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Free-edge effect in cross-ply laminated plates under a uniform extension

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Abstract. Based on the basic equations of elasticity, free-edge effects on stresses in cross-ply laminated plates are found by using the state space method. The laminates are subjected to uniaxial-uniform extension plate, which is a typical example of general plane strain problem. The study takes into account material constants of all individual material layers and the state equation of a laminate is solved analytically in the through thickness direction. By this approach, a composite plate may be composed of an arbitrary number of orthotropic layers, each of which may have different material properties and thickness. The solution provides a continuous displacement and inter-laminar stress fields across all material interfaces and an approxiamte prediction to the singularity of stresses occurring in the boundary layer region of a free-edge. Numerical solutions are obtained and compared with the results obtained from an alternative numerical method.

Key words: laminated plates; uniform extension; free edge effects; state equation; analytical solution.

1. Introduction

It is well known that the behavior of the structures composed of advanced composite materials are considerably more complicated than isotropic ones. It has been recognized that composite materials are considerably more sensitive in edge effects than isotropic ones. The high level transverse normal and shear stresses acting in the region near free edges of a laminated composite is of great interest to composite manufacturers and designers because the stresses can cause edge delamination and then lead to material failure at a load that is far below the nominal failure load of the material. Thus, a better understanding of the edge effects on such materials is of particularly important to composite manufacturers and users. Considerable attention has been paid to free-edge effects in composites in the last few decades. Numerical, analytical and semi-analytical approaches have been used to characterize free edge effects in composite materials. The numerical approaches include finite element solutions (Wang and Crossman 1977, Nailadi and Adams 2002) and finite difference

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ones (Pipes and Pagano 1970, Bhaskar *et al.* 2000). The analytical analyses were largely based on higher order theories of bending (Pagano 1974, Wang and Choi 1982, Delale 1984, Becker 1993, Huang and Chen 1994, Lindemann and Becker 2000, Chue and Liu 2001, 2002). Wang *et al.* (2000) combined state space equation method with an eigen-expansion method and presented an analytical solution. Semi-analytical solutions combined the traditional finite element method with either an eigen-expansion method (Dong and Goetshel 1982) or the state space equation method (Ye and Sheng 2003) have also been used to study the problem.

Although finite element method is probably one of the most universal methods that can be applied to problems involving any cross section and lamination profile, it is quite computationally expensive since a large number of elements in the through-thickness direction are needed to model a multi-layered profile. Moreever, FE analyses may sometimes unreliable, especially in the case of free-edge analysis of multi-layered laminates where the requirement for many elements in the region of a free edge may lead the elements divided to high aspect ratios. Because of this, for laminates having regular cross sections, analytical methods are still preferable tools that can be used to obtain the solutions. These methods can also be used to replace expensive numerical calculations or experiments. However, due to the complex nature of anisotropy, many recently published results were almost exclusively confined to examining edge effects on the basis of higher order laminated plate or shell theories and imposing simple displacement and stress fields. The introduced simplifications may lead to significant errors in predicting inter-laminar stresses, particularly, near free edges of a laminate.

The state space method is considered to be an efficient and effective approach for analyzing laminated structures. The method has been successfully used in connection with static and dynamic analyses of laminates subjected to various load and boundary conditions (Fan and Ye 1990, Ye and Soldatos 1994, Sheng and Fan 1997, Fan 1998, Sheng 2000, Sheng and Ye 2002). The method takes into account all independent material constants and guarantees continuous fields of all transverse stresses across interfaces between material layers. A comprehensive account of the method can be found in Ye (2002). On the basis of the theory of elasticity and the state space equation method, this paper studies free-edge effects of stresses in laminated plates. The laminate considered in this paper is subjected to a uniform extension in one of the in-plane principal directions. Consequently, the problem is finally reduced to a general plane strain system. All material constants of every individual material layers are taken into account in the calculation and the state equation of the laminate is solved analytically in the through thickness direction. The composite plates may be composed of an arbitrary number of orthotropic layers, each of which may have different material properties and thickness. The solution provides a continuous inter-laminar stress field across interfaces and an approximate prediction to the singularity of stresses occurring in the boundary layer region of a free-edge. Numerical solutions are compared with the results obtained from an alternative numerical method.

