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Solids 3-D with bounded tensile strength under the action of thermal strains. Theoretical aspects and numerical procedures

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Abstract. This paper is devoted to illustrate some numerical procedures to solve the boundary equilibrium problems of three-dimensional solids that are subjected to thermal strains. The constitutive equations take into account the bounded tensile strength of the material and they are presented in the framework of non-linear elasticity and small strains. The associated equilibrium problem is solved numerically by means of the finite element method and the numerical techniques, i.e. the Newton-Raphson method and the secant method, are revised in order to assure the solution convergence of the discretized problem. Some numerical examples are illustrated.

Key words: masonry; thermal strain; bounded tensile strength; finite element.

1. Introduction

In the last decades, numerous authors have been devoted in modeling the response of no-tension materials or materials with bounded tensile strength having in mind the result application to the masonries. Only to quote the most recent, we recall the paper of Romano and Sacco (1884) and Sacco (1990) which proposed the constitutive equations for no-tension materials via a variational approach and suggested to solve the boundary equilibrium problem by means of the finite element method by adopting the secant procedure.

In the same years, Del Piero (1989) revised the model and proposed the constitutive equations for a masonry-like material under the hypotheses of small strains, no tensile strength and a normality postulate. On the further assumption concerning the symmetry of the elastic tensor, the existence of the strain energy density was proved. Lucchesi *et al.* (1994) proposed a numerical method to solve the non-linear equilibrium problem of an isotropic body made of masonry-like material using the finite element method. The authors used the Newton-Raphson procedure that is based on the tangent approach recognizing its fast convergence with respect to the secant method. Furthermore, Lucchesi *et al.* (1995) extended this method to solve the equilibrium problems for materials with bounded tensile strength.

Subsequently, the model of no-tension material subjected to thermal strains was elaborated by Padovani *et al.* (2000) and a complete model concerning the no-tension materials was presented by

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Lucchesi *et al.* (2000) in the framework of the thermodynamics and the thermoelasticity.

By following a recent work of Pimpinelli (2003), the aim of the paper is to present the constitutive equation of an isotropic body with bounded tensile strength subjected to thermal strains in the 3-D case and to illustrate the numerical procedures in order to solve the boundary equilibrium problem. Without some loss of generality, the dependence of the elastic moduli on the temperature is not made explicit.

Having in mind to solve the boundary equilibrium problem by means of the finite element method, we present the constitutive equation in a form such that it is assured *a priori* the positiveness of the tangent elastic matrix. Indeed, by following a suggestion of Padovani (2000) we consider an approximated material depending on a parameter δ varying from zero to one. This is accomplished modifying the constitutive equation and making linear the dependence of the stress on the anelastic part of the deformation by means of the elasticity tensor factorized by the parameter δ. In this way, the modified constitutive law describes the behavior of a family of isotropic materials which, in absence of the thermal loads and for $\delta = 0$, coincides with that described by Lucchesi *et al.* (1995) whereas for $\delta = 1$ coincides with the one linearly elastic. Of course, for $\delta \neq 0$, the constitutive equations associated to materials with bounded tensile strength or, in the limit case, with no-tensile strength, are verified only approximately. On the other hand, as we will show in the Section 3, the assumption of $\delta \neq 0$ renders the constitutive law strictly monotone. In virtue of this assumption, the numerical method based on the Newton-Raphson procedure possesses the indispensable requirements to be convergent (see also Pimpinelli (2003)). Furthermore, the results comparison to the secant method shows that the approximation is acceptable.

Thus, the paper is organized as follow. In Section 2 we illustrate briefly the constitutive equation for an isotropic material with bounded tensile strength that is subjected to thermal strain. The constitutive equations are presented in a form such that the positiveness of the elastic tangent matrix is assured *a priori*. Moreover, a shortly reference is made to the bi-modular materials whose behavior can be modeled by the proposed constitutive equations.

In Section 3 we discuss the numerical procedures to solve the equilibrium problem by means of the finite element method; the examined procedures are the tangent method (i.e., the Newton-Raphson method) and the secant method. Namely, we prove that the tangent method may be convergent only if $\delta \neq 0$ and, in view of the numerical instabilities emanating by the numerical investigation, we propose a new one expression for the tangent elastic matrix which depends on the strain eigenvectors. Moreover, the secant method is examined and we consider the secant tensor as proposed by Romano and Sacco (1884) and Sacco (1990) furnishing its form that does not depend on the strain eigenvectors. Furthermore, a new one secant tensor is proposed. Finally, in Section 4 we show some numerical examples.

2. The mechanical model and the modified constitutive law

In this section, we begin shortly to present the constitutive assumption for a material with bounded tensile strength that is subjected to the thermal strains. For the details of the presentation we remand to Pimpinelli (2003), Lucchesi *et al.* (2000) and Padovani *et al.* (2000).

We denote by *Lin* the space of the second order tensors equipped by the inner product whereas we denote by *Sym*, Sym^+ and Sym^- the subsets of *Lin* constituted by the symmetric, symmetric positive semi-definite and symmetric negative semi-definite tensors, respectively.

