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Stability of a cylindrical shell with an oblique end

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Abstract. The linearized buckling problem is considered for an isotropic clamped-clamped cylindrical shell with an oblique end. A theoretical solution based on the Budiansky shell theory is developed, and numerical results are determined using the differential quadrature method. In formulating the solutions, the surface of the shell is developed onto a plane, and the resulting irregular domain is then mapped, using blending functions, onto a square parent domain. The analysis is carried out in the parent domain. Convergence, validation, and parametric studies are conducted for a uniform external pressure loading. Results determined are compared with finite element results. The paper ends with an appropriate set of conclusions.

Key words: stability; cylindrical shell; differential quadrature method; finite element method.

1. Introduction

Cylindrical shells with oblique ends (Fig. 1) are frequently encountered in structural mechanics. Industrial applications include mitred pipe bends, 'hillside' nozzles in pressure vessels, and diagonal tubular members in offshore rigs. Formulating analytical or semi-analytical solutions for such shells is a major challenge. While solutions have been presented for the elastostatics problem (Gill 1970, Sobieszczanski 1970), and for the free vibration problem (Hu and Redekop 2003), a search of the literature has not uncovered a solution for the problem of buckling.

The finite element method (FEM) is preferred for most engineering analyses, but other methods can play useful roles, particularly in the carrying out of validation and parametric studies. The recently developed differential quadrature method (DQM) is available for such a role. Most DQM applications to date have been for regular geometries (ex. Bert and Malik 1996, Ng and Lam 1999), but a few applications have dealt with irregular geometries. Attempts have been made in some studies (ex. Wang *et al.* 1998, Chen 1999, 2000) to develop differential quadrature elements that can handle arbitrary geometries. Blending or mapping functions have been used in other studies (ex. Malik and Bert 1996, Shu *et al.* 2000) to deal with specific irregular geometries. In the latter category, applications so far have only been made to flat plate geometries.

In the current study, the DQM is applied to the stability problem of a cylindrical shell with an oblique end. The curved shell surface is developed onto a plane, and blending functions are then used to map the geometrically irregular domain onto a square parent domain. The blending

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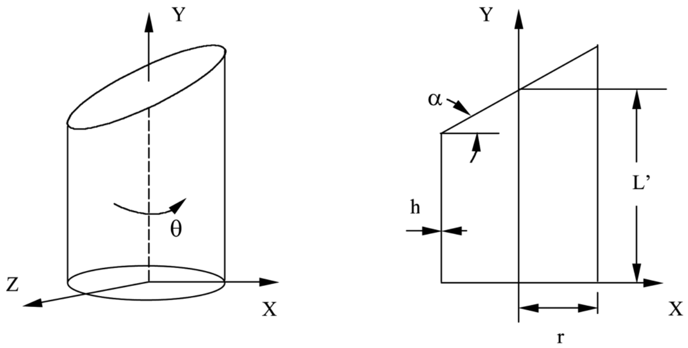

Fig. 1 Circular cylindrical shell with an oblique end

functions resemble those used earlier by Malik and Bert (1996) for a plate vibration problem. Special attention is paid to the boundary conditions on the elliptical oblique end of the shell. Convergence, validation, and parametric studies are conducted. The results are compared with results found using the finite element method (FEM), and conclusions are drawn.

2. Geometry and boundary conditions

The analyis is for a cylindrical shell having a radius r, mean height L', and thickness h (Fig. 1). The position of a typical point P on the shell mid-surface is given by the physical coordinates Y, θ. Displacement components u, v, w (respectively in the axial, circumferential, and normal directions), and stress resultants (Fig. 2) are defined in these physical coordinates. The base of the shell lies in a plane perpendicular to the shell axis, while the top lies in a plane that is oblique at an angle of α with the base of the shell. At both, the base and top of the shell, clamped support conditions are assumed. A uniform external normal pressure p acts on the surface of the shell.