2. State space equation approach for laminated plates

2.1 An single-layered plate

Consider an arbitrarily thick plate with a constant thickness h. The displacements in x, y and z directions are denoted by u, v and w, respectively. Suppose the plate is subjected to a uniform extension in one of the principal directions in the x-y plane. For example, the plate is subjected to a uniform strain in the y direction. If the plate is made of a linearly elastic orthotropic material whose material axes of orthotropy coincide with the axes of the adopted co-ordinate system, there exist the following fundamental equations.

(a) Stress-strain relations

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{Bmatrix} \tag{1}$$

where C_{ij} are stiffness coefficients.

(b) Equilibrium equations

$$\begin{cases} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} + \dfrac{\partial \sigma_{xz}}{\partial z} = 0 \\ \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{yz}}{\partial z} = 0 \\ \dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} = 0 \end{cases} \tag{2}$$

(c) Strain-displacement relations

$$\begin{cases} \varepsilon_{xx} = \dfrac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \dfrac{\partial v}{\partial y}, \quad \varepsilon_{zz} = \dfrac{\partial w}{\partial z} \\ \gamma_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}, \quad \gamma_{xz} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}, \quad \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{cases} \tag{3}$$

Considering that the plate is subjected to a uniform extension in the y direction, it is assumed that

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \varepsilon_0 = \text{constant}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \tag{4}$$

In addition, we can further assume that displacement v is only related to y, i.e., $v = v(y)$. As a result, we can conclude from Eqs. (1), (3) and (4) that $\sigma_{yz} = \sigma_{xy} = 0$ and the other six variables u, w, σ_{xz}, σ_{zz}, σ_{xx}, σ_{yy} are all independent of y.

To facilitate the following deduction process, let

$$\alpha = \partial/\partial x, \quad C_1 = -C_{13}/C_{33}, \quad C_2 = C_{11} - C_{13}^2/C_{33}, \quad C_3 = C_{12} - C_{13}C_{23}/C_{33},$$

$$C_4 = C_{22} - C_{23}^2/C_{33}, \quad C_5 = -C_{23}/C_{33}, \quad C_7 = 1/C_{33}, \quad C_8 = 1/C_{55} \tag{5}$$

From the third equation of Eq. (1) and Eq. (3), the following relation is obtained

$$\frac{\partial w}{\partial z} = C_1 \alpha u + C_7 \sigma_{zz} + C_5 \varepsilon_0 \tag{6}$$

Substituting Eq. (6) into Eq. (1), the two in-plane stresses can be calculated by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{Bmatrix} = \begin{bmatrix} C_2 \alpha - C_1 \\ C_3 \alpha - C_5 \end{bmatrix} \begin{Bmatrix} u \\ \sigma_{zz} \end{Bmatrix} + \begin{Bmatrix} C_3 \\ C_4 \end{Bmatrix} \varepsilon_0 \tag{7}$$

Inserting Eq. (7) into the first and third equations of Eq. (2) and considering Eq. (6) as well as the fifth equation of Eq. (1), we can obtain the following first order non-homogenous partial differential equation system

$$\frac{\partial}{\partial z} \begin{Bmatrix} u \\ w \\ \sigma_{xz} \\ \sigma_{zz} \end{Bmatrix} = \begin{bmatrix} 0 & -\alpha & C_8 & 0 \\ C_1 \alpha & 0 & 0 & C_7 \\ -C_2 \alpha^2 & 0 & 0 & C_1 \alpha \\ 0 & 0 & -\alpha & 0 \end{bmatrix} \begin{Bmatrix} u \\ w \\ \sigma_{xz} \\ \sigma_{zz} \end{Bmatrix} + \begin{Bmatrix} 0 \\ C_5 \varepsilon_0 \\ 0 \\ 0 \end{Bmatrix} \tag{8}$$

Assume the following displacement and stress fields for the solution of the problem;

$$u(x, z) = \bar{u}(x, z) + U^{(0)}(z)\left(1 - \frac{2x}{L_x}\right) \tag{9}$$

$$\begin{Bmatrix} \bar{u} \\ w \\ \sigma_{xz} \\ \sigma_{zz} \end{Bmatrix} = \sum_m \begin{Bmatrix} \bar{U}_m(z)\sin(\xi x) \\ W_m(z)\cos(\xi x) \\ X_m(z)\sin(\xi x) \\ Z_m(z)\cos(\xi x) \end{Bmatrix} \tag{10}$$

where $\xi = m\pi/L_x$, L_x is the length of the plate in the x direction and $U^{(0)}(z)$ is an unknown boundary displacement function that can be determined by imposing traction free conditions along free edges and we'll discuss this problem in details in later section.