For a material with bounded tensile strength under the action of thermal strains, the constitutive equations are:

$$(\boldsymbol{T} - \sigma\mathbf{1}) \in Sym^{-} \tag{1a}$$

$$\boldsymbol{E} - \boldsymbol{E}^{t} = \boldsymbol{E}^{e} + \boldsymbol{E}^{a} \tag{1b}$$

$$\boldsymbol{T} = \boldsymbol{C}(\boldsymbol{E} - \boldsymbol{E}^{t} - \boldsymbol{E}^{a}) \tag{1c}$$

$$(\boldsymbol{T} - \sigma\mathbf{1}) \cdot \boldsymbol{E}^{a} = 0 \tag{1d}$$

where Eq. (1a) represents the unilateral constraint on the stress $\boldsymbol{T}$, $\sigma \geq 0$ is the tensile strength of the material whereas Eq. (1b) is the additive decomposition of the infinitesimal strain $\boldsymbol{E}$ minus the thermal strain $\boldsymbol{E}^t$ into an elastic part $\boldsymbol{E}^e$ and into an anelastic part $\boldsymbol{E}^a$ semi-definite positive. Furthermore, Eq. (1c) imposes the linear dependence of the stress $\boldsymbol{T}$ on the elastic strain $\boldsymbol{E}^e$ by means of the elasticity forth order tensor $\boldsymbol{C}$ and Eq. (1d) is the normality postulate. If the material is isotropic and the temperature variation $\Delta\theta$ is small, we have:

$$\boldsymbol{C} = 2\mu\boldsymbol{I} + \lambda\mathbf{1} \otimes \mathbf{1} \tag{2a}$$

$$\boldsymbol{E}^{t} = \beta(\theta)\mathbf{1} = \alpha_t\Delta\theta\mathbf{1} \tag{2b}$$

where μ and λ are the Lame's moduli which do not depend on the temperature and satisfy the inequalities $\mu > 0$ and $2\mu + 3\lambda > 0$. Moreover, $\beta(\theta)$ is the thermal expansion, α_t is the linear coefficient of the thermal expansion, $\boldsymbol{I}$ is the forth order identity tensor over the elements of Sym whereas $\mathbf{1}$ is the identity tensor on the ordinary vector space.

By following the scheme of the proof shown by Lucchesi *et al.* (1995), it is possible to demonstrate that $\boldsymbol{T}$ and $\boldsymbol{E}^a$ are coaxial and by the isotropic properties of the elastic tensor $\boldsymbol{C}$, $\boldsymbol{T}$, $\boldsymbol{E}^e$, $\boldsymbol{E}$ and $\boldsymbol{E}^t$ are coaxial too.

In order to assure the positeveness of the tangent elastic tensor and by using the arguments illustrated by Pimpinelli (2003) we modify the constitutive Eqs. (1) such that:

$$\boldsymbol{T} = (1 - \delta)\boldsymbol{C}(\boldsymbol{E} - \boldsymbol{E}^{t} - \boldsymbol{E}^{a}) + \delta\boldsymbol{C}(\boldsymbol{E} - \boldsymbol{E}^{t}) \tag{3a}$$

$$[\boldsymbol{C}(\boldsymbol{E} - \boldsymbol{E}^{t} - \boldsymbol{E}^{a}) - \sigma\mathbf{1}] \in Sym^{-} \tag{3b}$$

$$[\boldsymbol{C}(\boldsymbol{E} - \boldsymbol{E}^{t} - \boldsymbol{E}^{a}) - \sigma\mathbf{1}] \cdot \boldsymbol{E}^{a} = 0 \tag{3c}$$

where δ is a parameter which varies from zero to one and the stress $\boldsymbol{T}$ depends also on the anelastic strain $\boldsymbol{E}^a$. The resulting stress-strain law is illustrated in Fig. 1 in the uniaxial case.

As remarked by Pimpinelli (2003), the condition $(\boldsymbol{T} - \sigma\mathbf{1}) \in Sym^{-}$ is verified only approximately depending on the choice of the parameter δ. By choosing δ very close to zero, the effect on the material response is very small as we will see in the following by defining the extra stresses.

By using the representation theorem for the isotropic functions, there exist three scalar functions β_0, β_1 and β_2 of the principal invariants of $\boldsymbol{E}$ such that:

$$\boldsymbol{T} = \beta_0\mathbf{1} + \beta_1\boldsymbol{E} + \beta_2\boldsymbol{E}^{2} \tag{4}$$

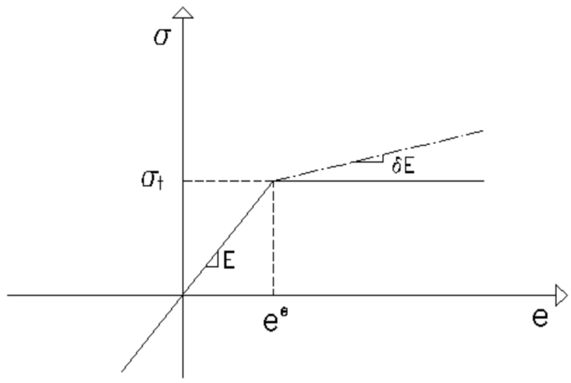

Fig. 1 The uniaxial stress-strain law for a material with bounded tensile strength

where the coefficients β_i, $i = 0, 1, 2$ depend in a non-linear way on the eigenvalues of $\boldsymbol{E}$ and on the thermal expansion $\beta(\theta)$, respectively. Furthermore, in view of Eqs. (2), we have:

$$\boldsymbol{T} = (1-\delta)[2\mu(\boldsymbol{E}-\boldsymbol{E}^a) + \lambda tr(\boldsymbol{E}-\boldsymbol{E}^a)\mathbf{1}] + \delta[2\mu\boldsymbol{E} + \lambda tr(\boldsymbol{E})\mathbf{1}] - \sigma^0\mathbf{1} \tag{5a}$$

$$\boldsymbol{T} = (1-\delta)(\beta_0\mathbf{1} + \beta_1\boldsymbol{E} + \beta_2\boldsymbol{E}^2) + \delta(\gamma_0\mathbf{1} + \gamma_1\boldsymbol{E}) \tag{5b}$$

