$= \frac{2\times10^{8}\times7.659\times10^{-5}/3.9624}{2\times10^{8}\times7.659\times10^{-5}/3.9624 + 2\times10^{8}\times5.286\times10^{-5}/3.9624} = 0.5917$$

and from Eq. (19), the distribution factors for the lower ends of column $C_{21}$ and $C_{22}$ are $\mu_{l,\,21} = \mu_{l\,,22} = 1 - \mu_{u,\,11} = 0.4083$.

(3) The end-fixity factors for the lower end of the columns are $r_{l,\,11} = r_{l,\,12} = 1$ as the column bases are rigidly connected to the foundation. The beam-to-column rotational-restraining stiffness contributed by beam $B_{11}$ to the upper ends of columns $C_{11}$ and $C_{12}$ can be computed from Eqs. (17) as

$$R_{u,\,11} = R_{u,\,12} = \mu_{u,\,11}R_{bu,\,11} = 0.5917\times55256.69 = 32695.38 \text{ kN-m/rad}$$

The corresponding end-fixity factors are given by Eqs. (4):

$$r_{u,\,11} = r_{u,\,12} = \frac{1}{1+\dfrac{3EI_{c,\,11}}{R_{u,\,11}L_{c,\,11}}} = \frac{1}{1+\dfrac{3\times2\times10^{8}\times7.659\times10^{-5}}{32695.38\times3.9624}} = 0.7382$$

(4) The column lateral stiffness modification coefficients $\beta_{0,\,1j}$ and $\beta_{1,\,1j}$ can now be evaluated from Eqs. (25) as

$$\beta_{0,\,11} = \beta_{0,\,12} = \frac{r_{l,\,11}+r_{u,\,11}+r_{u,\,11}r_{l,\,11}}{4-r_{l,\,11}r_{u,\,11}} = \frac{1+0.7382+0.7382}{4-1\times0.7382} = 0.7592$$

$$\beta_{1,\,11} = \beta_{1,\,12} = \frac{8(5+r_{u,\,11}^{2})-(34-r_{u,\,11})r_{u,\,11}r_{l,\,11}+(8+r_{u,\,11}+3r_{u,\,11}^{2})r_{l,\,11}^{2}}{30(4-r_{l,\,11}r_{u,\,11})^{2}}$$

$$= \frac{8(5+0.7382^2)-(34-0.7382)\times 0.7382+(8+0.7382+3\times 0.7382^2)}{30\times(4-0.7382)^2} = 9.455\times 10^{-2}$$

(5) Compute critical load multiplier and column effective length factors:

$$\sum_{j=1}^{2}\frac{P_{u,ij}\beta_{1,ij}}{L_{c,ij}} = \frac{2P_{u,11}}{L_{c,11}}\beta_{1,11} = \frac{2\times 320.4\times 0.09455}{3.9624} = 15.2907 \text{ kN/m}$$

$$\sum_{j=1}^{2}\frac{EI_{c,1j}\beta_{0,1j}}{L_{c,1j}^3} = \frac{2\times 2\times 10^8\times 7.659\times 10^{-5}\times 0.7592}{3.9624^3} = 373.8638 \text{ kN/m}$$

Substituting the foregoing values into Eq. (28) yields

$$\lambda_{1cr} = \sum_{j=1}^{2}\frac{EI_{c,1j}\beta_{0,1j}}{L_{c,1j}^3}\Bigg/\sum_{j=1}^{2}\frac{P_{u,1j}\beta_{1,1j}}{L_{c,1j}} = \frac{373.8638}{15.2907} = 24.45;$$

Based on Eq. (30), the storey-based effective length factors of the columns in Storey 1 are

$$K_{11} = K_{12} = \frac{\pi}{L_{c,11}}\sqrt{\frac{EI_{c,11}}{\lambda_{1cr}P_{u,11}}} = \frac{3.1416}{3.9624}\times\sqrt{\frac{2\times 10^8\times 7.659\times 10^{-5}}{24.45\times 320.4}} = 1.11$$

**Storey 2:**

(1) The rotational stiffnesses of beam $B_{21}$ are the same as that of beam $B_{11}$,

$$R_{bu,21} = R_{bu,11} = 55256.7 \text{ kN-m/rad}$$

(2) The distribution factors for the upper ends of column $C_{21}$ and $C_{22}$ are

$$\mu_{u,21} = \mu_{u,22} = \frac{EI_{c,21}/L_{c,21}}{EI_{c,21}/L_{c,21}+EI_{c,31}/L_{c,31}}$$

$$= \frac{2\times 10^8\times 5.286\times 10^{-5}/3.9624}{2\times 10^8\times 5.286\times 10^{-5}/3.9624+2\times 10^8\times 5.286\times 10^{-5}/3.9624} = 0.5$$

and from Eq. (19), the distribution factors for the lower ends of column $C_{31}$ and $C_{32}$ are $\mu_{l,31} = \mu_{l,32} = 1 - \mu_{u,21} = 0.5$.