Introducing Eq. (9) into Eq. (8) yields

$$\frac{\partial}{\partial z}\{\boldsymbol{F}\} = [\boldsymbol{G}]\{\boldsymbol{F}\} + \{\boldsymbol{B}\} \tag{11}$$

where

$$\{\boldsymbol{B}\} = \left[-\frac{dU^{(0)}(z)}{dz}\left(1 - \frac{2x}{L_x}\right) \quad C_5 \varepsilon_0 - \frac{2C_1}{L_x} U^{(0)}(z) \quad 0 \quad 0 \right]^T \tag{12}$$

$$\{\boldsymbol{F}\} = [\bar{u} \;\; w \;\; \sigma_{xz} \;\; \sigma_{zz}]^T \tag{13}$$

and $[\boldsymbol{G}]$ is the 4×4 matrix shown in Eq. (8). Substituting Eq. (10) into Eq. (11) and expanding also the x co-ordinate in vector $\{\boldsymbol{B}\}$ into a Fourier series, as follows

$$x = -\frac{2L_x}{\pi}\sum_m \frac{\cos m\pi}{m}\sin\frac{m\pi x}{L_x} \tag{14}$$

one has the following non-homogenous state equation for an arbitrary value of m

$$\frac{d}{dz}\{\boldsymbol{F}_m(z)\} = [\boldsymbol{G}]\{\boldsymbol{F}_m(z)\} + \{\boldsymbol{B}_m(z)\} \tag{15}$$

where

$$\{\boldsymbol{F}_m(z)\} = [\bar{U}_m(z) \;\; W_m(z) \;\; X_m(z) \;\; Z_m(z)]^T \tag{16}$$

$$[\boldsymbol{G}] = \begin{bmatrix} 0 & \xi & C_8 & 0 \\ C_1\xi & 0 & 0 & C_7 \\ C_2\xi^2 & 0 & 0 & -C_1\xi \\ 0 & 0 & -\xi & 0 \end{bmatrix} \tag{17}$$

$$\{\boldsymbol{B}_0(z)\} = \left[0 \quad C_5\varepsilon_0 - \frac{2C_1}{L_x}U^{(0)}(z) \quad 0 \quad 0\right]^T \tag{18}$$

$$\{\boldsymbol{B}_m(z)\} = \left[-\frac{2}{m\pi}(1+\cos m\pi)\frac{dU^{(0)}(z)}{dz} \quad 0 \quad 0 \quad 0\right]^T \tag{19}$$

The solution of Eq. (15) can easily be found as Fan (1998), Ye (2002)

$$\begin{aligned}\{\boldsymbol{F}_m(z)\} &= e^{[\boldsymbol{G}]z}\{\boldsymbol{F}_m(0)\} + \int_0^z e^{[\boldsymbol{G}](z-\tau)}\{\boldsymbol{B}_m(\tau)\}d\tau \\ &= [\boldsymbol{D}_m(z)]\{\boldsymbol{F}_m(0)\} + \{\boldsymbol{H}_m(z)\} \qquad z \in [0, h]\end{aligned} \tag{20}$$

In particular, at $z = h$,

$$\{\boldsymbol{F}_m(h)\} = [\boldsymbol{D}_m(h)]\{\boldsymbol{F}_m(0)\} + \{\boldsymbol{H}_m(h)\} \tag{21}$$

where $[\boldsymbol{D}_m(h)]$ is called transfer matrix that can be calculated either analytically or numerically. For example, let λ_1, λ_2, λ_3, λ_4 be the eigenvalues of $[\boldsymbol{G}]$ and $[\boldsymbol{P}]$ be a matrix composed of the associated eigenvectors. From linear algebra, we have.

$$[\boldsymbol{D}_m(h)] = [\boldsymbol{P}]\begin{bmatrix} e^{\lambda_1 h} & & \\ & \ddots & \\ & & e^{\lambda_4 h}\end{bmatrix}[\boldsymbol{P}]^{-1} \tag{22}$$

$$\{\boldsymbol{H}_m(h)\} = [\boldsymbol{P}]\left(\int_0^h \begin{bmatrix} e^{\lambda_1(h-\tau)} & & \\ & \ddots & \\ & & e^{\lambda_4(h-\tau)}\end{bmatrix}\{\boldsymbol{B}_m(\tau)\}d\tau\right)[\boldsymbol{P}]^{-1} \tag{23}$$