The shell mid-surface, which in the physical coordinate system is irregular, and continuous in the circumferential direction, is first considered developed onto a plane (Fig. 3). In the development

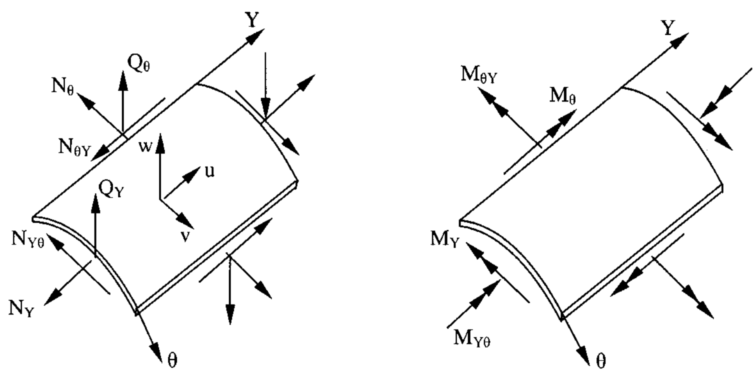

Fig. 2 Displacements and stress resultants

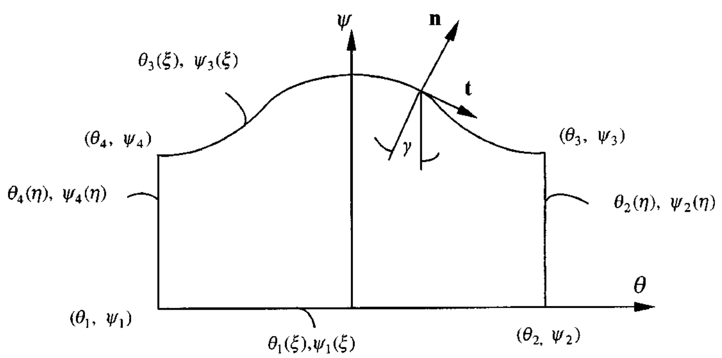

Fig. 3 Cylindrical surface mapped onto a plane

process two artificial boundary lines are created at the former $\theta = \pm 180^{\circ}$ line, disrupting the circumferential continuity. Conditions must be enforced on these two lines, comparable to the continuity conditions existing over the $\theta = \pm 180^{\circ}$ line in the original shell. Planar coordinates X, Y which have dimensions of length are used to describe positions on the developed surface. Non-dimensional coordinates for this surface are defined as $\theta = X/r$, $\psi = Y/r$. The governing domain equations for the shell are written in these non-dimensional coordinates.

A square parent domain is defined in the natural coordinates ξ, η with $1 \geq \xi \geq -1$, $1 \geq \eta \geq -1$. Blending functions (Malik and Bert 1996) are then used to develop mapping relations between the natural coordinates ξ, η and the developed coordinates θ, ψ. The blending functions are given by

$$s = \frac{1}{2}[(1-\eta)\,\bar{s}_1(\xi) + (1+\xi)\,\bar{s}_2(\eta) + (1+\eta)\,\bar{s}_3(\xi) + (1-\xi)\,\bar{s}_4(\eta)]$$
$$-\frac{1}{4}[(1-\xi)(1-\eta)s_1 + (1+\xi)(1-\eta)s_2 + (1+\xi)(1+\eta)s_3 + (1-\xi)(1+\eta)s_4] \qquad (1)$$

where $s = \theta, \psi$. The $\bar{\theta}_i(\xi)$, $\bar{\theta}_i(\eta)$, $\bar{\psi}_i(\xi)$, $\bar{\psi}_i(\eta)$ expressions are the parametric equations for the edges of the developed surface, and the θ_i, ψ_i are the non-dimensional Cartesian coordinates of the corner points of the developed surface. Using (1), the relations between the two sets of coordinates for the present case are obtained as

$$\theta = \pi\xi; \quad \psi = \zeta(1+\eta) \qquad (2)$$

where $\zeta = a + b\cos\phi$, $a = L'/2r$, $b = 0.5\tan\alpha$. The product $\pi\xi$ is represented for clarity by ϕ, and the variables ϕ, η are used subsequently to describe the mapped domain.

