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# Exact natural frequencies of structures consisting of two-part beam-mass systems

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**Abstract.** Using two different, but related approaches, an exact dynamic stiffness matrix for a two-part beam-mass system is developed from the free vibration theory of a Bernoulli-Euler beam. The first approach is based on matrix transformation while the second one is a direct approach in which the kinematical conditions at the interfaces of the two-part beam-mass system are satisfied. Both procedures allow an exact free vibration analysis of structures such as a plane or a space frame, consisting of one or more two-part beam-mass systems. The two-part beam-mass system described in this paper is essentially a structural member consisting of two different beam segments between which there is a rigid mass element that may have rotatory inertia. Numerical checks to show that the two methods generate identical dynamic stiffness matrices were performed for a wide range of frequency values. Once the dynamic stiffness matrix is obtained using any of the two methods, the Wittrick-Williams algorithm is applied to compute the natural frequencies of some frameworks consisting of two-part beam-mass systems. Numerical results are discussed and the paper concludes with some remarks.

**Key words:** dynamic stiffness method; beam-mass systems; free vibration; Wittrick-Williams algorithm.

## 1. Introduction

Beam-mass systems with varying degrees of complexities have been analysed by many investigators using different methods. Dowell (1979) appears to be one of the earlier investigators who studied some general properties of combined dynamical systems involving beams, springs, and lumped masses. He made some useful observations for different component systems connected at more than one points and provided solutions, which are particularly useful when establishing upper and lower bounds of natural frequencies of complex vibrating systems. Some years later, Nicholson and Bergman (1986) investigated the free vibration behaviour of combined dynamical systems by using the classical method of separation of variables. They used Green's function when solving the generalised differential equations which eventually yielded the characteristic equation for the natural frequencies of the system. However, one of the drawbacks of their method is that the convergence towards an accurate result was somehow slow. Interestingly, Ercoli and Laura (1987) carried out an analytical as well as experimental investigation on continuous beams having elastically mounted

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masses. From a theoretical standpoint, they obtained solution using different variational approaches. They corroborated their theoretical predictions by experimental results. Liu *et al.* (1988) on the other hand, used the Laplace transformation technique to formulate the frequency equations for beams carrying intermediate concentrated masses. Their investigation covered both uniform and non-uniform beams with one, two or three intermediate concentrated masses. Later, Wu and Lin (1990) employed a technique which combines both analytical and numerical methods to study the free vibration behaviour of uniform cantilever beams with point masses. Other contributors in this field include Larrondo *et al.* (1992), Gurgoze (1996), Wu and Zhou (1998, 1999). In most of these earlier works, the presence of a mass connected to a beam has often been assumed to be of negligible size and concentrated at a point. Such simple models may lead to large errors in the modal analysis if the mass has a sizeable dimension which is a significant proportion of that of the beam. A classic example is an engine mounted on a high aspect ratio aircraft wing of a commercial airliner. Of course, the wing may be idealised as an assembly of beams whereas the engine may be assumed to be a lumped mass possessing almost infinite stiffness compared to that of the wing. Clearly, the size of the engine may not be small enough to be regarded as a point mass when carrying out an accurate free or forced vibration analysis of the combined wing-engine system satisfactorily. It appears that this particular type of problems has been addressed only recently by Kopmaz and Telli (2002) and Banerjee and Sobey (2003). The theory developed by these investigators has only been applied to one-dimensional structures in a limited context. The solution was restricted to a single two-part beam-mass system with specific boundary conditions at the ends. Essentially, the works of Kopmaz and Telli (2002), and Banerjee and Sobey (2003) account for the dynamic behaviour of a two-part beam-mass system consisting of two different beam segments between which lies a rigidly connected mass/inertia element of finite length.

The purpose of this paper is to extend the above investigations substantially so that a two-part beam-mass system can be used in a framework. In order to achieve this, an exact dynamic stiffness matrix of a two-part beam-mass system is developed from the free vibration theory of a Bernoulli-Euler beam. The main advantage of the dynamic stiffness method is that it puts the analysis in a much more general context in which a two-part beam-mass system can be a structural element so as to form an integral part (or a component) of an overall final structure.

The dynamic stiffness matrix of a two-part beam-mass system is developed in this paper by employing two different approaches. The first approach is that of the transfer matrix method (Lee 2000, Syngellakis and Younes 1991, Tanaka *et al.* 1981) whereas the second one is a direct approach which relies on satisfying the kinematical conditions at the joints of the combined system. In the transfer matrix approach, the displacements and forces at one end of the two-part beam-mass system are progressively transferred to the next adjacent end using suitable transformation. For harmonic oscillation, the expressions for the displacements and forces are obtained from the exact solutions of the governing differential equations of the combined system. The dynamic stiffness matrix is finally developed by relating the forces and displacements at the two end-nodes of the freely vibrating combined system. In the direct approach, the two-part beam-mass system is idealised using two different coordinate systems. The Y-axes for both coordinate systems are vertical. However, the X-axis for the left-hand beam element is from left to right whereas the corresponding X-axis for the right-hand one is from right to left. By approaching the problem from both sides of the coordinate systems, and satisfying the kinematical conditions at the joints between the beams and the rigid mass, the dynamic stiffness matrix is derived. In both approaches, the displacement and force vectors at one end of the combined system is related to those of the other.