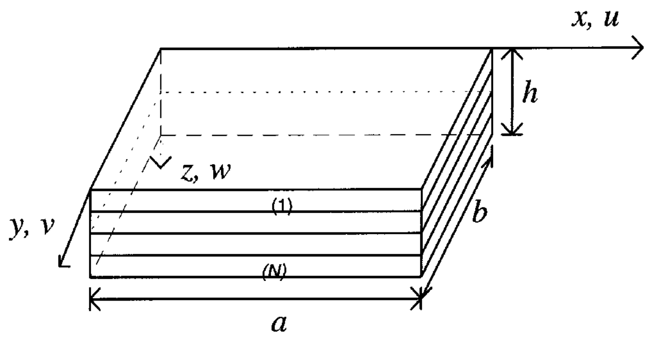


Fig. 1 Nomenclature of a laminated rectangular plate

2.2 A thick laminated plate

Consider a thick laminated plate which is composed of N orthotropic layers (See Fig. 1), the state space equation of the j-th layer is obtained as

$$\frac{d}{dz}\{\boldsymbol{F}_m(z)\}_j = [\boldsymbol{G}_j]\{\boldsymbol{F}_m(z)\}_j + \{\boldsymbol{B}_m(z)\}_j \tag{24}$$

In order to find the final solution of the problem, we must solve first the unknown displacement function, $U^{(0)}(r)$, appearing in Eqs. (18) and (19). If the layers of the laminate are all sufficiently thin, it is reasonable to assume that the displacement $U^{(0)}(r)$ within the thin layer is linearly distributed in the z direction, i.e.,

$$U_j^{(0)}(z) = U_j^-\left(1-\frac{z}{h_j}\right) + U_j^+\frac{z}{h_j} \quad z \in [0, h_j], \quad j = 1, 2, \ldots, N \tag{25}$$

where U_j^- and U_j^+ are the values of $U_j^{(0)}(z)$ at the upper and bottom surfaces of the j-th thin layer. If a layer is not sufficiently thin, we can divide it into several thin sub-layers. Hence, two types of material interfaces are distinguished in the laminate, i.e., the fictitious interfaces which separate sub-layers with the same material properties and the real ones that separate sub-layers composed of different materials. Upon choosing a suitably large value of N, each individual sub-layers becomes sufficiently thin and, as a result, Eq. (25) is considered to be adequate. The solution of Eq. (24) for an arbitrary sub-layer can easily be found from Eqs. (20)-(23). With appropriate continuity requirements imposed at all the real and fictitious interfaces, an approximate solution for the entire laminate can be obtained. Also, the solution can be found to the required accuracy by increasing the total number of the thin layers.

Inserting Eq. (25) into Eqs. (18) and (19), vector $\{\boldsymbol{B}_m(z)\}_j$ in Eq. (24) can be expressed as

$$\{\boldsymbol{B}_0(z)\}_j = \left[0 \quad C_5\varepsilon_0 - \frac{2C_1}{L}\left(U_j^-\left(1-\frac{z}{h_j}\right) + U_j^+\frac{z}{h_j}\right) \quad 0 \quad 0\right]^T, \quad z \in [0, h_j] \tag{26}$$

$$\{\boldsymbol{B}_m(z)\}_j = \left[\frac{4}{m\pi}\frac{U_j^- - U_j^+}{h_j} \quad 0 \quad 0 \quad 0\right]^T, \quad (m = 2, 4, \ldots) \tag{27}$$

The solution of Eq. (24) is

$$\{\boldsymbol{F}_m(h_j)\}_j = [\boldsymbol{D}_m(h_j)]_j\{\boldsymbol{F}_m(0)\}_j + \{\boldsymbol{H}_m(h_j)\}_j \tag{28}$$