(3) The beam-to-column rotational-restraining stiffnesses contributed by beam $B_{11}$ to the lower ends of columns $C_{21}$ and $C_{22}$ and the corresponding end-fixity factors can be obtained as

$$R_{l,21} = R_{l,22} = \mu_{l,21}R_{bu,11} = 0.4083\times 55256.69 = 22561.31 \text{ kN-m/rad}$$

$$r_{l,21} = r_{l,22} = \frac{1}{1+\dfrac{3EI_{c,21}}{R_{u,21}L_{c,21}}} = \frac{1}{1+\dfrac{3\times 2\times 10^8\times 5.286\times 10^{-5}}{22561.31\times 3.9624}} = 0.7382$$

The beam-to-column rotational-restraining stiffness for the upper ends of columns $C_{21}$ and $C_{22}$ are

$$R_{u,21} = R_{u,22} = \mu_{u,21}R_{bu,21} = 0.5 \times 55256.7 = 27628.35 \text{ kN-m/rad}$$

from which the corresponding end-fixity factors are given by

$$r_{u,21} = r_{u,22} = \frac{1}{1+\dfrac{3EI_{c,21}}{R_{u,21}L_{c,21}}} = \frac{1}{1+\dfrac{3\times 2\times 10^8\times 5.286\times 10^{-5}}{27628.35\times 3.9624}} = 0.7754$$

(4) The column lateral stiffness modification coefficients $\beta_{0,2j}$ and $\beta_{1,2j}$ can be evaluated from Eqs. (25) as

$$\beta_{0,21} = \beta_{0,22} = \frac{r_{l,21}+r_{u,21}+r_{u,21}r_{l,21}}{4-r_{l,21}r_{u,21}} = \frac{0.7381+0.7754+0.7381\times 0.7754}{4-0.7381\times 0.7754} = 0.6085$$

$$\beta_{1,21} = \beta_{1,22} = \frac{8(5+r_{u,21}^2)-(34-r_{u,21})r_{u,21}r_{l,21}+(8+r_{u,21}+3r_{u,21}^2)r_{l,21}^2}{30(4-r_{l,21}r_{u,21})^2}$$

$$= \frac{8(5+0.7754^2)-(34-0.7754)\times 0.7754\times 0.7381+(8+0.7754+3\times 0.7754^2)\times 0.7381^2}{30\times(4-0.7754\times 0.7381)^2}$$

$$= 8.9534\times 10^{-2}$$

(5) Compute critical load multiplier and column effective length factors:

$$\sum_{j=1}^{2}\frac{P_{u,2j}\beta_{1,2j}}{L_{c,2j}} = \frac{2P_{u,21}}{L_{c,21}}\beta_{1,21} = \frac{2\times 222.5\times 0.089534}{3.9624} = 10.0552$$

$$\sum_{j=1}^{2}\frac{EI_{c,2j}\beta_{0,2j}}{L_{c,2j}^3} = \frac{2\times 2\times 10^8\times 5.286\times 10^{-5}\times 0.6085}{3.9624^3} = 206.8106$$

Thus, the critical load multiplier for Storey 2 is

$$\lambda_{2cr} = \sum_{j=1}^{2}\frac{EI_{c,2j}\beta_{0,2j}}{L_{c,2j}^3}\Bigg/\sum_{j=1}^{2}\frac{P_{u,2j}\beta_{1,2j}}{L_{c,2j}} = \frac{206.8106}{10.0552} = 20.57$$

and the associated storey-based factors can be calculated from Eq. (30) as

$$K_{21} = K_{22} = \frac{\pi}{L_{c,21}}\sqrt{\frac{EI_{c,21}}{\lambda_{2cr}P_{u,21}}} = \frac{3.1416}{3.9624}\times\sqrt{\frac{2\times 10^8\times 5.286\times 10^{-5}}{20.57\times 222.5}} = 1.21$$