where Eq. (5a) is a direct consequence of Eq. (3a) and Eq. (5b) is a consequence of Eq. (4). Moreover, the coefficients γ_0 and γ_1 and defined by:

$$\gamma_0 = \lambda tr[\boldsymbol{E} - \beta(\theta)\mathbf{1}] - 2\mu\beta(\theta) \tag{6a}$$

$$\gamma_1 = 2\mu \tag{6b}$$

$$\sigma^0 = (2\mu + 3\lambda)\beta(\theta) \tag{6c}$$

Thus, fixed the thermal strain $\boldsymbol{E}^t$ and given a strain $\boldsymbol{E}$, the key problem is to find the anelastic strain $\boldsymbol{E}^a$ and the stress $\boldsymbol{T}$, under the conditions Eqs. (3b), (c) and enforcing Eq. (5a). To this scope let us denote by (e_1, e_2, e_3) the eigenvalues of $\boldsymbol{E}$ such that $e_1 \le e_2 \le e_3$ and by (a_1, a_2, a_3) the eigenvalues of $\boldsymbol{E}^a$ (principal anelastic strains) which are assumed non-negative. Furthermore, we denote by I_1, I_2 and I_3 the invariants of $\boldsymbol{E}$, i.e.:

$$I_1 = tr(\boldsymbol{E}) = e_1 + e_2 + e_3 \tag{7a}$$

$$I_1 = \boldsymbol{E}\cdot\boldsymbol{E} = e_1^2 + e_2^2 + e_3^2 \tag{7b}$$

$$I_3 = \boldsymbol{E}^2\cdot\boldsymbol{E} = e_1^3 + e_2^3 + e_3^3 \tag{7c}$$

Then, setting $\alpha = \dfrac{\lambda}{\mu} \ge 0$, $\varepsilon = \dfrac{\sigma}{\mu} \ge 0$, $\eta = \dfrac{\sigma^0}{\mu}$, the Eq. (3c), projected in the strain principal reference frame, is split into a system of three equations, namely:

$$[2(e_1 - a_1) + \alpha(e_1 + e_2 + e_3 - a_1 - a_2 - a_3) - (\varepsilon + \eta)]a_1 = 0 \tag{8}$$

$$[2(e_2 - a_2) + \alpha(e_1 + e_2 + e_3 - a_1 - a_2 - a_3) - (\varepsilon + \eta)]a_2 = 0 \tag{9}$$

$$[2(e_3 - a_3) + \alpha(e_1 + e_2 + e_3 - a_1 - a_2 - a_3) - (\varepsilon + \eta)]a_3 = 0 \tag{10}$$

By following Lucchesi *et al.* (1995), Lucchesi *et al.* (2000) and Padovani *et al.* (2000), the condition $a_1 = a_2 = a_3 = 0$ defines a subset of *Sym* in which the behavior of the material is linearly elastic and the condition Eq. (3b) determines it.

Specifically, in the domain:

$$\Re_1 = \begin{Bmatrix} \boldsymbol{E} \in Sym;\ 2e_1 + \alpha I_1 - (\varepsilon + \eta) \le 0 \\ 2e_2 + \alpha I_1 - (\varepsilon + \eta) \le 0 \\ 2e_3 + \alpha I_1 - (\varepsilon + \eta) \le 0 \end{Bmatrix} \tag{11}$$

the behavior of the material is linearly elastic and the following relations hold:

$$t_i = 2\mu e_i + \lambda I_1 - \sigma^0 \qquad i = 1, 2, 3 \tag{12a}$$

$$\beta_0 = \gamma_0 = \lambda I_1 - \sigma^0 \tag{12b}$$

$$\beta_1 = \gamma_1 = 2\mu, \quad \beta_2 = 0 \tag{12c}$$

where by t_i, $i = 1, 2, 3$ we denote the principal stress of $\boldsymbol{T}$. Furthermore, the coefficients β_i, $i = 0, 1, 2$ are determined equating Eq. (12a) to the components of Eq. (4) in the principal reference frame constituted by the eigenvectors of $\boldsymbol{E}$.

The conditions $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$ define a domain dominated by the anelastic strains. The system obtained by Eqs. (8), (9) and (10) determines the value of a_1, a_2 and a_3 and the semi-positiveness of the anelastic strain $\boldsymbol{E}^a$ determines the domain:

$$\Re_2 = \left\{ \boldsymbol{E} \in Sym;\ e_1 - \frac{\varepsilon + \eta}{2 + 3\alpha} \ge 0 \right\} \tag{13a}$$

$$a_i = e_i - \frac{\varepsilon + \eta}{2 + 3\alpha}, \qquad i = 1, 2, 3 \tag{13b}$$

$$t_i = (1 - \delta)\sigma + \delta(2\mu e_i + \lambda I_1 - \sigma^0), \qquad i = 1, 2, 3 \tag{13c}$$

$$\beta_0 = \sigma, \quad \gamma_0 = \lambda I_1 - \sigma^0, \quad \beta_1 = \beta_2 = 0, \quad \gamma_1 = 2\mu \tag{13d}$$

Let us denote by $\Delta t_i(\delta) = t_i(\delta) - t_i(0)$, $i = 1, 2, 3$ the principal extra stress. Thus, we have:

$$\Delta t_i(\delta) = \delta[2\mu e_i + \lambda I_1 - (\sigma^0 + \sigma)], \qquad i = 1, 2, 3 \tag{14}$$

In the particular case, defined by $\alpha = 0$, i.e., when the Poisson's modulus vanishes, we obtain $\Delta t_i(\delta) = \delta[2\mu e_i - (\sigma^0 + \sigma)]$, $i = 1, 2, 3$ where the role played by δ results evident.