According to the continuity conditions at all interfaces, one has

$$\{\boldsymbol{F}_m(h_j)\}_j = \{\boldsymbol{F}_m(0)\}_{j+1} \tag{29}$$

By using Eqs. (28) and (29) recursively, a relationship between the state vectors on the upper and bottom surfaces of the plate is established as follows

$$\{\boldsymbol{F}_m(h_N)\}_N = [\bar{\boldsymbol{D}}_m]_N\{\boldsymbol{F}_m(0)\}_1 + \{\bar{\boldsymbol{H}}_m\} \tag{30}$$

where
$$[\bar{\boldsymbol{D}}_m]_N = \left(\prod_{j=N}^{1}[\boldsymbol{D}_m(h_j)]_j\right) \tag{31}$$

$$\{\bar{\boldsymbol{H}}_m\} = \left(\prod_{j=N}^{2}[\boldsymbol{D}_m]_j\right)\{\boldsymbol{H}_m\}_1 + \left(\prod_{j=N}^{3}[\boldsymbol{D}_m]_j\right)\{\boldsymbol{H}_m\}_2 + \ldots + \{\boldsymbol{H}_m\}_N \tag{32}$$

$\{\boldsymbol{F}_m(h_N)\}_N$ and $\{\boldsymbol{F}_m(0)\}_1$ are, respectively, the state vectors at the upper and bottom surfaces of the laminate. Upon using the traction free conditions at the upper and bottom surfaces, the following stress conditions are obtained:

$$[X_m(h_N), Z_m(h_N)]_N^T = (0,0)^T, \quad [X_m(0), Z_m(0)]_1^T = (0,0)^T \tag{33}$$

Substituting Eq. (33) into Eq. (30) yields the following linear algebra equations:

$$\begin{bmatrix} \bar{D}_{31} & \bar{D}_{32} \\ \bar{D}_{41} & \bar{D}_{42} \end{bmatrix} \begin{Bmatrix} \bar{U}_m \\ W_m \end{Bmatrix}_1 = -\begin{Bmatrix} \bar{H}_{m3} \\ \bar{H}_{m4} \end{Bmatrix} \tag{34}$$

where $\bar{D}_{ij}$ and $\bar{H}_{mi}$ are the relevant elements in $[\bar{\boldsymbol{D}}_m]$ and $\{\bar{\boldsymbol{H}}_m\}$, respectively. Eq. (34) is a set of linear algebra equation in terms of the two displacement components, $\bar{U}_m$ and W_m, at the upper surface. The free terms of Eq. (34), $\bar{H}_{m3}$ and $\bar{H}_{m4}$, contain $2 \times N$ unknown constants, U_j^- and U_j^+ $(j = 1, 2, \ldots, N)$ (j=1, 2,…N), introduced in Eq. (25). These constants are determined by introducing boundary conditions.

3. Free edge boundary conditions and solution of the laminate

The unknown constants U_j^- and U_j^+ (j=1, 2, … N) contained in Eq. (34) can be determined according to the free edge conditions at $x = 0$, L_x as follows

$$\sigma_{xz} = \sigma_{xy} = \sigma_{zz} = 0 \tag{35}$$

In order to impose the above traction free conditions at the two free edges, we introduce Eq. (9) into Eq. (7). As a result, the two in-plane stresses are expressed as follows:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \end{Bmatrix} = \begin{bmatrix} C_2\alpha - C_1 \\ C_3\alpha - C_5 \end{bmatrix} \begin{Bmatrix} \bar{u} \\ \sigma_{zz} \end{Bmatrix} + \begin{Bmatrix} C_3 \\ C_4 \end{Bmatrix} \varepsilon_0 - \begin{Bmatrix} C_2 \\ C_3 \end{Bmatrix} \frac{2U^{(0)}}{a} \tag{36}$$

After substituting Eq. (13) into Eq. (35), we obtain for the j-th sub-layer

$$\sigma_{xx} = \sum_m [C_2\xi\bar{U}_m(z) - C_1 Z_m(z)]_j \cos\xi x + [C_3]_j \varepsilon_0 - \left[\frac{2C_2}{a} U^{(0)}(z)\right]_j \tag{37}$$

It can be seen from Eqs. (4) and (10) that $\sigma_{xz} = \sigma_{xy} = 0$ are satisfied automatically at the free edges ($x = 0$ and $x = L_x$). The remaining boundary condition to be satisfied at the free edges is $\sigma_{xx} = 0$. Due to symmetry, we only need to impose the condition at $x = 0$. Thus, from Eq. (37), we obtain the following condition

$$\sum_m [C_2\xi\bar{U}_m(z) - C_1 Z_m(z)]_j + [C_3]_j \varepsilon_0 - \left[\frac{2C_2}{a} U^{(0)}(z)\right]_j = 0 \tag{38}$$