**Storey 3:**

The procedure of calculating Storey 3 is similar to that of in Storey 2. The beam-to-column rotational-restraining stiffnesses for the lower ends of columns $C_{31}$ and $C_{32}$ and the corresponding end fixity factors are $R_{l,31} = R_{l,32} = R_{u,21} = 27628.35$ kN-m/rad and $r_{l,31} = r_{l,32} = r_{u,21} = 0.7754$. For the upper ends of columns $C_{31}$ and $C_{32}$, the beam-to-column rotational-restraining stiffnesses and corresponding end fixity factors are

$$R_{u,31} = R_{u,32} = R_{bu,31} = 6\frac{EI_{b,31}}{L_{b,31}} = 6\frac{2\times10^8\times12.112\times10^{-5}}{7.65} = 19074.02\text{ kN-m/rad}$$

$$r_{u,31} = r_{u,32} = \frac{1}{1+\dfrac{3EI_{c,31}}{R_{u,31}L_{c,31}}} = \frac{1}{1+\dfrac{3\times2\times10^8\times5.286\times10^{-5}}{19074.02\times3.9624}} = 0.7044$$

The critical load multiplier for Storey 3 is found as $\lambda_{3cr} = 35.55$, while the storey-based effective length factor for the columns are $K_{31} = K_{32} = 1.23$.

Comparing with the critical load multiplier found for each storey, the critical storey of the frame in lateral instability is Storey 2 with the load multiplier $\lambda_{cr} = \lambda_{2cr} = 20.57$. The column effective length factor associated with $\lambda_{cr}$ are obtained from Eq. (31) are $K_{11} = K_{12} = 1.21$, $K_{21} = K_{22} = 1.21$, and $K_{31} = K_{32} = 1.61$.

*Geometrical Stiffness Distribution (GSD) approach*

The foregoing calculation of obtaining the critical load multiplier of the frame ($\lambda_{cr} = 20.57$) based on the CSD approach is identical to that of the first iteration for the GSD approach. Therefore, the second iteration of the GSD approach is illustrated herein. Due to the space limitation, only the detail calculation associated with the columns in Storey 1 is presented. However, the resulting effective length factors for columns in Storeys 2 and 3 are summarized in Table 2.

**Storey 1:**

(1) The rotational stiffnesses of beam $B_{11}$ are obtained previously as $R_{bu,11} = R_{bu,21} = 55256.69$ kN-m/rad.

(2) Evaluate stiffness distribution factors: Based on Eq. (29), the critical axial forces for columns in Storeys 1 and 2 are

$$P_{11} = P_{12} = \lambda_{\min}P_{u,11} = 20.57\times(124.6+97.9+97.9) = 6590.63\text{ kN}$$

$$P_{21} = P_{22} = \lambda_{\min}P_{u,21} = 20.57\times(124.6+97.9) = 4576.83\text{ kN}$$

Then, the corresponding parameters, $\phi_{ij}$, are obtained from Eq. (5) as

$$\phi_{11} = \phi_{12} = \sqrt{\frac{P_{11}L_{c,11}}{EI_{c,11}/L_{c,11}}} = \sqrt{\frac{6590.63\times3.9624}{3865.84}} = 2.5991$$

$$\phi_{21} = \phi_{22} = \sqrt{\frac{P_{21}L_{c,21}}{EI_{c,21}/l_{C,21}}} = \sqrt{\frac{4576.83\times3.9624}{2668.08}} = 2.6071$$

Note that the end-fixity factors are unity at column bases ($r_{l,11} = r_{l,12} = 1$) and take $R_{u,21} \approx R_{bu,21} = 55256.69$ kN-m/rad. The stiffness distribution factors associated with the upper end of columns $C_{11}$ and $C_{12}$ can be computed based on Eq. (21) as

$$\mu_{u,11} = \mu_{u,12} = \frac{\dfrac{EI_{c,11}}{L_{c,11}}\dfrac{3r_{l,11}-(1-r_{l,11})\phi_{11}\tan\phi_{11}}{1-r_{l,11}+3r_{l,11}\dfrac{\tan\phi_{11}}{\phi_{11}}}}{\dfrac{EI_{c,11}}{L_{c,11}}\dfrac{3r_{l,11}-(1-r_{l,11})\phi_{11}\tan\phi_{11}}{1-r_{l,11}+3r_{l,11}\dfrac{\tan\phi_{11}}{\phi_{11}}}+\dfrac{EI_{c,21}}{L_{c,21}}\dfrac{R_{bu,212}/(EI_{c,21}/L_{c,21})-\phi_{21}\tan\phi_{21}}{1+\dfrac{R_{u,21}}{(EI_{c,21}/L_{c,21})}\dfrac{\tan\phi_{21}}{\phi_{21}}}}$$

$$= \frac{-16667.24}{-16667.24-16037.72} = 0.5096$$

(3) The beam-to-column rotational-restraining stiffness contributed by beam $B_{11}$ to the upper ends of columns $C_{11}$ and $C_{12}$ can be computed from Eqs. (17) as

$$R_{u,11} = R_{u,12} = \mu_{u,11}R_{bu,11} = 0.5096\times 55256.7 = 28160.15 \text{ kN-m/rad}$$

and the corresponding end-fixity factors can be found as

$$r_{u,11} = r_{u,12} = \frac{1}{1+3(EI_{c,11}/L_{c,11})/R_{u,11}} = \frac{1}{1+3\times 3865.84/28160.15} = 0.7083$$