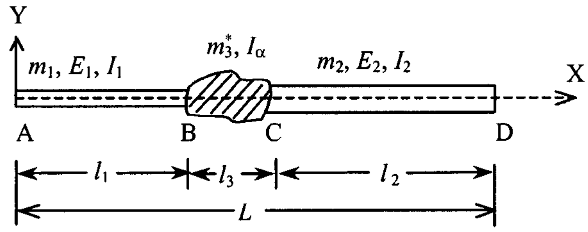


Fig. 1 Notation and coordinate system of a two-part beam-mass system for the transfer matrix method

The algorithm of Wittrick and Williams (Wittrick and Williams 1971, Williams and Wittrick 1983 Williams and Howson 1997) is finally applied to the resulting dynamic stiffness matrix to yield natural frequencies of frameworks consisting of two-part beam-mass systems.

## 2. Theory

A two-part beam-mass system is shown in a right-handed rectangular Cartesian coordinate system in Fig. 1. The central element, which connects two beam elements at its end, is a rigid body with mass $m_3^*$, length $l_3$, and mass moment of inertia $I_\alpha$ about its centroidal axis. The lengths of the two beam elements are $l_1$ and $l_2$ respectively, whereas the mass per unit length, extensional rigidity and bending rigidity of the two beam elements are $m_1$, $E_1A_1$, $E_1I_1$, and $m_2$, $E_2A_2$, $E_2I_2$, respectively. The total length of the whole assembly is $L$ as shown in the figure.

### *2.1 Transfer matrix approach*

The transfer matrix approach is used here to analyse the free vibratory motion of the combined system. The method essentially focuses on the derivation of a relationship between the forces and displacements at the left-hand end A, with those at the right-hand end D of the combined system (see Fig. 1).

The state vector for this problem in general form is defined as

$$\mathbf{S} = [u \;\; v \;\; \theta \;\; P \;\; S \;\; M]^T \tag{1}$$

where $u$ is the axial displacement, $v$ is the transverse bending displacement, $\theta$ is the anti-clockwise (tangential) bending rotation, $P$ is the axial force, $S$ is the shear force and $M$ is the bending moment at any cross-section of the beam-mass system. Note that the superscript $T$ denotes a transpose.

The transfer matrix method allows the state vector at the point B to be expressed in terms of that at A, (see Fig. 1) as follows

$$\mathbf{S}_\mathrm{B} = \mathbf{T}_1\mathbf{S}_\mathrm{A} \tag{2}$$

where $\mathbf{S}_\mathrm{A}$ and $\mathbf{S}_\mathrm{B}$ are the state vectors at points A and B respectively, and $\mathbf{T}_1$ is the corresponding transfer matrix relating the two.

Likewise, the state vector at the point D ($\mathbf{S}_\mathrm{D}$) can be determined in terms of that at C ($\mathbf{S}_\mathrm{C}$) by using the transfer matrix $\mathbf{T}_2$ as follows

$$\mathbf{S}_D = \mathbf{T}_2\mathbf{S}_C \tag{3}$$

For the central member which is a rigid body, the state vector $\mathbf{S}_C$ at C can similarly be expressed in terms of $\mathbf{S}_B$ at B by using the transfer matrix $\mathbf{T}_3$ to give

$$\mathbf{S}_C = \mathbf{T}_3\mathbf{S}_B \tag{4}$$

Finally, the force displacement relationship between the two ends of the freely vibrating combined system is obtained in the form of the following matrix relationship.

$$\mathbf{S}_D = \mathbf{T}\mathbf{S}_A = \mathbf{T}_2\mathbf{T}_3\mathbf{T}_1\mathbf{S}_A \tag{5}$$

where $\mathbf{T}_i(i = 1, 2, 3)$ are the three transfer matrices corresponding to each part of the two-part beam mass system.

The equations of motion in free longitudinal and flexural vibration for the two beam elements shown in Fig. 1 are,

$$E_iA_i\frac{\partial^2 u_i}{\partial x_i^2} = m_i\frac{\partial^2 u_i}{\partial t^2} \tag{6}$$

and

$$E_iI_i\frac{\partial^4 v_i}{\partial x_i^4} + m_i\frac{\partial^2 v_i}{\partial t^2} = 0 \tag{7}$$

where $i$ ($i$ = 1, 2) denotes the left-hand and right-hand beam members, respectively.

For harmonic oscillation with circular (angular) frequency $\omega$, the displacements $u_i$ and $v_i$ can be expressed as

$$\left.\begin{aligned} u_i(x_i, t) &= U_i(x_i)e^{i\omega t} \\ v_i(x_i, t) &= V_i(x_i)e^{i\omega t} \end{aligned}\right\} \tag{8}$$

where $U_i$ and $V_i$ are the amplitudes of longitudinal and flexural displacements in free vibration respectively.