Next, setting $a_1 = 0$, $a_2 \neq 0$ and $a_3 \neq 0$, and solving the system provided by Eqs. (9) and (10), we find the anelastic strains and by Eq. (5a) the principal stresses. Thus, we have:

$$\mathfrak{R}_3 = \left\{ \boldsymbol{E} \in Sym;\ e_1 - \frac{\varepsilon + \eta}{2 + 3\alpha} < 0,\ \alpha e_1 + 2(1 + \alpha)e_2 - (\varepsilon + \eta) > 0 \right\} \tag{15a}$$

$$a_2 = e_2 + \frac{\alpha}{2(1 + \alpha)}e_1 - \frac{(\varepsilon + \eta)}{2(1 + \alpha)} \tag{15b}$$

$$a_3 = e_3 + \frac{\alpha}{2(1 + \alpha)}e_1 - \frac{(\varepsilon + \eta)}{2(1 + \alpha)} \tag{15c}$$

$$t_1 = (1 - \delta)\left\{ \frac{\mu}{(1 + \alpha)}[(2 + 3\alpha)e_1 + \alpha\varepsilon - \eta] \right\} + \delta(2\mu e_1 + \lambda I_1 - \sigma^0) \tag{15d}$$

$$t_2 = (1 - \delta)\sigma + \delta(2\mu e_2 + \lambda I_1 - \sigma^0) \tag{15e}$$

$$t_3 = (1 - \delta)\sigma + \delta(2\mu e_3 + \lambda I_1 - \sigma^0) \tag{15f}$$

where the definition of the domain $\mathfrak{R}_3$ is obtained by using in Eqs. (15) the inequalities $(t_1 - \sigma) < 0$ (with $\delta = 0$) and $a_2 > 0$. Next, setting $E = \mu(2 + 3\alpha)/(1 + \alpha)$ (the Young's modulus of the masonry), and using Eq. (5b) and Eqs. (15d), (e), (f) we obtain the coefficients β_i, $i = 0, 1, 2$:

$$\beta_0 = \mu\frac{e_1^2(1 + \alpha)\varepsilon - e_1\{e_3(1 + \alpha)\varepsilon - e_2[e_3(2 + 3\alpha) - (1 + \alpha)\varepsilon]\} + e_2e_3(\alpha\varepsilon - \eta)}{(e_2 - e_1)(e_3 - e_1)(1 + \alpha)}$$

$$\beta_1 = \frac{E(e_2 + e_3)}{(e_2 - e_1)(e_3 - e_1)}\left(\frac{\varepsilon + \eta}{2 + 3\alpha} - e_1\right)$$

$$\beta_2 = \frac{E}{(e_2 - e_1)(e_3 - e_1)}\left(e_1 - \frac{\varepsilon + \eta}{2 + 3\alpha}\right) \tag{16}$$

For the domain $\mathfrak{R}_3$, the extra stresses are:

$$\Delta t_1(\delta) = \delta\left\{ 2\mu e_1 + \lambda I_1 - \sigma^0 - \frac{\mu}{(1 + \alpha)}[(2 + 3\alpha)e_1 + \alpha\varepsilon - \eta] \right\}$$

$$\Delta t_2(\delta) = \delta[2\mu e_2 + \lambda I_1 - (\sigma + \sigma^0)]$$

$$\Delta t_3(\delta) = \delta[2\mu e_3 + \lambda I_1 - (\sigma + \sigma^0)] \tag{17}$$

and, in the case $\alpha = 0$, we obtain $\Delta t_1(\delta) = 0$ by which the expression of t_1 in Eqs. (15) is exact.

As above, by setting $a_1 = a_2 = 0$ and $a_3 \neq 0$, it is possible to define the domain $\mathfrak{R}_4$:

$$\mathfrak{R}_4 = \left\{ \begin{array}{ll} \boldsymbol{E} \in Sym; & \alpha e_2 + 2(1 + \alpha)e_1 - (\varepsilon + \eta) < 0 \\ & \alpha e_1 + 2(1 + \alpha)e_2 - (\varepsilon + \eta) \le 0 \\ & \alpha I_1 + 2e_3 - (\varepsilon + \eta) > 0 \end{array} \right\}$$

$$a_3 = e_3 + \frac{\alpha}{(2 + \alpha)}(e_1 + e_2) - \frac{(\varepsilon + \eta)}{(2 + \alpha)}$$

$$t_1 = (1-\delta)\left\{\frac{\mu}{(2+\alpha)}[4(1+\alpha)e_1 + 2\alpha e_2 + \alpha\varepsilon - 2\eta]\right\} + \delta(2\mu e_1 + \lambda I_1 - \sigma^0)$$

$$t_2 = (1-\delta)\left\{\frac{\mu}{(2+\alpha)}[4(1+\alpha)e_2 + 2\alpha e_1 + \alpha\varepsilon - 2\eta]\right\} + \delta(2\mu e_2 + \lambda I_1 - \sigma^0)$$

$$t_3 = (1-\delta)\sigma + \delta(2\mu e_3 + \lambda I_1 - \sigma^0) \tag{18}$$

Then, by setting $\psi = 2\mu/(2+\alpha)$, we obtain:

$$\beta_0 = -\psi\frac{2\alpha e_1^2 e_3 + e_1\{e_2[2e_3(2+3\alpha)-(2+\alpha)\varepsilon] + e_3(\alpha\varepsilon - 2\eta - 2e_3\alpha)\} + e_3(e_2-e_3)(2e_2\alpha + \alpha\varepsilon - 2\eta)}{2(e_3-e_1)(e_3-e_2)}$$

$$\beta_1 = \frac{\psi}{(e_3-e_1)(e_3-e_2)}[\alpha I_2 + 2e_3^2 + (2+3\alpha)e_1e_2 - (e_1+e_2)(\varepsilon+\eta)]$$

$$\beta_2 = -\frac{\psi}{(e_3-e_1)(e_3-e_2)}[\alpha I_1 + 2e_3 - (\varepsilon+\eta)] \tag{19}$$