(4) The column lateral stiffness modification coefficients $\beta_{0,1j}$ and $\beta_{1,1j}$ can be obtained from Eqs. (25) as

$$\beta_{0,11} = \beta_{0,12} = \frac{1+0.7083+0.7083}{4-1\times 0.7083} = 0.7341$$

$$\beta_{1,11} = \frac{8(5+0.7083^2)-(34-0.7083)(0.7083)+(8+0.7083+3\times 0.7083^2)}{30(4-0.7083)^2} = 9.4279\times 10^{-2}$$

(5) Compute the critical load multiplier and column effective length factors:

$$\sum_{j=1}^{2}\frac{P_{u,ij}\beta_{1,ij}}{L_{c,ij}} = \frac{2P_{u,11}}{L_{c,11}}\beta_{1,11} = \frac{2\times 320.4\times 0.094279}{3.9624} = 15.2469 \text{ kN/m}$$

$$\sum_{j=1}^{2}\frac{EI_{c,1j}\beta_{0,1j}}{L_{c,1j}^3} = \frac{2R_{cu,11}}{L_{c,11}^2}\beta_{0,11} = \frac{2\times 3865.84\times 0.7341}{3.9624^2} = 361.5254 \text{ kN/m}$$

Substituting the foregoing values into Eq. (28) yields

$$\lambda_{1cr} = \sum_{j=1}^{2}\frac{EI_{c,1j}\beta_{0,1j}}{L_{c,1j}^3}\Bigg/\sum_{j=1}^{2}\frac{P_{u,1j}\beta_{1,1j}}{L_{c,1j}} = \frac{361.5254}{15.2469} = 23.71$$

Therefore, the effective length factors for columns $C_{11}$ and $C_{12}$ are

$$K_{11} = K_{12} = \pi\sqrt{\frac{EI_{c,11}/L_{c,11}}{\lambda_{1cr}P_{u,11}L_{c,11}}} = 3.1416\times\sqrt{\frac{3865.84}{23.71\times 320.4\times 3.9624}} = 1.13$$

Repeating the foregoing procedure for storeys 2 and 3, it can be found that $\lambda_{2cr}$ = 22.71, $K_{21} = K_{22}$ = 1.15 and $\lambda_{3cr}$ = 26.24, $K_{31} = K_{32}$ = 1.43 for storeys 2 and 3, respectively. Therefore, the critical load multiplier of the frame in lateral instability is $\lambda_{cr} = \lambda_{2cr}$ = 22.71, and the corresponding column effective length factors are $K_{11} = K_{12}$ = 1.15, $K_{21} = K_{22}$ = 1.15, and $K_{31} = K_{32}$ = 1.53.

For the purpose of comparison, the column effective length factors of this example obtained from the CSD, FSD and GSD approaches and those obtained from other methods reported by Shanmugam *et al*. (1995) are presented in Table 2. It is found the results obtained from the CSD and FSD approaches are not different from this example, and they are in good agreement with those of LeMessurier's method when Eq. (30) is used for computing column effective length factors. It can be seen from Table 2 that among different methods of evaluation of column effective length factors, the GSD approach associated with using Eq. (31) yields the most accurate results to those of system buckling analysis.

**Two-bay two-storey frame**

The second illustration is a two-bay two-storey frame as shown in Fig. 8, which is investigated by Lui (1992), where the beam-to-column connections of are rigid. The moment of inertia of each member is shown in Fig. 8, in which $I = 8.3246 \times 10^{-5}$ m$^4$. For this example, the FSD approach is selected for computing column effective length factors, and the detailed calculation for columns in the first storey is present below to illustrate the procedure.

(1) Compute the rotational stiffnesses of beams in the first storey from Eqs. (16):

$$R_{bu,112} = R_{bu,121} = R_{bl,212} = R_{bl,221} = 6\frac{EI_{b,112}}{L_{b,112}} = 6\frac{2\times10^{8}\times87.4915\times10^{-5}}{6.0168} = 174494 \text{ kN-m/rad}$$

where the last subscript denotes whether the beam is located on the left or right side of the column. For example, the last subscript 2 in $R_{bu,112}$ denotes that the beam is on the right side of column $C_{11}$ as that shown Fig. 8.