It has been mentioned in the previous section that Eq. (34) contains unknown constants U_j^- and U_j^+ (j=1, 2, …, N). To solve for the constants, we first consider the continuity of $U^{(0)}(r)$ at the interface between the j-th and the (j+1)-th sub-layers. From Eq. (25), the following relationship is obtained

$$U_j^+ = U_{j+1}^- \quad (j = 1, 2, \ldots, N-1) \tag{39}$$

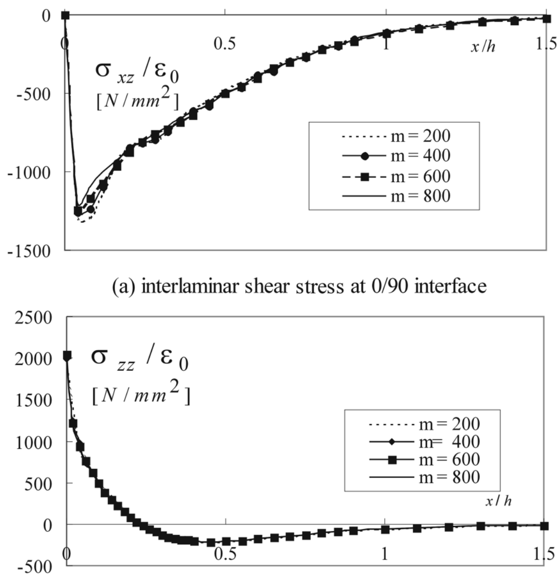
Hence, there exist only ($N + 1$) independent unknown constants. In order to solve these constants and also the two displacement components of the upper surface (see Eq. 34), the traction free condition (38) must be satisfied at the edges of all interfaces, including the fictitious and material interfaces. This can be done by introducing z-coordinates of the interfaces, z_j, into Eq. (38). This process yields ($N + 1$) independent linear algebra equations. Along with the two equations from Eq. (34), the two displacement components and the ($N + 1$) unknown constants can finally be solved. Once the equation system is solved, all the displacements and stresses can be obtained by bring back the solutions to the state space equations shown in the previous sections.

4. Examples of free edge effects in laminated plate

To validate the present method, numerical calculations are carried out for a four-layered cross-ply laminate ([0/90]s) and the following elastic constants are assumed

$$\begin{gathered} C_{11} = C_{33} = 15300 \text{ N/mm}^2 \quad C_{22} = 140000 \text{ N/mm}^2 \quad C_{44} = C_{55} = 5900 \text{ N/mm}^2 \\ C_{12} = C_{23} = 3900 \text{ N/mm}^2 \quad C_{13} = 3300 \text{ N/mm}^2 \end{gathered} \tag{40}$$

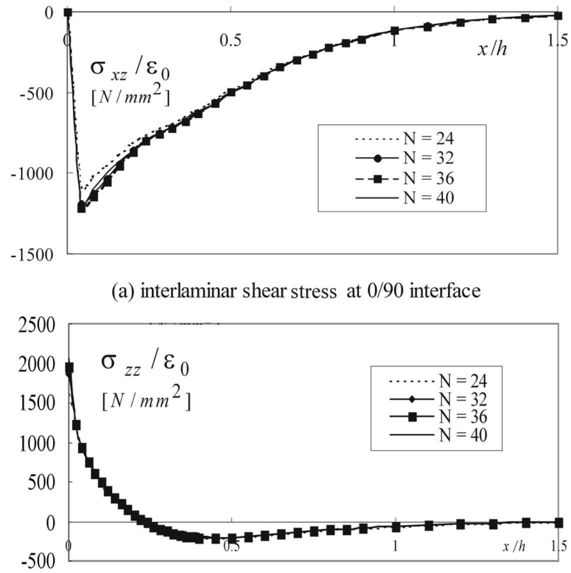
The geometry of the plate is $a/h = 10$. The plate has two opposite free edges in the x direction and the other two edges are subjected to a uniform extension that induces a constant strain ε_0 in the y direction. In order to obtain an accurate results near the free edges, we must take sufficiently large



(a) interlaminar shear stress at 0/90 interface

(b) interlaminar normal stress at 0/90 interface

Fig. 2 Convergence of interlaminar stresses for [0/90]s laminate against different m with $N = 40$