Here, the extra stresses are:

$$\Delta t_1(\delta) = \delta\left\{(2\mu e_1 + \lambda I_1 - \sigma^0) - \frac{\mu}{(2+\alpha)}[4(1+\alpha)e_1 + 2\alpha e_2 + \alpha\varepsilon - 2\eta]\right\}$$

$$\Delta t_2(\delta) = \delta\left\{(2\mu e_2 + \lambda I_1 - \sigma^0) - \frac{\mu}{(2+\alpha)}[4(1+\alpha)e_2 + 2\alpha e_1 + \alpha\varepsilon - 2\eta]\right\}$$

$$\Delta t_3(\delta) = \delta[2\mu e_3 + \lambda I_1 - (\sigma + \sigma^0)] \tag{20}$$

and, by setting $\alpha = 0$, we obtain $\Delta t_1(\delta) = \Delta t_2(\delta) = 0$.

Concluding, the error supplied by δ on the principal stress depends on its assumed small value and on the Poisson's modulus which for the masonry is small too.

2.1 Remark

The problem to model elastic materials with different behavior in tension and compression is not new in the literature. Such a type of problem was studied by Green and Mkrtichian (1977) in the framework of non-linear elasticity, Curnier *et al.* (1995) in the case of small strain and anisotropic materials, Padovani (2000) in the isotropic case and plane strain, Exadaktylos *et al.* (1999) for non-linear materials, small strain and uniaxial stress.

We would emphasize that, in the case of no-tensile strength and in absence of thermal strain, the modified constitutive Eq. (5a) is not only the expression of a numerical stabilization procedure but it is able to describe the behavior of the bi-modular materials where the stress-strain law is shown in Fig. 2 in the case of the uniaxial strain.

When the Poisson's modulus vanishes ($\alpha = 0$), Eq. (5a) provides exact results for the material response in function of the assumed value of the parameter δ and these results are in accord to those

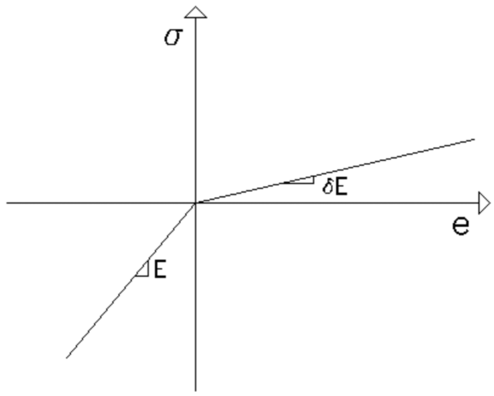

Fig. 2 The uniaxial stress-strain law for a bi-modular material

supplied by Padovani (2000). When the Poisson's modulus is not equal to zero, Eq. (5a) describes the behavior of a model which may be adopted as an alternative to that one provided by Padovani (2000).

3. The numerical solution procedures

In this section, we illustrate the numerical procedures which can be adopted to solve the boundary equilibrium problem of a masonry subjected to thermal strain by using the finite element method. In the framework of the iterative methods, we consider the Newton-Raphson method based on the tangent elastic operator and the direct method, i.e., the secant approach. For such methods, some properties of the elastic operator will be discussed in order to analyze the convergence of the numerical method.

3.1 The Newton-Raphson method

As well known, the Newton-Raphson method is based on the residual load redistribution and uses the tangent elastic stiffness. For the isotropic materials, the tangent operator was obtained by Lucchesi *et al.* (1995) by deriving Eq. (4) respect to the total strain $\boldsymbol{E}$. Here, we consider the derivative of Eq. (5b) made respect to $\boldsymbol{E}$. Thus setting:

$$\begin{aligned}
\alpha_1 &= (1-\delta)\frac{\partial\beta_0}{\partial I_1} + \delta\lambda & \alpha_2 &= (1-\delta)2\frac{\partial\beta_0}{\partial I_2} \\
\alpha_3 &= (1-\delta)2\frac{\partial\beta_1}{\partial I_2} & \alpha_4 &= (1-\delta)3\frac{\partial\beta_0}{\partial I_3} \\
\alpha_5 &= (1-\delta)3\frac{\partial\beta_1}{\partial I_3} & \alpha_6 &= (1-\delta)3\frac{\partial\beta_2}{\partial I_3} \\
\alpha_7 &= (1-\delta)\beta_1 + \delta 2\mu & \alpha_8 &= (1-\delta)\beta_2
\end{aligned} \tag{21}$$

we obtain:

$$D_{\boldsymbol{E}}\boldsymbol{T} = \alpha_1 \mathbf{1}\otimes\mathbf{1} + \alpha_2(\mathbf{1}\otimes\boldsymbol{E}+\boldsymbol{E}\otimes\mathbf{1}) + \alpha_3\boldsymbol{E}\otimes\boldsymbol{E} + \alpha_4(\mathbf{1}\otimes\boldsymbol{G}+\boldsymbol{G}\otimes\mathbf{1}) + \alpha_5(\boldsymbol{E}\otimes\boldsymbol{G}+\boldsymbol{G}\otimes\boldsymbol{E}) + \alpha_6\boldsymbol{G}\otimes\boldsymbol{G} + \beta_7\boldsymbol{I} + \beta_8(\mathbf{1}\,\overline{\underline{\otimes}}\,\boldsymbol{E} + \boldsymbol{E}\,\overline{\underline{\otimes}}\,\mathbf{1}) \tag{22}$$

where $\boldsymbol{G} = \boldsymbol{E}^2$ and $\overline{\underline{\otimes}}$ is the tensor product between two second-order tensors such that:

$$(\boldsymbol{A}\,\overline{\underline{\otimes}}\,\boldsymbol{B})\boldsymbol{C} = \boldsymbol{A}\boldsymbol{C}\boldsymbol{B}^T \ \forall\ \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in Lin \tag{23}$$