Using the chain rule of calculus, the transformation of derivatives from the θ, ψ system to the ϕ, η system can be determined as

$$\frac{\partial}{\partial\theta} = \frac{\partial}{\partial\phi} + \frac{b(1+\eta)\sin\phi}{\zeta}\frac{\partial}{\partial\eta}; \quad \frac{\partial}{\partial\psi} = \frac{1}{\zeta}\frac{\partial}{\partial\eta} \qquad (3)$$

These relations correspond to those given by Gill (1970) and used for the static analysis of mitred bends. Relations for higher order and mixed derivatives are readily developed from these basic relations.

For clamped supports at the base ($Y = 0$), the boundary equations are given by

$$u = 0; \;\; v = 0; \;\; w = 0; \;\; \frac{1}{r}\frac{\partial w}{\partial \psi} = 0 \tag{4}$$

where u, v, and w again represent the physical displacement components. For the clamped support conditions at the oblique top of the shell, the boundary relations are given (Shu 2000) as

$$u = 0; \;\; v = 0; \;\; w = 0; \;\; \frac{1}{r}\left(\frac{\partial w}{\partial \theta}\sin\gamma + \frac{\partial w}{\partial \psi}\cos\gamma\right) = 0 \tag{5}$$

where γ is the angle between the normal **n** to the shell boundary and the axial coordinate line (Fig. 3).

3. Finite element method

The commercial FEM program ADINA (2002) was used to provide an alternate solution. Flat four-noded twenty-four degree-of-freedom shell elements are available in this program for the solution of shell stability problems. Analyses were carried out in the physical space, with no account made of any symmetry in the geometry. Meshes were selected generally to yield elements that were roughly square in plan form. In the enforcement of clamped boundary conditions, restraints on the boundary nodes were made for the three translational and three rotational degrees of freedom. The FEM results were checked for convergence, and FEM values cited in the following represent such converged values.

4. Budiansky shell theory

To determine the buckling loads for the shell, the Budiansky shell theory is employed (Budiansky 1968). This theory is an extension of the Sanders linear shell bending theory which is considered one of the most accurate of the first order shell theories. The governing equations are given by

$$K[L]\{U\} + \lambda[\hat{L}]\{U\} = \{Q\} \tag{6}$$

where the symmetric arrays $[L]$ and $[\hat{L}]$ are given by

$$\begin{aligned} L_{11} &= \frac{\partial^2}{\partial\psi^2} + k_1\frac{\partial^2}{\partial\theta^2}; \;\; L_{12} = k_2\frac{\partial^2}{\partial\psi\partial\theta}; \;\; L_{13} = \nu\frac{\partial}{\partial\psi} + k_3\frac{\partial^3}{\partial\psi\partial\theta^2} \\ L_{22} &= k_4\frac{\partial^2}{\partial\psi^2} + k_5\frac{\partial^2}{\partial\theta^2}; \;\; L_{23} = \frac{\partial}{\partial\theta} - k_6\frac{\partial^3}{\partial\psi^2\partial\theta} - k\frac{\partial^3}{\partial\theta^3} \\ L_{33} &= 1 + k\left(\frac{\partial^4}{\partial\psi^4} + 2\frac{\partial^4}{\partial\psi^2\partial\theta^2} + \frac{\partial^4}{\partial\theta^4}\right) \end{aligned} \tag{7}$$

and

$$\hat{L}_{11} = n_\psi \frac{\partial^2}{\partial\psi^2} + n_\theta \frac{\partial^2}{\partial\theta^2} + 2n_{\psi\theta}\frac{\partial^2}{\partial\psi\partial\theta}$$

$$\hat{L}_{12} = 0;\ \hat{L}_{13} = -rp\frac{\partial}{\partial\psi}$$

$$\hat{L}_{22} = n_\psi \frac{\partial^2}{\partial\psi^2} + n_\theta\left(\frac{\partial^2}{\partial\theta^2} - 1\right) + 2n_{\psi\theta}\frac{\partial^2}{\partial\psi\partial\theta} + rp \tag{8}$$

$$\hat{L}_{23} = 2n_\theta\frac{\partial}{\partial\theta} + 2n_{\psi\theta}\frac{\partial}{\partial\psi} - rp\frac{\partial}{\partial\theta}$$

$$\hat{L}_{33} = -n_\psi \frac{\partial^2}{\partial\psi^2} + n_\theta\left(-\frac{\partial^2}{\partial\theta^2} + 1\right) - 2n_{\psi\theta}\frac{\partial^2}{\partial\psi\partial\theta} - rp$$