Substituting the Eqs. (8) into Eqs. (6) and (7) gives

$$\frac{d^2U_i}{dx_i^2} = -\alpha_i^2U_i \tag{9}$$

$$\frac{d^4V_i}{dx_i^4} = \beta_i^4V_i \tag{10}$$

where

$$\alpha_i^2 = m_i\omega^2/E_iA_i \tag{11}$$

$$\beta_i^4 = m_i\omega^2/E_iI_i \tag{12}$$

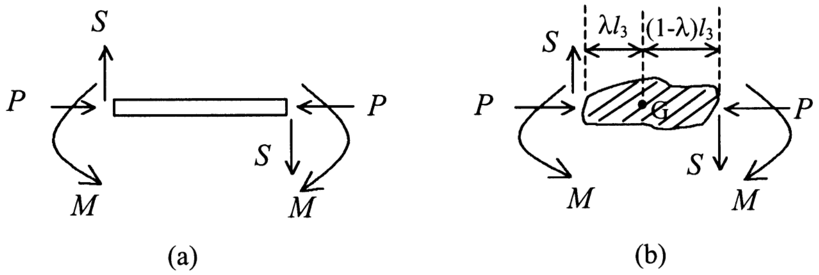


Fig. 2 Sign convention for positive axial force (*P*), shear force (*S*) and bending moment (*M*) (a) for a beam element, (b) for a rigid mass element

The solutions of differential Eqs. (9) and (10) are in the usual notation given by

$$U_i = f_i \cos\alpha_i x_i + g_i \sin\alpha_i x_i \tag{13}$$

$$V_i = a_i \cosh\beta_i x_i + b_i \sinh\beta_i x + c_i \cos\beta_i x_i + d_i \sin\beta_i x_i \tag{14}$$

where $f_i$, $g_i$, $a_i$, $b_i$, $c_i$ and $d_i$ ($i$ = 1, 2) are two sets of six arbitrary constants for the two beam elements respectively.

The bending rotation, axial force, shear force and bending moment of the beam element can be written as (see Fig. 2(a) for sign convention)

$$\theta_i = \frac{dV_i}{dx_i} = \beta_i(a_i \sinh\beta_i x_i + b_i \cosh\beta_i x_i - c_i \sin\beta_i x_i + d_i \cos\beta_i x_i) \tag{15}$$

$$P_i = -E_i A_i \frac{dU_i}{dx_i} = E_i A_i \alpha_i (f_i \sin\alpha_i x_i - g_i \cos\alpha_i x_i) \tag{16}$$

$$S_i = E_i I_i \frac{d^3 V_i}{dx_i^3} = E_i I_i \beta_i^3 (a_i \sinh\beta_i x_i + b_i \cosh\beta_i x_i + c_i \sin\beta_i x_i - d_i \sin\beta_i x_i) \tag{17}$$

$$M_i = -E_i I_i \frac{d^2 V_i}{dx_i^2} = -E_i I_i \beta_i^2 (a_i \cosh\beta_i x_i + b_i \sinh\beta_i x_i - c_i \cos\beta_i x_i - d_i \sin\beta_i x_i) \tag{18}$$

For each beam element, the right-hand state vector can be written in terms of the left-hand end state vector by substituting appropriate properties and boundary conditions for each element. For instance, the transfer matrix $\mathbf{T}_1$ for the beam element AB relating the state vector at A to that at B can be obtained as follows.

Substituting $x_1 = 0$ into Eqs. (13)-(18) gives the state vector at A as

$$\left.\begin{array}{ll} U_A = f_1 & P_A = -E_1 A_1 g_1 \alpha_1 \\ V_A = a_1 + c_1 & S_A = E_1 I_1 \beta_1^3 (b_1 - d_1) \\ \theta_A = \beta_1 (b_1 + d_1) & M_A = -E_1 I_1 \beta_1^2 (a_1 - c_1) \end{array}\right\} \tag{19}$$

At $x_1 = l_1$ for the end B, the state vector at B can be obtained from Eqs. (13)-(18) to give,

$$\left.\begin{aligned}
U_B &= f_1\cos\alpha_1 l_1 + g_1\sin\alpha_1 l_1 \\
V_B &= a_1\cosh\beta_1 l_1 + b_1\sinh\beta_1 l_1 + c_1\cos\beta_1 l_1 + d_1\sin\beta_1 l_1 \\
\theta_B &= \beta_1(a_1\sinh\beta_1 l_1 + b_1\cosh\beta_1 l_1 - c_1\sin\beta_1 l_1 + d_1\cos\beta_1 l_1) \\
P_B &= E_1A_1\alpha_1(f_1\sin\alpha_1 l_1 - g_1\cos\alpha_1 l_1) \\
S_B &= E_1I_1\beta_1^3(a_1\sinh\beta_1 l_1 + b_1\cosh\beta_1 l_1 + c_1\sin\beta_1 l_1 - d_1\cos\beta_1 l_1) \\
M_B &= -E_1I_1\beta_1^2(a_1\cosh\beta_1 l_1 + b_1\sinh\beta_1 l_1 - c_1\cos\beta_1 l_1 - d_1\sin\beta_1 l_1)
\end{aligned}\right\} \quad (20)$$

Now the six constants ($f_1$, $g_1$, $a_1$, $b_1$, $c_1$ and $d_1$) in Eqs. (19) and (20) can be eliminated to form the transfer matrix $\mathbf{T}_1$ relating the state vectors at B and A (see Fig. 1).