We have recomputed the derivatives of the coefficients β_i, $i = 0, 1, 2$ made respect to the invariants I_i, $i = 1, 2, 3$ of $\boldsymbol{E}$ finding that they depend on the thermal strain only by the coefficient β_2; thus, the expressions shown in the paper of Lucchesi *et al.* (1995) still hold.

As announced in the previous Section, we prove that the tangent elastic tensor defined by Eq. (22) is positive definite when $\delta > 0$. By following Ogden (1997), Chapter 6 and Appendix, the positiveness of the tangent elastic tensor is equivalent to require the positiveness of the engineering tangent elasticity matrix $\boldsymbol{D}_{ref}$ expressed in the principal frame constituted by the eigenvectors of the strain $\boldsymbol{E}$ or the stress $\boldsymbol{T}$:

$$[\boldsymbol{D}_{ref}] = \begin{bmatrix} \frac{\partial t_1}{\partial e_1} & \frac{\partial t_1}{\partial e_2} & \frac{\partial t_1}{\partial e_3} & 0 & 0 & 0 \\ \frac{\partial t_2}{\partial e_1} & \frac{\partial t_2}{\partial e_2} & \frac{\partial t_2}{\partial e_3} & 0 & 0 & 0 \\ \frac{\partial t_3}{\partial e_1} & \frac{\partial t_3}{\partial e_2} & \frac{\partial t_3}{\partial e_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\frac{t_1-t_2}{e_1-e_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\frac{t_1-t_3}{e_1-e_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\frac{t_2-t_3}{e_2-e_3} \end{bmatrix} \tag{24}$$

where t_i, $i = 1, 2, 3$ and e_i, $i = 1, 2, 3$ are the principal stress and strain, respectively.

Thus, by the inequalities on the Lame's moduli and on α, it is easy to show that in the region $\Re_1$ the elastic tensor is positive definite. In the region $\Re_2$, by using Eq. (13c), we obtain:

$$[\boldsymbol{D}_{ref}] = \delta\mu \begin{bmatrix} (2+\alpha) & \alpha & \alpha & 0 & 0 & 0 \\ \alpha & (2+\alpha) & \alpha & 0 & 0 & 0 \\ \alpha & \alpha & (2+\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{25}$$

Then, it is evident that the elastic tensor is positive definite if and only if $\delta > 0$. Next, we consider the strain $\boldsymbol{E}$ belonging to $\Re_3$; we obtain:

$$
\begin{aligned}
D_{ref}^{11} &= (1-\delta)E + \delta\mu(2+\alpha) \\
D_{ref}^{12} &= D_{ref}^{21} = \delta\mu\alpha, \qquad D_{ref}^{13} = \delta\mu\alpha \\
D_{ref}^{22} &= \delta\mu(2+\alpha), \qquad D_{ref}^{23} = \delta\mu\alpha \\
D_{ref}^{33} &= \delta\mu(2+\alpha) \\
D_{ref}^{44} &= \mu(1-\delta)\frac{(2+3\alpha)e_1-(\varepsilon+\eta)}{2(e_1-e_2)(1+\alpha)} + \mu\delta > 0 \\
D_{ref}^{55} &= \mu(1-\delta)\frac{(2+3\alpha)e_1-(\varepsilon+\eta)}{2(e_1-e_2)(1+\alpha)} + \mu\delta > 0 \\
D_{ref}^{66} &= \mu\delta > 0
\end{aligned}
\tag{26}
$$

and, when $\boldsymbol{E}$ belongs to $\Re_4$, we have:

$$
\begin{aligned}
D_{ref}^{11} &= (1-\delta)\frac{4\mu(1+\alpha)}{2+\alpha} + \delta\mu(2+\alpha) \\
D_{ref}^{12} &= D_{ref}^{21} = (1-\delta)\frac{2\mu\alpha}{2+\alpha} + \delta\mu\alpha, \qquad D_{ref}^{13} = D_{ref}^{31} = \delta\mu\alpha \\
D_{ref}^{22} &= (1-\delta)\frac{4\mu(1+\alpha)}{2+\alpha} + \delta\mu(2+\alpha), \qquad D_{ref}^{23} = D_{ref}^{32} = \delta\mu\alpha \\
D_{ref}^{33} &= \delta\mu(2+\alpha), \qquad D_{ref}^{44} = \mu > 0 \\
D_{ref}^{55} &= \frac{(1-\delta)\mu}{(2+\alpha)}\left[\frac{4(1+\alpha)e_1+2\alpha e_2-2(\varepsilon+\eta)}{2(e_1-e_3)}\right] + \mu\delta > 0 \\
D_{ref}^{66} &= \frac{1}{2}\frac{t_2-t_3}{e_2-e_3} = \frac{(1-\delta)\mu}{(2+\alpha)}\left[\frac{4(1+\alpha)e_2+2\alpha e_1-2(\varepsilon+\eta)}{2(e_2-e_3)}\right] + \mu\delta > 0
\end{aligned}
\tag{27}
$$

In these case too, we see that the tangent elastic tensor is positive definite if and only if $\delta > 0$. From a numerical point of view, if a node of the mesh is connected by elements for which the elasticity matrix is not definite positive then the global stiffness matrix become singular and the equation system cannot be solved.