The factor K is given by $K = Eh/(1 - \nu^2)$, where ν and E are Poisson's ratio and Young's modulus, respectively. The constants k_i are given by

$$k_1 = \frac{1-\nu}{2}\left(1 + \frac{k}{4}\right);\ k_2 = \frac{1+\nu}{2} - \frac{1-\nu}{2}\frac{3k}{4};\ k_3 = \frac{1-\nu}{2}k$$

$$k_4 = \frac{1-\nu}{2}\left(1 + \frac{9k}{4}\right);\ k_5 = 1 + k;\ k_6 = \frac{3-\nu}{2} \tag{9}$$

where the geometric factor k is defined as $k = \frac{1}{12}\left(\frac{h}{r}\right)^2$. The buckling parameter λ gives directly the critical buckling pressure ($\lambda = p_{cr}$). The vector of displacements $\{U\}$ is represented by $\{u\ v\ w\}^t$, while the vector of loads $\{Q\}$ is given by $\{0\ 0\ r^2\ p\}^t$, where p is the pressure. The quantities n_ψ, n_θ, and $n_{\psi\theta}$ represent, respectively, the axial, circumferential, and in-plane shear membrane stress resultants in the shell. The buckling parameter λ together with the matrix $[\hat{L}]$ represent the Budiansky additions to the basic Sanders' equations, which are incorporated in the matrix $[L]$.

For the determination of the prebuckling stress states, relations between the stress resultants and displacement components are required. In the Budiansky theory, these relations are given by

$$n_\psi = \frac{K}{r}\left[\frac{\partial u}{\partial\psi} + \nu\left(\frac{\partial v}{\partial\theta} + w\right)\right]$$

$$n_\theta = \frac{K}{r}\left[\frac{\partial v}{\partial\theta} + w + \nu\frac{\partial u}{\partial\psi}\right] \tag{10}$$

$$n_{\psi\theta} = K\frac{1-\nu}{2r}\left[\frac{\partial v}{\partial\psi} + \frac{\partial u}{\partial\theta}\right]$$

For the conditions of clamped supports at the base ($Y = 0$) and at the top of the shell, the boundary Eqs. (4), (5) become (Shu 2000)

$$u = 0; \; v = 0; \; w = 0; \; \frac{\partial w}{\partial \eta} = 0 \tag{11}$$

Substitution of the coordinate and derivative transfer relations (2-3) into the Eqs. (6)-(8) and (11), leads to domain and boundary equations, in the parent coordinate system, which govern the problem.

5. Differential quadrature method

Following the DQM approach (Shu 2000), a grid of sampling points is first defined in the parent domain. The derivatives which appear in the domain and boundary equations are replaced by linear series involving products of the displacements at the sampling points of the grid with known weighting coeffcients. Application of this procedure leads ultimately to a set of linear equations in terms of the displacements at the sampling points and the parameter λ. The solution of these equations for zero loading yields the eigenvalue and, thus, the buckling load.

For a function $f(x)$ of a single variable, the series used to replace the r-th derivative of the function at the sampling point x_i is taken as

$$\left.\frac{d^r f(x)}{dx^r}\right|_{x_i} = \sum_{h=1}^{M} A_{ih}^{(r)} f(x_h) \tag{12}$$

where the $A_{ih}^{(r)}$ are the weighting coefficients of the r-th order derivative in the x direction for the i-th sampling point, $f(x_h)$ is the value of $f(x)$ at sampling point x_h, and M is the number of sampling points in the x direction. For a function of two variables $g(x, y)$, the series for the $(r + s)$-th partial derivative at the sampling point x_i, y_j is taken as

$$\left.\frac{\partial^{(r+s)} g(x, y)}{\partial x^r \partial y^s}\right|_{x_i, y_j} = \sum_{h=1}^{M} A_{ih}^{(r)} \sum_{k=1}^{N} B_{jk}^{(s)} g(x_h, y_k) \tag{13}$$

where $B_{jk}^{(s)}$ and N describe the series for the y direction, and $g(x_h, y_k)$ is the value of $g(x, y)$ at sampling point x_h, y_k.