Following the same procedure $\mathbf{T}_2$ can be derived for the right-hand beam element as well.

Thus, $\mathbf{T}_i(i = 1, 2)$ in general can be expressed as

$$\mathbf{T}_i = \begin{bmatrix} T_{11} & 0 & 0 & T_{14} & 0 & 0 \\ 0 & T_{22} & T_{23} & 0 & T_{25} & T_{26} \\ 0 & T_{32} & T_{33} & 0 & T_{35} & T_{36} \\ T_{41} & 0 & 0 & T_{44} & 0 & 0 \\ 0 & T_{52} & T_{53} & 0 & T_{55} & T_{56} \\ 0 & T_{62} & T_{63} & 0 & T_{65} & T_{66} \end{bmatrix} \quad (21)$$

For both beam elements, the components of the matrix $\mathbf{T}_i(i = 1, 2)$ can be expressed by substituting appropriate beam parameters. The elements of $\mathbf{T}_i$ are as follows

$$\left.\begin{aligned}
T_{11} &= T_{44} = \cos\alpha_i l_i \\
T_{14} &= -\sin\alpha_i l_i/(E_iA_i\alpha_i) \\
T_{22} &= T_{33} = T_{55} = T_{66} = (\cosh\beta_i l_i + \cos\beta_i l_i)/2 \\
T_{23} &= -T_{65} = (\sinh\beta_i l_i + \sin\beta_i l_i)/(2\beta_i) \\
T_{25} &= (\sinh\beta_i l_i - \sin\beta_i l_i)/(2E_iI_i\beta_i^3) \\
T_{26} &= -T_{35} = -(\cosh\beta_i l_i - \cos\beta_i l_i)/(2E_iI_i\beta_i^2) \\
T_{32} &= -T_{56} = \beta_i(\sinh\beta_i l_i - \sin\beta_i l_i)/2 \\
T_{36} &= -(\sinh\beta_i l_i + \sin\beta_i l_i)/(2E_iI_i\beta_i) \\
T_{41} &= E_iA_i\alpha_i\sin\alpha_i l_i \\
T_{52} &= E_iI_i\beta_i^3(\sinh\beta_i l_i + \sin\beta_i l_i)/2 \\
T_{53} &= -T_{62} = E_iI_i\beta_i^2(\cosh\beta_i l_i - \cos\beta_i l_i)/2 \\
T_{63} &= -E_iI_i\beta_i(\sinh\beta_i l_i - \sin\beta_i l_i)/2
\end{aligned}\right\} \quad (22)$$

The rigid element, which may be considered to be a non-uniform rigid mass of length $l_3$ with its centre of gravity located at a distance $\lambda\, l_3$ from one end (see Fig. 2b).

The equations of motion follow from the equilibrium and compatibility conditions of the element as follows (see Figs. 1 and 2)

$$\left.\begin{aligned} U_C &= U_B \\ V_C &= V_B + l_3\theta_B \\ \theta_C &= \theta_B \\ P_B - P_C &= -m_3^*\omega^2 U_B \\ S_B - S_C &= -\omega^2[\lambda V_C + (1-\lambda)V_B]m_3^* \\ M_C - M_B + S_B\lambda l_3 + S_C(1-\lambda)l_3 &= \omega^2 I_\alpha \theta_B \end{aligned}\right\} \tag{23}$$

The state vector $\mathbf{S}_C$ at the end C can be written in terms of that at the end B by using the above conditions. In matrix notation, this transfer matrix $\mathbf{T}_3$ is

$$\mathbf{T}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & l_3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ m_3^*\omega^2 & 0 & 0 & 1 & 0 & 0 \\ 0 & m_3^*\omega^2 & \omega^2 m_3^*\lambda l_3 & 0 & 1 & 0 \\ 0 & \mu_1 & \mu_2 & 0 & -l_3 & 1 \end{bmatrix} \tag{24}$$

where

$$\left.\begin{aligned} \mu_1 &= -\omega^2 m_3^*(1-\lambda)l_3 \\ \mu_2 &= -\omega^2[m_3^*\lambda(1-\lambda)l_3^2 - I_\alpha] \end{aligned}\right\} \tag{25}$$

Using the matrices given by Eqs. (21) and (24), the final matrix $\mathbf{T}$ (see Eq. 5) of the combined system can be obtained as

$$\mathbf{T} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{26}$$

where each of the **A**, **B**, **C**, and **D** matrices is a 3 × 3 sub-matrix.