Furthermore, we observe that if the positiveness of $\boldsymbol{D}_{ref}$ holds, then, for the convexity of *Sym*, Eq. (5a) is strictly monotone in the interior points of $\Re_i$, $i = 1, 2, 3, 4$, i.e.:

$$
(\boldsymbol{T}^* - \boldsymbol{T}) \cdot (\boldsymbol{E}^* - \boldsymbol{E}) > 0, \;\; \boldsymbol{E}^* \neq \boldsymbol{E} \in Sym \tag{28}
$$

that is an equivalent condition to assure that the energy is a strictly convex function in the regions $\Re_i$, $i = 1, 2, 3, 4$, of *Sym*.

Finally, defining the derivative of Eq. (5b) respect to $\boldsymbol{E}$:

$$D_{\boldsymbol{E}}\boldsymbol{T} = (1-\delta)D_{\boldsymbol{E}}(\beta_0\mathbf{1} + \beta_1\boldsymbol{E} + \beta_2\boldsymbol{E}^2) + \delta(\lambda I_1\mathbf{1}\otimes\mathbf{1} + 2\mu\boldsymbol{I}) \tag{29}$$

we note that the first part of Eq. (29) is positive semi-definite whereas the second part is positive definite if $\delta > 0$. Therefore, if $\delta > 0$, then $D_{\boldsymbol{E}}\boldsymbol{T}$ is positive definite.

Lucchesi *et al.* (1996) proposed a different expression of the tangent elastic tensor which deduction is based on the derivatives of the eigenvalues and eigenvectors of the stress tensor $\boldsymbol{T}$ respect to the total strain $\boldsymbol{E}$. The obtained expression, which requires a strong mathematical apparatus, depends explicitly on the eigenverctors of the strain $\boldsymbol{E}$ and on the principal strains e_i, $i = 1, 2, 3$. An alternative result can be obtained in a very simple way.

Having in mind that the stress tensor $\boldsymbol{T}$ and the strain $\boldsymbol{E}$ are coaxial, the key idea is to use the tangent elastic matrix described in the principal reference system frame constituted by the eigenvectors of the strain $\boldsymbol{E}$ and rotate it in the given reference system frame.

Let us denote by $\boldsymbol{l}$, $\boldsymbol{m}$ and $\boldsymbol{n}$ the eigenvectors corresponding to the principal strains e_i, $i = 1, 2, 3$ and denote by $\boldsymbol{Q}$ the matrix which maps the vector of the engineering components of the strain $\boldsymbol{\varepsilon}$ onto the engineering vector of the principal strains $\boldsymbol{e}$ such that $\boldsymbol{e} = \boldsymbol{Q}\boldsymbol{\varepsilon}$. Thus:

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1^2 & l_2^2 & l_3^2 & l_1l_2 & l_1l_3 & l_2l_3 \\ m_1^2 & m_2^2 & m_3^2 & m_1m_2 & m_1m_3 & m_2m_3 \\ n_1^2 & n_2^2 & n_3^2 & n_1n_2 & n_1n_3 & n_2n_3 \\ 2l_1m_1 & 2l_2m_2 & 2l_3m_3 & l_2m_1+m_2l_1 & l_1m_3+m_1l_3 & l_2m_3+m_2l_3 \\ 2l_1n_1 & 2l_2n_2 & 2l_3n_3 & l_2n_1+n_2l_1 & l_1n_3+n_1l_3 & l_2n_3+n_2l_3 \\ 2m_1n_1 & 2m_2n_2 & 2m_3n_3 & m_2n_1+n_2m_1 & m_1n_3+n_1m_3 & m_2n_3+n_2m_3 \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{13} \\ 2E_{23} \end{bmatrix} \tag{30}$$

Next, let us define by $\boldsymbol{Q}^T$ the matrix which maps the engineering components of the principal stress $\boldsymbol{t}$ onto the vector of the engineering components of the stress $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma} = \boldsymbol{Q}^T\boldsymbol{t}$. Then:

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{13} \\ T_{23} \end{bmatrix} = \begin{bmatrix} l_1^2 & m_1^2 & n_1^2 & 2l_1m_1 & 2l_1n_1 & 2m_1n_1 \\ l_2^2 & m_2^2 & n_2^2 & 2l_2m_2 & 2l_2n_2 & 2m_2n_2 \\ l_3^2 & m_3^2 & n_3^2 & 2l_3m_3 & 2l_3n_3 & 2m_3n_3 \\ l_1l_2 & m_1m_2 & n_1n_2 & l_1m_2+m_1l_2 & l_1n_2+n_1l_2 & m_1n_2+n_1m_2 \\ l_1l_3 & m_1m_3 & n_1n_3 & l_1m_3+m_1l_3 & l_1n_3+n_1l_3 & m_1n_3+n_1m_3 \\ l_2l_3 & m_2m_3 & n_2n_3 & l_2m_3+m_2l_3 & l_2n_3+n_2l_3 & m_2n_3+n_2m_3 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{31}$$

Finally, if $\boldsymbol{D}_{ref}$ is the tangent elasticity matrix expressed in the principal reference frame constituted by the eigenvectors of the strain $\boldsymbol{E}$, then the tangent engineering elasticity matrix $\boldsymbol{D}$ expressed in the given reference system frame is:

$$\boldsymbol{D} = \boldsymbol{Q}^T\boldsymbol{D}_{ref}\boldsymbol{Q} \tag{32}$$

3.2 The secant method

The secant method for no-tension materials was presented firstly by Romano and Sacco (1984) and Sacco (1990) which provided an explicit form for the secant operator. For it, an alternative form was presented by Alfano *et al.* (2000) in the isotropic case of the plane stress/strain.