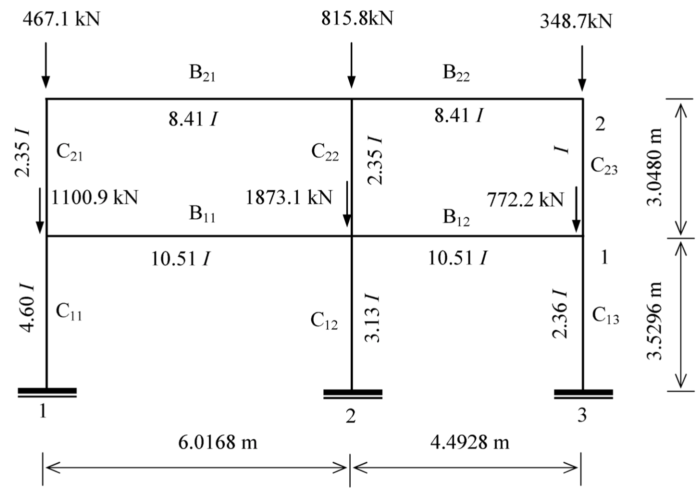


Fig. 8 Example 2: Two-bay two-storey frame

Similarly, it can be found that $R_{bu,122} = R_{bu,131} = 233685$ kN-m/rad, $R_{bu,212} = R_{bu,221} = 139629$ kN-m/rad, and $R_{bu,222} = R_{bu,231} = 186992$ kN-m/rad. Hence, the beam-to-column restraining stiffnesses are: $R_{bu,11} = R_{bu,112} = 174494$ kN-m/rad, $R_{bu,12} = R_{bu,121} + R_{bu,122} = 408179$ kN-m/rad, $R_{bu,13} = R_{bu,131} = 233685$ kN-m/rad, $R_{bu,21} = R_{bu,212} = 139629$ kN-m/rad, $R_{bu,22} = R_{bu,221} + R_{bu,222} = 326621$ kN-m/rad, $R_{bu,23} = R_{bu,231} = 186992$ kN-m/rad.

(2) Evaluate the stiffness distribution factors: as the end-fixity factors are unity at the column bases ($r_{l,11} = r_{l,12} = r_{l,13} = 1$) and the beam-to-column restraining stiffness at the upper end of column $C_{21}$ is $R_{u,21} = R_{bu,21} = 139629$ kN-m/rad, the distribution factors associated with the FSD approach for the upper end of column $C_{11}$ can be obtained from Eq. (22) as,

$$\mu_{u,11} = \frac{\dfrac{EI_{c,11}}{L_{c,11}}\dfrac{3r_{l,11}}{1+2r_{l,11}}}{\dfrac{EI_{c,11}}{L_{c,11}}\dfrac{3r_{l,11}}{1+2r_{l,11}} + \dfrac{EI_{c,21}}{L_{c,21}}\dfrac{1}{1+EI_{c,21}/R_{u,21}L_{c,21}}} = \frac{\dfrac{EI_{c,11}}{L_{c,11}}}{\dfrac{EI_{c,11}}{L_{c,11}} + \dfrac{EI_{c,21}}{L_{c,21}}\dfrac{1}{1+EI_{c,21}/R_{u,21}L_{c,21}}}$$

$$= \frac{\dfrac{2\times10^{8}\times38.2933\times10^{-5}}{3.5296}}{\dfrac{2\times10^{8}\times38.2933\times10^{-5}}{3.5296} + \dfrac{2\times10^{8}\times19.5628\times10^{-5}}{3.048}\dfrac{1}{1+12836/139629}} = \frac{21698}{21698+11756} = 0.6486$$

and from Eq. (19) the stiffness distribution factor for the lower end of column $C_{21}$ is $\mu_{l,21} = 1 - \mu_{u,11} = 0.3514$. Similarly, the distribution factors corresponding to the other columns can be found as $\mu_{u,12} = 0.5445$, $\mu_{l,22} = 0.4555$, $\mu_{u,13} = 0.6772$, $\mu_{l,23} = 0.3228$.