Now the relationship between forces and displacements can be rearranged with the help of Eq. (5) to give

$$\mathbf{F} = \mathbf{K}\boldsymbol{\delta} \tag{27}$$

where

$$\left.\begin{aligned} \boldsymbol{\delta} &= [U_A\;\; V_A\;\; \theta_A\;\; U_D\;\; V_D\;\; \theta_D]^T \\ \mathbf{F} &= [P_A\;\; S_A\;\; M_A\;\; P_D\;\; S_D\;\; M_D]^T \end{aligned}\right\} \tag{28}$$

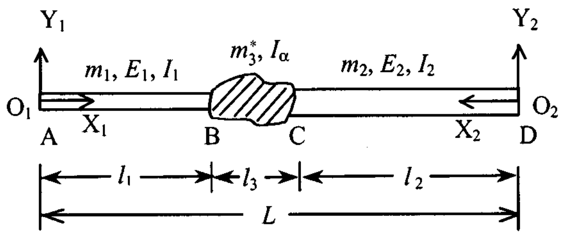


Fig. 3 Notation and coordinate system of a two-part beam-mass system for the direct method

**δ** and **F** above are the displacement and force vectors at the two ends A and D of the combined system (see Fig. 1). Note that for presentational purposes, the column vectors for nodal displacements and nodal forces are represented by their corresponding transpose. The required frequency dependent dynamic stiffness matrix for the combined system can be expressed after some matrix manipulation as

$$\mathbf{K} = \begin{bmatrix} -\mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1} \\ \mathbf{C}-\mathbf{D}\mathbf{B}^{-1}\mathbf{A} & \mathbf{D}\mathbf{B}^{-1} \end{bmatrix} \tag{29}$$

### *2.2 Direct approach*

Two coordinate systems, namely $O_1X_1Y_1$ and $O_2X_2Y_2$ shown in Fig. 3, are chosen for the left-hand and right-hand beam elements respectively. Axial and bending stiffnesses are uncoupled and they are obtained by separate consideration of axial and bending motion of the combined system.

The equations of motion in free longitudinal and flexural vibration for the two beam elements AB and DC are given by Eqs. (6) and (7). Assuming harmonic oscillation as in Eqs. (8), and introducing the non-dimensional length $\xi_i$ so that

$$\xi_i = x_i / l_i \tag{30}$$

Eqs. (9) and (10) can be re-written in non-dimensional form as shown below

$$\frac{d^2U_i}{d\xi_i^2} + \gamma_i^2 U_i = 0 \tag{31}$$

$$\frac{d^4V_i}{d\xi_i^4} - k_i^4 V_i = 0 \tag{32}$$

where

$$\gamma_i^2 = \alpha_i^2 l_i^2 \tag{33}$$

$$k_i^4 = \beta_i^4 l_i^4 \tag{34}$$

The solutions of differential Eqs. (31) and (32) are given by

$$U_i(\xi_i) = \hat{f}_i \cos\gamma_i\xi_i + \hat{g}_i \sin\gamma_i\xi_i \tag{35}$$

$$V_i(\xi_i) = \hat{a}_i \cosh k_i\xi_i + \hat{b}_i \sinh k_i\xi + \hat{c}_i \cos k_i\xi_i + \hat{d}_i \sin k_i\xi_i \tag{36}$$

where $\hat{f}_i, \hat{g}_i, \hat{a}_i, \hat{b}_i, \hat{c}_i, \hat{d}_i$ $(i = 1, 2)$ are two sets of six different arbitrary constants for the two beam elements AB and DC respectively.

The bending rotation, axial force, shear force and bending moment of the beam elements can be expressed as

$$\theta_i(\xi_i) = \frac{k_i}{l_i}(\hat{a}_i \sinh k_i\xi_i + \hat{b}_i \cosh k_i\xi_i - \hat{c}_i \sin k_i\xi_i + \hat{d}_i \cos k_i\xi_i) \tag{37}$$

$$P_i(\xi_i) = \frac{E_iA_i\gamma_i}{l_i}(\hat{f}_i \sin\gamma_i x_i - \hat{g}_i \cos\gamma_i x_i) \tag{38}$$

$$S_i(\xi_i) = \frac{E_iI_ik_i^3}{l_i^3}(\hat{a}_i \sinh k_i\xi_i + \hat{b}_i \cosh k_i\xi_i + \hat{c}_i \sin k_i\xi_i - \hat{d}_i \cos k_i\xi_i) \tag{39}$$

$$M_i(\xi_i) = -\frac{E_iI_ik_i^2}{l_i^2}(\hat{a}_i \cosh k_i\xi_i + \hat{b}_i \sinh k_i\xi_i - \hat{c}_i \cos k_i\xi_i - \hat{d}_i \sin k_i\xi_i) \tag{40}$$

At the intersections at $x_1 = l_1$ and $x_2 = l_2$ (i.e., $\xi_1 = 1$ and $\xi_2 = 1$) for points B and C, see Fig. 3, the following geometric and dynamic matching conditions must apply.

Continuity of slope:

$$\frac{1}{l_1}V_B'(1) = -\frac{1}{l_2}V_C'(1) \tag{41}$$

Compatibility of longitudinal and flexural displacements:

$$U_B(1) = -U_C(1) \tag{42}$$

$$V_B(1) + \frac{l_3}{l_1}V_B'(1) = V_C(1) \tag{43}$$

Equilibrium equations for axial and transverse motions:

$$P_C(1) = P_B(1) - m_3^*\omega^2 U_B(1) \tag{44}$$

$$\frac{E_1I_1}{l_1^3}V_B'''(1) + \frac{E_2I_2}{l_2^3}V_C'''(1) = -m_3^*\omega^2\left[V_B(1) + \frac{l_3}{2l_1}V_B'(1)\right] \tag{45}$$

Equilibrium equation of rotational motion:

$$-\frac{E_1I_1}{l_1^2}V_B''(1) + \frac{E_2I_2}{l_2^2}V_C''(1) + \frac{l_3}{2}\frac{E_2I_2}{l_2^2}V_C'''(1) - \frac{l_3}{2}\frac{E_1I_1}{l_1^3}V_B'''(1) = -\frac{I_\alpha\omega^2}{l_1}V_B'(1) \tag{46}$$

Now it is possible to relate the two sets of the constants $\hat{f}_i, \hat{g}_i, \hat{a}_i, \hat{b}_i, \hat{c}_i$ and $\hat{d}_i$. Hence, the derivation of dynamic stiffness matrix of the system essentially involves elimination of six constants instead of the twelve.