The weighting coefficient are determined a-priori with the help of an assumed grid and a set of trial functions. In the present study for the axial (ψ) direction, the well known Chebyshev-Gauss-Lobatto spacing of sampling points with δ points is used, and a series of polynomial trial functions is selected. For such a scheme, explicit formulas are available (Bert and Malik 1996) for the weighting coeffcients $A_{ih}^{(r)}$.

For the circumferential (θ) direction, equally spaced sampling points are used, and a series of trigonometric trial functions is selected. Continuity over the artificial boundary lines $\theta = \pm 180^{\circ}$, created in the mapping, is then automatically satisfied, without explicit boundary equations. Explicit formulas for the weighting coeffcients $B_{jk}^{(s)}$ are available for such series (Mirfakhraei and Redekop 1998, Shu 2001). In a variation of the current approach a power series was employed in the circumferential direction. Boundary conditions were then satisfied at $\theta = \pm 180^{\circ}$, corresponding to the continuity conditions. This modified approach led to a notably slower convergence in the solution, and was not pursued further for the current problem.

At each sampling point of the DQM grid, either the DQM analogues of the boundary or domain equations are represented. For shells, there are four boundary conditions, while there are only three governing equations. One of the boundary conditions is then enforced at an adjacent domain point instead of a domain equation. Such a point, labelled a 'δ point', is taken a short distance ($\delta \cong 10^{-5}$) from the boundary point (Bert and Malik 1996). The δ approach follows one of the older DQM techniques. It proved entirely suitable for the boundary conditions considered in the current study.

The quadrature rules (12-13) for the derivatives in the domain equations are inserted into the governing domain equations, and the evaluation is carried out at a typical domain sampling point. The DQM analogues for the domain equations applicable to the buckling problem are