(a) interlaminar shear stress at 0/90 interface

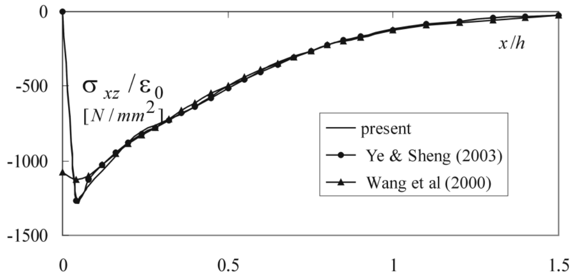
(b) interlaminar normal stress at 0/90 interface

Fig. 3 Convergence of interlaminar stresses for [0/90]s laminate against different N with $m = 600$

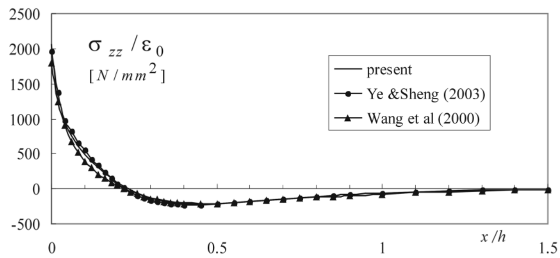
numbers of series term m in Eq. (13) and the thin layers N in Eq. (25). Fig. 2 and Fig. 3 show, respectively, the convergence rate of interlaminar stresses against m and N. It can be seen from the figures that the convergence of σ_{zz} is faster than σ_{xz}. Even for the shear stress, the difference between the results form $m = 200$ and $m = 800$ are not significant, considering the fact that stress singularity occurs in the region. All the results in the reminder of the paper are obtained using $N = 40$ and $m = 600$.

The distributions of the intrelaminar stresses $\sigma_{xz}/\varepsilon_0$ and $\sigma_{zz}/\varepsilon_0$ at 0/90 interface in the vicinity of the free edge are compared with those obtained by Wang *et al.* (2000) and Ye and Sheng (2003) and shown in Fig. 4. According to the classic plate theory, the shear stress σ_{xz} and normal stress σ_{zz} are uniformly zero for the symmetrically laminated plate with cross-ply layers. Fig. 4 shows, however, that these stresses approach to zero only for $x > 1.5h$. Hence, the boundary layer zone in this case is localized to the region of a distance of about $1.5h$ from the free edge. Fig. 5 shows the distribution of in-plane stress $\sigma_{xx}/\varepsilon_0$ and the comparisons with those obtained by Wang *et al.* (2000). It is also observed the stress $\sigma_{xx}/\varepsilon_0$ approaches a constant value in the region of $x > 1.5h$. Hence, we can conclude from above results that the stress disturbance area only occurs near the free edge. It can be seen from the figures that the present results agree very well with the comparisons except σ_{xz} in the region very close to the free edge. This is due to that fact that Wang's results do not satisfy the traction free conditions at $x = 0$, where Wang *et al.* (2000) using a linear combination of eigen expansions to express stresses and displacements and then determine unknown coefficients using the principle of virtual work. Thus, the free-edge boundary conditions were only satisfied in the sense of average stresses. According to the Saint-Venant principle this approximation may not give idea result for stresses near a free edge.

Finally, we consider the influences of lamination on the distribution of σ_{xz} and σ_{zz} of four plates