(3) The beam-to-column rotational-restraining stiffnesses contributed by beams $B_{11}$ and $B_{12}$ to columns $C_{11}$, $C_{21}$ and $C_{13}$ can be computed respectively from Eqs. (18) as

$$R_{u,11} = \mu_{u,11}R_{bu,11} = 0.6486\times174494 = 113177 \text{ kN-m/rad}$$

$$R_{u,12} = \mu_{u,12}R_{bu,12} = 0.5445\times408179 = 2222253 \text{ kN-m/rad}$$

and

$$R_{u,13} = \mu_{u,13}R_{bu,13} = 0.6772\times233685 = 158251 \text{ kN-m/rad}$$

The corresponding end-fixity factors can be obtained from Eq. (4) as $r_{u,11} = 0.6439$, $r_{u,12} = 0.8338$, and $r_{u,13} = 0.8257$.

Table 3 Example 2: Parameters associated with columns in Storey 1

| Column $1j$ | $r_{u,1j}$ | $r_{l,1j}$ | $\beta_{0,1j}$ | $\beta_{1,1j}\times10^{-2}$ | $P_{u,1j}\beta_{1,1j}/L_{c,1j}$ | $EI_{c,1j}\beta_{0,1j}/L_{c,1j}^3$ |
|---|---|---|---|---|---|---|
| 1 | 0.6349 | 1.000 | 0.6745 | 9.386 | 41.696 | 1174.78 |
| 2 | 0.8338 | 1.000 | 0.8425 | 9.585 | 73.023 | 998.52 |
| 3 | 0.8257 | 1.000 | 0.8353 | 9.572 | 30.397 | 746.40 |
| $\Sigma P_{u,1j}\beta_{1,1j}/L_{c,1j}, \Sigma EI_{c,1j}\beta_{0,1j}/L_{c,1j}^3$ | | | | | 145.115 | 2919.70 |

Table 4 Example 2: Column effective length factors

| Column | System buckling | Alignment chart | LeMessurier | Lui | CSD Eq. (30) | FSD Eq. (30) | GSD Eq. (30) | CSD Eq. (31) | FSD Eq. (31) | GSD Eq. (31) |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_{11}$ | 1.36 | 1.19 | 1.40 | 1.39 | 1.39 | 1.39 | 1.34 | 1.39 | 1.39 | 1.34 |
| $C_{12}$ | 0.86 | 1.06 | 0.88 | 0.86 | 0.88 | 0.87 | 0.85 | 0.88 | 0.87 | 0.85 |
| $C_{13}$ | 1.15 | 1.07 | 1.18 | 1.18 | 1.18 | 1.17 | 1.14 | 1.18 | 1.17 | 1.14 |
| $C_{21}$ | 2.06 | 1.25 | 1.40 | 1.58 | 1.39 | 1.40 | 1.75 | 2.11 | 2.10 | 2.04 |
| $C_{22}$ | 1.56 | 1.11 | 1.06 | 1.21 | 1.06 | 1.06 | 1.32 | 1.59 | 1.59 | 1.54 |
| $C_{23}$ | 1.56 | 1.12 | 1.05 | 1.20 | 1.05 | 1.06 | 1.32 | 1.59 | 1.59 | 1.54 |

(4) The column lateral stiffness modification coefficients $\beta_{0,1j}$, and $\beta_{1,1j}$ are calculated based on Eqs. (25) and tabulated in Table 3.

(5) Computing the critical load multiplier and column effective length factors: based on the information provided in Table 3, the critical load multiplier $\lambda_{1cr}$ can be found from Eq. (28) as

$$\lambda_{1cr} = \sum_{j=1}^{3} \frac{EI_{c,1j}\beta_{0,1j}}{L_{c,1j}^{3}} \Bigg/ \sum_{j=1}^{3} \frac{P_{u,1j}\beta_{1,1j}}{L_{c,1j}} = \frac{2919.70}{145.115} = 20.12$$

Thus, the effective length factors for columns $C_{11}$, $C_{12}$, and $C_{13}$ can be evaluated based on Eq. (30) as $K_{11} = 1.39$, $K_{12} = 0.87$, and $K_{13} = 1.17$, respectively.

(6) Repeat steps (2) to (5) for columns in the second storey; the corresponding column effective length factors are obtained and presented in Table 4.

For the reason of comparison, the column effective length factors calculated based on the CSD, FSD and GSD approaches and that of other methods reported by Lui (1992) are also presented in Table 4. It can be seen from Table 4 that when Eq. (30) is used for computing column effective length factors, the results of the CSD and FSD approaches are in good agreement with those of LeMessurier's method while the GSD approach associated with Eq. (31) yields the most accurate results to those of system buckling analysis.