$$
\begin{gathered}
\mu_{11a}\sum A_{ih}^{(2)}U_{hj} + \mu_{11b}\sum A_{ih}^{(1)}U_{hj} + \mu_{11c}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}U_{hk} + \mu_{11d}\sum B_{jk}^{(2)}U_{ik} \\
+ \mu_{12a}\sum A_{ih}^{(2)}V_{hj} + \mu_{12b}\sum A_{ih}^{(1)}V_{hj} + \mu_{12c}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}V_{hk} + \mu_{13a}\sum A_{ih}^{(3)}W_{hj} \\
+ \mu_{13b}\sum A_{ih}^{(2)}W_{hj} + \mu_{13c}\sum A_{ih}^{(1)}W_{hj} + \mu_{13d}\sum A_{ih}^{(2)}\sum B_{jk}^{(1)}W_{hk} \\
+ \mu_{13e}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}W_{hk} + \mu_{13f}\sum A_{ih}^{(1)}\sum B_{jk}^{(2)}W_{hk} + \lambda[c_{11a}\sum A_{ih}^{(2)}U_{hj} \\
+ c_{11b}\sum A_{ih}^{(1)}U_{hj} + c_{11c}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}U_{hk} + c_{11d}\sum B_{jk}^{(2)}U_{ik} \\
+ \mu_{13a}\sum A_{ih}^{(1)}W_{hj}] = 0 \\[1ex]
\mu_{21a}\sum A_{ih}^{(2)}U_{hj} + \mu_{21b}\sum A_{ih}^{(1)}U_{hj} + \mu_{21c}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}U_{hk} + \mu_{22a}\sum A_{ih}^{(2)}V_{hj} \\
+ \mu_{22b}\sum A_{ih}^{(1)}V_{hj} + \mu_{22c}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}V_{hk} + \mu_{22d}\sum B_{jk}^{(1)}V_{ik} + \mu_{23a}\sum A_{ih}^{(3)}W_{hj} \\
+ \mu_{23b}\sum A_{ih}^{(2)}W_{hj} + \mu_{23c}\sum A_{ih}^{(1)}W_{hj} + \mu_{23d}\sum A_{ih}^{(2)}\sum B_{jk}^{(1)}W_{hk} \\
+ \mu_{23e}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}W_{hk} + \mu_{23f}\sum A_{ih}^{(1)}\sum B_{jk}^{(2)}W_{hk} + \mu_{23g}\sum B_{jk}^{(3)}W_{ik} \\
+ \mu_{23h}\sum B_{jk}^{(1)}W_{ik} + \lambda[c_{22a}\sum A_{ih}^{(2)}V_{hj} + c_{22b}\sum A_{ih}^{(1)}V_{hj} + c_{22c}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}V_{hk} \\
+ c_{22d}\sum B_{jk}^{(2)}V_{ik} + c_{23a}\sum A_{ih}^{(1)}W_{hj} + c_{23b}\sum B_{jk}^{(1)}W_{hk} + D_2] = 0 \\[1ex]
\mu_{31a}\sum A_{ih}^{(3)}U_{hj} + \mu_{31b}\sum A_{ih}^{(2)}U_{hj} + \mu_{31c}\sum A_{ih}^{(1)}U_{hj} + \mu_{31d}\sum A_{ih}^{(2)}\sum B_{jk}^{(1)}U_{hk} \\
+ \mu_{31e}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}U_{hk} + \mu_{31f}\sum A_{ih}^{(1)}\sum B_{jk}^{(2)}U_{hk} + \mu_{32a}\sum A_{ih}^{(3)}V_{hj} \\
+ \mu_{32b}\sum A_{ih}^{(2)}V_{hj} + \mu_{32c}\sum A_{ih}^{(1)}V_{hj} + \mu_{32d}\sum A_{ih}^{(2)}\sum B_{jk}^{(1)}V_{hk} + \mu_{32e}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}V_{hk} \\
+ \mu_{32f}\sum A_{ih}^{(1)}\sum B_{jk}^{(2)}V_{hk} + \mu_{32g}\sum B_{jk}^{(3)}V_{ik} + \mu_{32h}\sum B_{jk}^{(1)}V_{ik} \\
+ \mu_{33a}\sum A_{ih}^{(4)}W_{hj} + \mu_{33b}\sum A_{ih}^{(3)}W_{hj} + \mu_{33c}\sum A_{ih}^{(2)}W_{hj} + \mu_{33d}\sum A_{ih}^{(1)}W_{hj}
\end{gathered}
\tag{14}
$$

$$+ \mu_{33e}\sum A_{ih}^{(3)}\sum B_{jk}^{(1)}W_{hk} + \mu_{33f}\sum A_{ih}^{(2)}\sum B_{jk}^{(1)}W_{hk} + \mu_{33g}\sum A_{ih}^{(2)}\sum B_{jk}^{(2)}W_{hk}$$

$$+ \mu_{33h}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}W_{hk} + \mu_{33i}\sum A_{ih}^{(1)}\sum B_{jk}^{(2)}W_{hk} + \mu_{33j}\sum A_{ih}^{(1)}\sum B_{jk}^{(3)}W_{hk}$$

$$+ \mu_{33k}\sum B_{jk}^{(4)}W_{ik} + W_{ij} + \lambda[c_{31a}\sum A_{ih}^{(1)}U_{hj} + c_{32a}\sum A_{ih}^{(1)}V_{hj} + c_{32b}\sum B_{jk}^{(1)}V_{ik}$$

$$+ c_{33a}\sum A_{ih}^{(2)}W_{hj} + c_{33b}\sum A_{ih}^{(1)}W_{hj} + c_{33c}\sum A_{ih}^{(1)}\sum B_{jk}^{(1)}W_{hk} + c_{33d}\sum B_{jk}^{(2)}W_{ik}$$

$$+ D_3] = r^2 p$$

where the μ_{lmn} signify known functions of the material and geometric properties and the geometric coordinates. The quantities c_{lmn}, D_i are known functions of the material and geometric properties, the geometric coordinates, and the prestress loading. The three sets of functions are given in full by Hu (2003). The $A_{ij}^{(r)}$ and $B_{ij}^{(r)}$, respectively, are known weighting coeffcients applicable to the ψ and θ directions. The U_{ij}, V_{ij}, W_{ij} are the unknown displacements at the sampling points, and p is again the normal pressure. Similar DQM analogue equations may be written for the boundary conditions.