Applying the boundary conditions for the axial and bending displacements, bending rotations, axial forces, shear forces and bending moments, and noting that $\xi_i$ are zeros at end A and D for $i = 1$ and $i = 2$ respectively, the following equations can be obtained

$$\left.\begin{aligned} U_A &= \hat{f}_1 & P_A &= -\frac{E_1 A_1 \gamma_1}{l_1}\hat{g}_1 \\ V_A &= \hat{a}_1 + \hat{c}_1 & S_A &= \frac{E_1 I_1 k_1^3}{l_1^3}(\hat{b}_1 - \hat{d}_1) \\ \theta_A &= \frac{k_1}{l_1}(\hat{b}_1 + \hat{d}_1) & M_A &= -\frac{E_1 I_1 k_1^2}{l_1^2}(\hat{a}_1 - \hat{c}_1) \end{aligned}\right\} \tag{47}$$

and

$$\left.\begin{aligned} U_D &= \hat{f}_2 & P_D &= -\frac{E_2 A_2 \gamma_2}{l_2}\hat{g}_2 \\ V_D &= \hat{a}_2 + \hat{c}_2 & S_D &= \frac{E_2 I_2 k_2^3}{l_2^3}(\hat{b}_2 - \hat{d}_2) \\ \theta_D &= \frac{k_2}{l_2}(\hat{b}_2 + \hat{d}_2) & M_D &= -\frac{E_2 I_2 k_2^2}{l_2^2}(\hat{a}_2 - \hat{c}_2) \end{aligned}\right\} \tag{48}$$

Eqs. (47) and (48) can now be written in the following matrix forms,

$$\boldsymbol{\delta} = \mathbf{RC} \tag{49}$$

$$\mathbf{F} = \mathbf{QC} \tag{50}$$

where $\boldsymbol{\delta}$ and $\mathbf{F}$ have already been defined in Eqs. (28), $\mathbf{C}$ is the unknown constant vector given by

$$\mathbf{C} = [\hat{f}_1, \hat{g}_1, \hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1]^T \tag{51}$$

The matrices $\mathbf{R}$ and $\mathbf{Q}$ in Eqs. (49) and (50) are obtained with the help of boundary conditions in Eqs. (47)-(48) and the matching conditions in Eqs. (41)-(46). Thus the dynamic stiffness matrix of the two-part beam-mass system $\mathbf{K}$ can be derived by eliminating the constant vector $\mathbf{C}$ from Eqs. (49) and (50) and in this way relating the amplitudes of the forces $\mathbf{F}$ to those of the displacements $\boldsymbol{\delta}$ at the ends. In matrix notation, this is represented by Eq. (27) with

$$\mathbf{K} = \mathbf{QR}^{-1} \tag{52}$$

where $\mathbf{K}$ is the required frequency dependent $6 \times 6$ dynamic stiffness matrix of the two-part beam-mass system.

## 3. Solution procedures for the natural frequencies

The dynamic stiffness matrix described by Eqs. (29) or (52) can now be used to compute the natural frequencies and mode shapes of a two-part beam-mass system with various end conditions or a structure consisting of such systems. An accurate and reliable method of calculating the natural frequencies and mode shapes of a structure consisting of two-part beam-mass systems using the dynamic stiffness method, is to apply the algorithm of Wittrick and Williams (1971) which has featured in numerous papers (see for example, Williams and Wittrick 1983, Banerjee 1997). Before applying the algorithm the dynamic stiffness matrices of all individual elements in a structure are to be assembled to form the overall dynamic stiffness matrix $\mathbf{K}_f$ of the final (complete) structure, which may, of course, consist of a single element. The algorithm monitors the Sturm sequence condition of $\mathbf{K}_f$ in such a way that there is no possibility of missing a frequency (or mode) of the structure. This is, of course, not possible in the conventional finite element method. The algorithm (unlike its proof) is very simple to use. The procedure is briefly summarised as follows.

According to the Wittrick-Williams algorithm, $j$, the number of natural frequencies passed, as $\omega$ is increased from zero to $\omega^*$, is given by

$$j = j_0 + s\{\mathbf{K}_f\} \tag{53}$$

where $\mathbf{K}_f$, the overall dynamic stiffness matrix of the final structure whose elements all depend on $\omega$, is evaluated at $\omega = \omega^*$; $s\{\mathbf{K}_f\}$ is the number of negative elements on the leading diagonal of $\mathbf{K}_f^{\Delta}$, $\mathbf{K}_f^{\Delta}$ is the upper triangular matrix obtained by applying the usual form of Gauss elimination to $\mathbf{K}_f$, and $j_0$ is the number of natural frequencies of the structure still lying between $\omega = 0$ and $\omega = \omega^*$ when the displacement components to which $\mathbf{K}_f$ corresponds are all zeros. (Note that the structure can still have natural frequencies when all its nodes are clamped, because exact member equations allow each individual member to displace between nodes with an infinite number of degrees of freedom, and hence infinite number of natural frequencies between nodes.)