## 7. Conclusions

This paper presents a study on elastic stability analysis of multi-storey unbraced frames based on the concept of storey-based buckling. A practical method of evaluating the effective length factors for columns in multi-storey frames is proposed by means of decomposing the multi-storey frame into a series of single-storey PR frames and applying the storey-based stability analysis procedure (Xu and Liu 2002) to each single-storey PR frame. In this study, the lateral stiffness of a multi-storey frame is derived and expressed as the product of the lateral stiffness of each individual storey, which make it possible to investigate the lateral stability of the multi-storey frame through examining the stability of each individual storey. The end rotational stiffness of an axially loaded column is derived, and rotational stiffness interaction between the upper and lower columns that are connected to each other is investigated. The study concludes that while calculating the column end rotational stiffness, the rotational stiffness interaction among columns in the same column line and in different stories is insignificant and can be neglected for the reason of engineering practice.

To facilitate the frame decomposition, the different approaches of distributing beam-to-column rotational-restraining stiffnesses between the upper and lower columns are investigated. The proposed three decomposition approaches, namely the GSD, FSD and CSD approaches, are characterized by the means of distributing beam-to-column rotational-restraining stiffnesses between the upper and lower columns. Among the three decomposition approaches, GSD accounts for the effect of axial force on column end rotational stiffness and therefore, provides more accurate results than that the other two simplified approaches, in which the effect of the axial force is neglected. However, the drawback associated with the GSD approach is that numerical iterations are required. This study proposed a procedure, which involves only two iterations to obtain accurate results while using the GSD approach.

After decomposing the multi-storey frame into a series of single-storey PR frames, the procedure proposed by Xu and Liu (2002) is applied to each decomposed single-storey PR frame to evaluate column effective length factors. Numerical examples are then presented to illustrate the effectiveness of the proposed procedure. The results obtained from the proposed approaches are compared with those of system buckling analysis, alignment chart method and methods proposed by other researchers (LeMessurier 1977, Lui 1992). It is found that the results obtained from the GSD approach provide better accuracy than the other methods. Among the three approaches proposed in this study, the CSD approach is simplest and provides reasonable accuracy for the column effective length factors; therefore, it is recommended for engineering practice. In the case that more refined results are desired, the GSD approach can be applied to obtain higher accuracy.

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## Appendix

It is shown in this Appendix that the stability equation associated with alignment chart can be obtained from the CSD model in the case of single column subassemblage. The stability equation which is used to develop the alignment chart for evaluating the effective length factor for column *ij* in unbraced frames is (AISC 1999)

$$\frac{\phi_{ij}}{\tan\phi_{ij}} - \frac{G_{l,ij}G_{u,ij}\phi_{ij}^2 - 36}{6(G_{l,ij} + G_{u,ij})} = 0 \tag{A1}$$

where $\phi_{ij}$ is defined in Eq. (5) and the column-to-beam stiffness ratios associated with the lower and upper joints of the column are defined respectively as

$$G_{l,ij} = \left(\sum_{k=i-1}^{i}\frac{EI_{c,kj}}{L_{c,kj}}\right)\Bigg/\left(\sum_{k=1}^{2}\frac{EI_{bl,ijk}}{L_{bl,ijk}}\right) \tag{A2a}$$

$$G_{u,ij} = \left(\sum_{k=i}^{i+1}\frac{EI_{c,kj}}{L_{c,kj}}\right)\Bigg/\left(\sum_{k=1}^{2}\frac{EI_{bu,ijk}}{L_{bu,ijk}}\right) \tag{A2b}$$

For a single PR column *ij* as shown in Fig. (6b), the stability equation for the column buckling in lateral sway mode such that the lateral stiffness of the column vanishes that can be expressed as follows based on Eq. (2):

$$\beta_{ij} = \frac{\phi_{ij}^3}{12}\frac{a_{1,ij}\phi_{ij}\cos\phi_{ij} + a_{2,ij}\sin\phi_{ij}}{18r_{l,ij}r_{u,ij} - a_{3,ij}\cos\phi_{ij} + a_{4,ij}\phi_{ij}\sin\phi_{ij}} = 0 \tag{A3}$$

where coefficients $a_{1,ij}$, $a_{2,ij}$ and $a_{3,ij}$ are defined in Eqs. (3). Since Eq. (A3) is true for any values of $r_{l,ij}$ and $r_{u,ij}$ between zero and one, thus the numerator in Eq. (A3) must satisfy the following:

$$a_{1,ij}\phi_{ij}\cos\phi_{ij} + a_{2,ij}\sin\phi_{ij} = 0 \tag{A4}$$

Considering column *ij* is in an unbraced frame with rigid beam-to-column connections buckles in lateral sway mode, this yields $r_{k,1} = r_{k,2} = 1$ and $\nu = 1$ in Eqs. (16). Therefore, the beam-to-column rotational-restraining stiffnesses $R_{l,ij}$ and $R_{u,ij}$ at the lower and upper ends of the column can be obtained from Eqs. (16) as

$$R_{bl,ij} = 6\sum_{k=1}^{2}\frac{EI_{bl,ijk}}{L_{bl,ijk}} \tag{A5a}$$

$$R_{bu,ij} = 6\sum_{k=1}^{2}\frac{EI_{bu,ijk}}{L_{bu,ijk}} \tag{A5b}$$

Form Eq. (23), the stiffness distribution factors associated with the CSD approach for the upper end of the column is

$$\mu_{u,ij} = \frac{EI_{c,ij}/L_{c,ij}}{EI_{c,ij}/L_{c,ij} + EI_{c,(i+1)j}/L_{c,(i+1)j}} \tag{A6a}$$

Similar to the derivation of Eq. (23), the stiffness distribution factors associated with the CSD approach for the lower end of the column can be obtained as

$$\mu_{l,ij} = \frac{EI_{c,ij}/L_{c,ij}}{EI_{c,ij}/L_{c,ij} + EI_{c,(i-1)j}/L_{c,(i-1)j}} \tag{A6b}$$

Substituting Eqs. (A5) and (A6) into Eqs. (17), the rotational-restraining stiffness of the column ends are

$$R_{u,ij} = 6\frac{EI_{c,ij}}{L_{c,ij}}\left(\sum_{k=1}^{2}\frac{EI_{bu,ijk}}{L_{bu,ijk}}\right)\Bigg/\left(\sum_{k=1}^{i+1}\frac{EI_{c,kj}}{L_{c,kj}}\right) \tag{A7a}$$

$$R_{l,ij} = 6\frac{EI_{c,ij}}{L_{c,ij}}\left(\sum_{k=1}^{2}\frac{EI_{bl,ijk}}{L_{bl,ijk}}\right)\Bigg/\left(\sum_{k=i-1}^{i}\frac{EI_{c,kj}}{L_{c,kj}}\right) \tag{A7b}$$

Considering Eqs. (A2), Eqs. (A7) can be expressed in terms of *G* factors as

$$R_{l,ij} = \frac{6EI_{c,ij}/L_{c,ij}}{G_{l,ij}} \tag{A8a}$$

$$R_{u,ij} = \frac{6EI_{c,ij}/L_{c,ij}}{G_{u,ij}} \tag{A8b}$$

Thus, the corresponding end-fixity factors for the column *ij* can be obtained from Eq. (4) as

$$r_{u,ij} = \frac{1}{1 + \frac{G_{u,ij}}{2}} \tag{A9a}$$

$$r_{l,ij} = \frac{1}{1 + \frac{G_{l,ij}}{2}} \tag{A9b}$$

Eqs. (A9) define the relationship between column end-fixity factors and the stiffness ratios. Substituting Eqs. (A9) into Eqs. (3a and 3b), the coefficients $a_{1,ij}$ and $a_{2,ij}$ can be obtained as

$$a_{1,ij} = \frac{6(G_{l,ij} + G_{u,ij})}{(2 + G_{u,ij})(2 + G_{l,ij})} \tag{A10a}$$

$$a_{2,ij} = \frac{(36 - G_{l,ij}G_{u,ij}\varphi_{ij}^2)}{(2 + G_{u,ij})(2 + G_{l,ij})} \tag{A10b}$$

By substituting Eqs. (A10) into Eq. (A4), the stability equation for the PR column *ij* shown in Fig. (6b), buckling in lateral sway mode can be expressed in terms of stiffness ratios other than the column end fixity factors as

$$\frac{6(G_{l,ij}+G_{u,ij})}{(2+G_{u,ij})(2+G_{l,ij})}\varphi_{ij}\cos(\varphi_{ij})+\frac{(36-G_{l,ij}G_{u,ij}\varphi_{ij}^2)}{(2+G_{u,ij})(2+G_{l,ij})}\sin(\varphi_{ij}) = 0 \tag{A11}$$

Eq. (A11) can be simplified as

$$6(G_{l,ij}+G_{u,ij})\varphi_{ij}\cos(\varphi_{ij})+(36-G_{l,ij}G_{u,ij}\varphi_{ij}^2)\sin(\varphi_{ij}) = 0 \tag{A12}$$

Thus, Eq. (A1) can be obtained by rearranging Eq. (A12).