6. Stability analysis

A two-step procedure is used in the stability analysis. In the first step, the prebuckling stress analysis, a static analysis is conducted to find the resultants n_ψ, n_θ, and $n_{\psi\theta}$ for a unit uniform external pressure. In the use of Eq. (13), the parameter λ is set to zero, and the displacements are found for the specified pressure $p = -1$ Pa. Resultants are then found from the displacements using Eq. (10). In the second step, the buckling analysis, the Eq. (13) with the load term $\{Q\}$ zero, are solved for λ and $\{U\}$.

As results for buckling of cylindrical shells with oblique ends are not available in the literature, comparison was made with results for shells with perpendicular ends. Such results are available from the study of Vodenticharova and Ansourian (1996), who used series solutions to obtain results from the Fluegge theory. Comparisons for the critical buckling pressures for ten of the cases studied by these authors are given in Table 1. Cases 6-10 represent very thin shells and are included mainly

Table 1 Convergence and validation for buckling problem ($p_{cr} \times 10^2$ Pa)

Case	L′/r	r/h	FEM	DQMA	DQM1	DQM2	DQM3	V – A
1	0.5	300	3509.	10177.	3608.	3608.	3608.	3617.9 (17)
2	1.0	300	1722.	2370.	1736.	1736.	1736.	1725.8 (13)
3	2.0	300	882.9	867.5	884.6	884.6	884.6	876.1 (10)
4	3.0	300	588.1	570.9	587.1	587.1	587.1	578.51 (8)
5	5.0	300	360.5	340.0	357.8	357.8	357.8	348.3 (6)
6	0.5	3000	11.32	799.9	51.00	12.14	11.03	10.946 (33)
7	1.0	3000	5.646	148.2	7.943	5.518	5.518	5.520 (24)
8	2.0	3000	2.846	16.75	2.791	2.791	2.791	2.782 (17)
9	3.0	3000	1.908	4.323	1.868	1.868	1.868	1.858 (14)
10	5.0	3000	1.156	1.182	1.126	1.126	1.126	1.114 (11)

to establish validation for the DQM at the thin end of the range. The shells cover a wide range of L'/r ratios. The results labelled FEM in the table were obtained using ADINA, the DQMA results were obtained using the DQM with a mesh of 22 × 22 sampling points, and the DQM1, DQM2, DQM3 results were obtained using graded DQM meshes of 40 × 12, 60 × 12, 66 × 12, respectively. Finally, the results labelled V-A give the critical pressures calculated by Vodenitcharova and Ansourian (1996) together with the corresponding circumferential wave numbers. All results correspond to shells having material properties of E = 200 GPa and ν = 0.3.

It is first noted that there is close agreement between the computed FEM results and the analytical V-A results. The latter results indicate that the buckling modes corresponding to the critical pressure generally involve a large number of waves in the circumferential direction, especially so for thin shells with low L'/r ratios. The DQMA results (22 × 22 mesh) correspond well with the FEM and V-A results only when the buckling mode represents relatively few waves (less than 12) in the circumferential direction. The DQM2 and DQM3 results involving fine rectangular grids favoring the circumferential direction do lead to results that compare favorably with all the V-A results. In particular, the DQM3 results, corresponding to the finest grid, are within 1% of the analytical results except for cases 4-5, thus, representing closer agreement to the V-A results than the FEM. The DQM1 results, representing a coarser rectangular grid compare favorably with all the V-A results except for cases 6-7.