Thus

$$j_0 = \sum j_m \tag{54}$$

where $j_m$ is the number of natural frequencies between $\omega = 0$ and $\omega = \omega^*$ for a component member with its ends fully clamped, while the summation extends over all members of the structure. This simple feature of the algorithm (coupled with the fact that successive trial frequencies can be chosen by the user to bracket a natural frequency) can be used to converge on any required natural frequency to any desired (or specified) accuracy. The paper by Williams and Howson (1977) provides the step by step procedure for determining the natural frequencies of frameworks using the dynamic stiffness matrix method.

## 4. Results and discussions

The dynamic stiffness matrix of the two-part beam-mass system was first numerically checked to machine accuracy for a wide range of frequency values using the above two formulations, namely the transfer matrix approach and the direct approach, to ensure that the two methods give the same results.

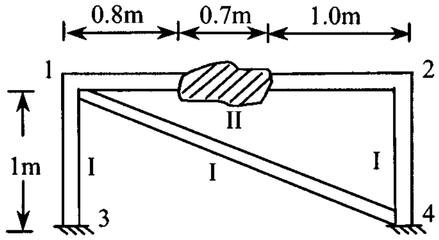


Fig. 4 A framework consisting of two-part beam-mass systems with nodes 1 to 4, and member types I and II

It should be recognised that a single two-part beam-mass system for a given boundary condition can be analysed for its free vibration characteristics by simply applying the boundary conditions of the whole system, and without resorting to the development of its dynamic stiffness matrix. The work of Kopmaz and Telli (2002) and that of Banerjee and Sobey (2003) are in fact examples of this relatively simple approach in which the authors have solved the governing differential equations of a two-part beam-mass system without developing the dynamic stiffness matrix. This approach is all right for simple problems, but is inadequate when studying the free vibration characteristics of frameworks consisting of two-part beam-mass systems. The present theory based on the dynamic stiffness method has no such limitation because it can handle a single two-part beam-mass system as well as a combination of them placed in any arbitrary orientations.

To demonstrate some general applications of the theory, two illustrative examples are chosen. The first example is a framework consisting of four structural elements of which three are uniform beam elements without any rigid mass attachment whereas the fourth one has a rigid mass forming a two-part beam-mass system. The geometrical details of this frame are shown in Fig. 4. The node numbering and element types are also shown in the figure. Note that elements with the same extensional rigidity *EA*, bending rigidity *EI*, and mass per unit length *m* constitute a single member type. Thus the elements connecting nodes 1-3, 1-4, and 2-4 sharing the same above properties, have been classified as member type I, see Fig. 4. Of course, the member connecting nodes 1 and 2 is a two-part beam-mass system for this example, which is considered to be member type II as shown. The data used for these two member types are as follows.

For member type I:

$EI = 4.0 \times 10^5$ Nm$^2$, $EA = 8.0 \times 10^7$ N, $m = 50$ kg/m.

For member type II:

$E_1 I_1 = E_2 I_2 = 4.0 \times 10^5$ Nm$^2$, $E_1 A_1 = E_2 A_2 = 8.0 \times 10^7$ N, $m_1 = m_2 = 50$ kg/m
$I_\alpha = 1.562$ kgm$^2$ $m_3^* = 75$ kg.

The first three natural frequencies for the frame were computed using the present theory and are shown in column 2 of Table 1. In order to examine the effect of the length of the rigid mass, a second set of results was obtained by assuming the rigid mass to be concentrated at a point on its centre of gravity. The results are shown in column 3 of the table. A comparison of results shown in columns 2 and 3 indicates that the size of the mass did not make much difference to the

Table 1 The first three natural frequencies of the framework shown in Fig. 4, with and without a two-part beam-mass system (TPBMS)

| Natural frequency number | Natural frequencies (Hz) | | |
|---|---|---|---|
| | Current method | | With no rigid mass |
| | With TPBMS | With point mass | |
| 1 | 25.04 | 25.16 | 35.39 |
| 2 | 36.21 | 40.53 | 42.77 |
| 3 | 43.74 | 48.60 | 59.64 |

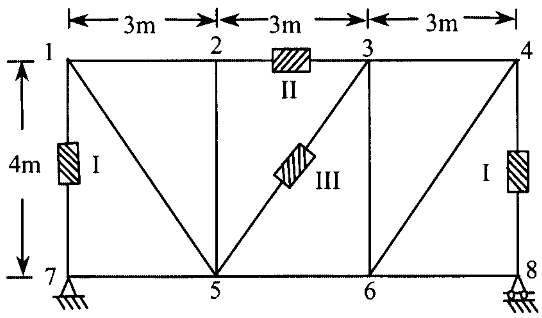


Fig. 5 A framework consisting of two-part beam-mass systems with nodes 1 to 8 and member types I, II and III

fundamental natural frequency, but it has altered the second and third natural frequencies by around 12%. The final set of results for this example was obtained by removing the rigid mass altogether. The natural frequencies without the rigid mass are shown in the final column of the table. The results indicate significant influence of the rigid mass on the three natural frequencies. Clearly, the presence of the rigid mass reduces the natural frequencies as expected.