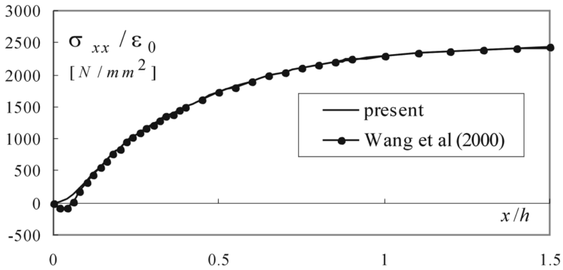


(a) interlaminar shear stress at the interface of [0/90]s

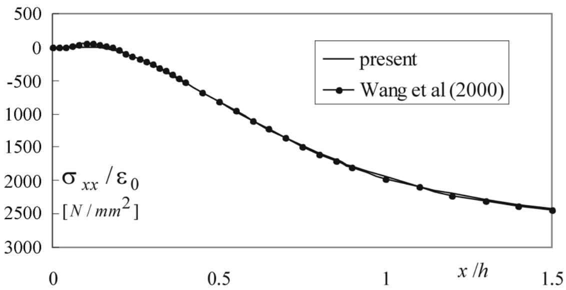


(b) interlaminar normal stress at the interface of [0/90]s

Fig. 4 Comparisons of interlaminar stresses for laminate [0/90]s

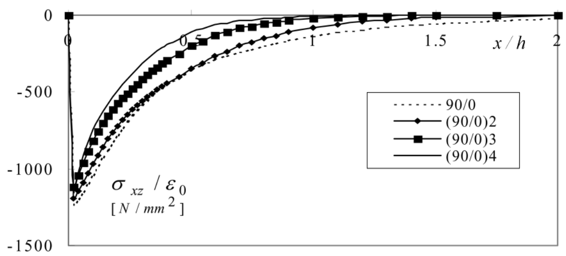


(a) in-plane stress at the middle interface of 0-layers

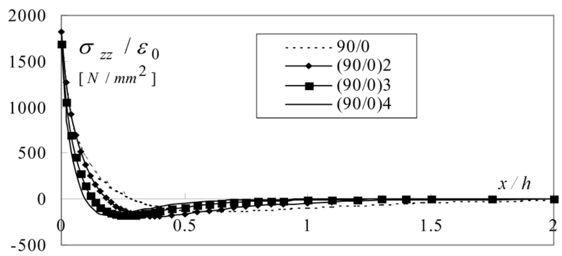


(b) in-plane stress at the middle interface of the laminate

Fig. 5 Comparisons of in-plane stresses for laminate [0/90]s

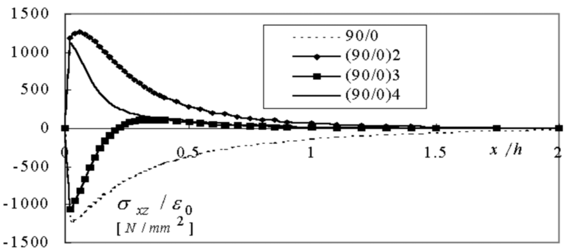


(a) interlaminar shear stress at upper 0/90 interface

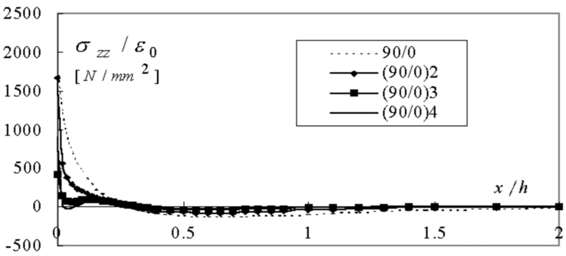


(b) interlaminar normal stress at upper 0/90 interface

Fig. 6 Distributions of interlaminar stresses along upper 0/90-interface against different lay-ups



(a) interlaminar shear stress at middle surface of the laminate



(a) interlaminar normal stress at middle surface of the plate

Fig. 7 Distributions of interlaminar stresses along the mid-surface against different lay-ups

having two, four, six and eight antisymmetric cross-ply layers, respectively. The laminates have the same material and geometric properties as shown in Eq. (40), The distribution of the stresses are shown in Figs. 6 and 7.

From Figs. 6 and 7 it can be seen that the boundary layer effect becomes less significant as the number of material layers increases. It is also interested to notice from Fig. 7 that the interfacial shear stresses at the middle interfaces of (0/90) and $(0/90)_2$ plates change directions along the entire length in the x direction, while as the number of material layers increases, e.g., for the $(0/90)_3$ and $(0/90)_4$ plates, the stress only changes direction in the vicinity of the free edge and then approaches a constant as it decays towards the interior zone of the composites.

5. Conclusions

An analytical method based on a state space representation of displacements and transverse stresses has been presented to solve free-edge stresses in cross-ply laminated plates subjected to a uniform extension. Numerical examples were given to show the applications of the method to the stress analysis of free edge effects.

Numerical tests and comparisons have been carried out to validate the method. By comparing with a three-dimensional eigen-expansion method (Wang *et al.* 2000), it was observed that the present method could provide accurate results. In the vicinity of free edges, the obtained results showed good approximation to stress singularities.

Since the recursive formulation (see Eqs. 30-32) was used to derive the state equations of laminated plates, the dimension of the final state equations (see Eq. 34) did not depend on the number of layers of the laminate. As a consequence, this method is particularly suitable to solve free-edge stress problems of multi-layered plates. The method always provides continuous distributions of both displacements and transverse stresses across the thickness of a laminate. The method and the obtained results can also be used to provide benchmark tests for validating new theories, numerical solutions and finite element codes, especially when dealing with stress singularities or material discontinuities.

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