A parametric study was conducted for the buckling of shells with an oblique end. The material properties for the shells were those of the validation study. In the analysis, the length ratio L'/r was varied from 1.5 to 2.5, the thickness ratio r/h from 50 to 200, and the obliquity angle α from 0^o to 45^o. As these shells are not particulary short or thin, the circumferential buckling mode number is expected to be relatively low and, thus, a DQM grid of 22 × 22 was used.

Comparisons of the critical pressures for the shells are given in Table 2. There is close agreement between the FEM and DQM results, with a maximum difference less than 4%. Critical pressures decrease rapidly with increase in the length ratio L'/r and very rapidly with increase in the thickness ratio r/h. The strongest dependence on the obliquity angle occurs for short thin shells. The

Table 2 Parametric study for buckling problem (p_{cr} × 10^5 Pa)

		r/h = 50		r/h = 100		r/h = 200	
L'/r	α	FEM	DQM	FEM	DQM	FEM	DQM
1.5	0^o	104.6	107.0	18.03	18.21	3.214	3.214
	15^o	94.43	97.80	16.52	16.77	2.910	2.933
	30^o	85.52	88.42	14.83	15.07	2.581	2.621
	45^o	76.83	79.15	13.09	13.39	2.257	2.295
2.0	0^o	75.63	78.40	13.55	13.65	2.453	2.443
	15^o	73.48	75.48	12.87	13.05	2.267	2.279
	30^o	69.09	70.71	11.97	12.12	2.091	2.096
	45^o	64.27	65.35	10.96	11.08	1.886	1.897
2.5	0^o	62.08	63.37	11.12	11.19	2.036	1.949
	15^o	60.50	61.75	10.72	10.73	1.876	1.872
	30^o	58.04	59.05	10.07	10.15	1.756	1.757
	45^o	55.26	55.82	9.423	9.490	1.621	1.624

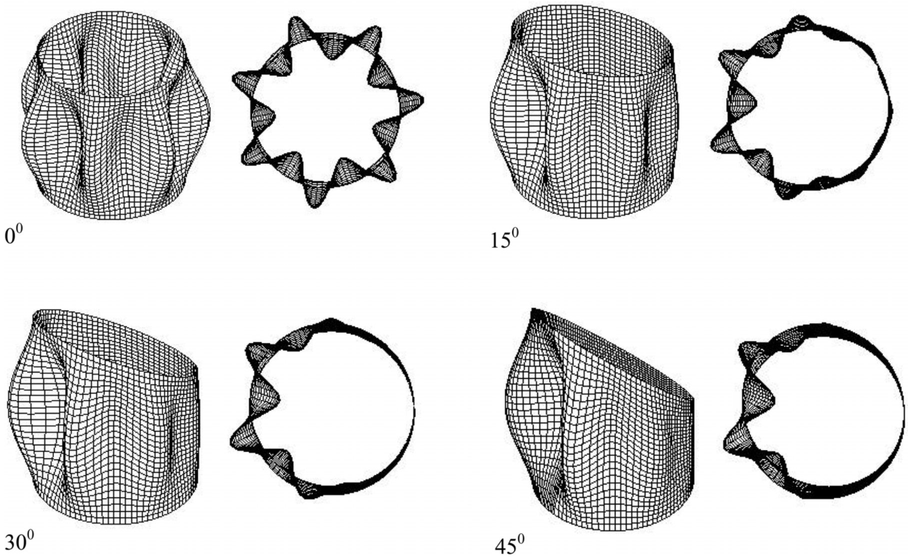

Fig. 4 Fundamental mode shapes of buckling (L/r = 2.0, r/h = 100)

fundamental mode shapes, obtained using the FEM, for the cases L'/r = 2.0, r/h = 100, α = 0°, 15°, 30°, 45° are given in Fig. 4.

7. Conclusions

The use of blending functions together with the differential quadrature method leads to an effective solution of the buckling problem of a cylindrical shell with an oblique end. Results obtained using this approach agree well with available published results and with finite element results. In this study, as in previous ones, there are indications that the differential quadrature method can give better accuracy than the finite element method. The approach presented herein thus offers a promising method for the buckling analysis of shells of irregular shape.

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