The second example is also a framework, but is very different from the first one (see Fig. 5). This particular problem was earlier solved by Williams and Howson (1977), but without any two-part beam-mass system attachments. For the purposes of demonstration of the present theory, four of the thirteen members of the original frame (earlier used by Williams and Howson 1977) were replaced by two-part beam-mass systems as shown in the figure. The structural parameters used in the analysis are as follows.

For member type I:

$E_1I_1 = E_2I_2 = 4.0\times10^6\ \mathrm{Nm}^2$, $E_1A_1 = E_2A_2 = 8.0\times10^8\ \mathrm{N}$,
$m_1 = m_2 = 30\ \mathrm{kg/m}$, $l_1 = l_2 = 3.0\ \mathrm{m}$,
$m_3^* = 22.5\ \mathrm{kg}$, $I_\alpha = 1.875\ \mathrm{kgm}^2$, $l_3 = 1.0\ \mathrm{m}$.

For member type II:

$E_1I_1 = E_2I_2 = 4.0\times10^6\ \mathrm{Nm}^2$, $E_1A_1 = E_2A_2 = 8.0\times10^8\ \mathrm{N}$,
$m_1 = m_2 = 30\ \mathrm{kg/m}$, $l_1 = l_2 = 1.15\ \mathrm{m}$,
$m_3^* = 15.75\ \mathrm{kg}$, $I_\alpha = 0.643\ \mathrm{kgm}^2$, $l_3 = 0.70\ \mathrm{m}$.

Table 2 The first five natural frequencies of the framework shown in Fig. 5, with and without two-part beam-mass systems (TPBMS)

| Natural frequency number | Natural frequencies (Hz) | | Difference (%) |
|---|---|---|---|
| | with TPBMS | without TPBMS [Williams and Howson 1977] | |
| 1 | 31.363 | 35.762 | 14 |
| 2 | 37.647 | 39.104 | 4 |
| 3 | 40.290 | 42.555 | 6 |
| 4 | 48.869 | 51.394 | 5 |
| 5 | 49.004 | 53.935 | 10 |

For member type III:

$$E_1I_1 = E_2I_2 = 4.0\times10^6\ \mathrm{Nm}^2, \qquad E_1A_1 = E_2A_2 = 8.0\times10^8\ \mathrm{N},$$
$$m_1 = m_2 = 30\ \mathrm{kg/m}, \qquad l_1 = l_2 = 1.9\ \mathrm{m},$$
$$m_3^* = 27.0\ \mathrm{kg}, \qquad I_\alpha = 3.240\ \mathrm{kgm}^2, \qquad l_3 = 1.2\ \mathrm{m}.$$

The properties used for the rest of the members are same as those reported by Williams and Howson (1977). Results are obtained for the first five natural frequencies of the frame using the present theory and are shown in Table 2 alongside the results of Williams and Howson (1977). The percentage difference shown indicates that the effect of the two-part beam-mass system can make significant differences to some of the natural frequencies, particularly for the first and fifth frequencies.

## 5. Conclusions

By using two different approaches, the dynamic stiffness matrix of a two-part beam-mass system has been developed and applied to frameworks. It has been shown that as a result of using the present theory, the finite size of a rigid mass possessing rotatory inertia can be accounted for, in the prediction of natural frequencies of frameworks accurately. The theory provides considerable scopes for parametric studies to enable vibration attenuation of complex vibrating structures to be made, by using two-part beam-mass systems and thus, placing the natural frequencies within appropriate and desirable bands. Numerical results for natural frequencies are given for two example frameworks and their significance has been discussed. The theory presented is expected to pave the way for further research in the development of dynamic stiffness formulation of complex structural systems combining both continuous and discrete elements.

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## Notation

$E_1A_1$, $E_2A_2$ : extensional rigidity of the two beam elements of the TPBMS
$E_1I_1$, $E_2I_2$ : bending rigidity of the two beam elements of the TPBMS
$I_\alpha$ : mass moment of inertia of the rigid mass about the centroidal axis
$l_1$, $l_2$ : lengths of the two beam elements of the TPBMS
$l_3$ : length of the rigid element
$L$ : the total length of the TPBMS

| | |
|---|---|
| $m_1$, $m_2$ | : mass per unit length of the two beam elements of the TPBMS |
| $m_3^*$ | : mass of the rigid element |
| $M$ | : bending moment |
| $P$ | : axial force |
| $S$ | : shear force |
| $\mathbf{S}_A$, $\mathbf{S}_B$ | : state vectors at points A and B respectively |
| $\mathbf{S}_C$, $\mathbf{S}_D$ | : state vectors at points C and D respectively |
| TPBMS | : two-part beam-mass system |
| $\mathbf{T}$ | : corresponding transfer matrix |
| $u$ | : axial displacement |
| $v$ | : transverse (bending) displacement |
| $\theta$ | : anti-clockwise (tangential) bending rotation |