The secant method is based on the definition of a symmetric fourth-order tensor (function of the actual strain $\boldsymbol{E}$) such that if $\boldsymbol{T}$ is the stress tensor corresponding to the strain $\boldsymbol{E}$ then:

$$\boldsymbol{T} = -\boldsymbol{C}\boldsymbol{E}^t + \boldsymbol{C}_{sec}\boldsymbol{E} \tag{33}$$

There is not a criterion on which to choice the more convenient form of the secant operator and, on the suggestion of Alfano *et al.* (2000), we consider the following expression:

$$\boldsymbol{C}_{sec} = \alpha_1 \mathbf{1} \otimes \mathbf{1} + \alpha_2(\mathbf{1} \otimes \boldsymbol{E} + \boldsymbol{E} \otimes \mathbf{1}) + \alpha_3 \boldsymbol{E} \otimes \boldsymbol{E} + \alpha_6 \boldsymbol{E}^2 \otimes \boldsymbol{E}^2 + \alpha_7 \boldsymbol{I} \tag{34}$$

where $\alpha_1 = \lambda$, $\alpha_7 = 2\mu$ and α_2, α_3 and α_6 to be determine. In view of the definition (33) and by the aid of Eq. (4) we have:

$$\beta_0 \mathbf{1} + \beta_1 \boldsymbol{E} + \beta_2 \boldsymbol{E}^2 = -\sigma^0 \mathbf{1} + \alpha_1 I_1 \mathbf{1} + \alpha_2(I_2 \mathbf{1} + I_1 \boldsymbol{E}) + \alpha_3 I_2 \boldsymbol{E} + \alpha_6 I_3 \boldsymbol{E}^2 + \alpha_7 \boldsymbol{E} \tag{35}$$

Thus, by equating term to term and solving the system, we obtain:

$$\begin{aligned} \alpha_2 &= \frac{1}{I_2}(\beta_0 + \sigma^0 - \lambda I_1) \\ \alpha_3 &= \frac{1}{I_2}\left[\beta_1 - 2\mu - \frac{I_1}{I_2}(\beta_0 + \sigma^0 - \lambda I_1)\right] \\ \alpha_6 &= \frac{\beta_2}{I_3} \end{aligned} \tag{36}$$

where the coefficients β_i, $i = 0, 1, 2$ have been determined previously.

It remains to show the conditions on which the elasticity secant tensor is positive definite. This is an open question to which, actually, the answer is only of a numerical type. Indeed, in the case of the bounded tensile strength, several numerical investigations, i.e., the extraction of the eigenvalues by the secant elasticity matrix, have shown the positiveness of the secant tensor.

Romano and Sacco (1984) and Sacco (1990) proposed the following form of the secant elasticity tensor:

$$\boldsymbol{C}_{sec} = \boldsymbol{C} - \frac{\boldsymbol{C}\boldsymbol{E}^a \otimes \boldsymbol{C}\boldsymbol{E}^a}{\boldsymbol{C}\boldsymbol{E}^a \cdot \boldsymbol{E}} \tag{37}$$

where it is easy to verify that the condition $\boldsymbol{T} = -\boldsymbol{C}\boldsymbol{E}^t + \boldsymbol{C}_{sec}\boldsymbol{E} = \boldsymbol{C}(\boldsymbol{E} - \boldsymbol{E}^a - \boldsymbol{E}^t)$ is fulfilled.

Notice that Eq. (37) holds for a general anisotropic material subjected to thermal strain and that its explicit form depends on the eigenvectors of the elastic strain $\boldsymbol{E}$ as was shown by Romano and Sacco (1984) in the case of the isotropy. In this case, it is possible to place Eq. (37) into an alternative elegant form. Indeed, let us consider the non linear function $\wp$ that maps the strain $\boldsymbol{E}$ onto the anelastic strain $\boldsymbol{E}^a$.

Since $\boldsymbol{E}$ and $\boldsymbol{E}^a$ are coaxial, then the function $\wp$ is isotropic and there exist three scalar functions ω_i, i = 0, 1, 2 of the invariants of $\boldsymbol{E}$ such that:

$$\boldsymbol{E}^a = \omega_0 \mathbf{1} + \omega_1 \boldsymbol{E} + \omega_2 \boldsymbol{E}^2 \tag{38}$$

Splitting Eq. (38) into the reference frame constituted by the eigenvectors of the strain $\boldsymbol{E}$ and equating to the principal anelastic strains, we obtain a system that determines the coefficients ω_i, $i = 0, 1, 2$. Thus, for the region $\Re_2$, by Eqs. (13b) we obtain the following system:

$$\begin{aligned} \omega_0 + \omega_1 e_1 + \omega_2 e_1^2 &= e_1 - \frac{\varepsilon + \eta}{2 + 3\alpha} \\ \omega_0 + \omega_1 e_2 + \omega_2 e_2^2 &= e_2 - \frac{\varepsilon + \eta}{2 + 3\alpha} \\ \omega_0 + \omega_1 e_3 + \omega_2 e_3^2 &= e_3 - \frac{\varepsilon + \eta}{2 + 3\alpha} \end{aligned} \tag{39}$$

by which:

$$\omega_0 = -\frac{\varepsilon + \eta}{2 + 3\alpha}, \quad \omega_1 = 1, \quad \omega_2 = 0 \tag{40}$$

Thus:

$$\text{If } \boldsymbol{E} \in \Re_3: \quad \omega_0 = \frac{e_1\{(e_1 - e_3)(\alpha e_1 - \varepsilon - \eta) - e_2[\alpha e_1 + 2e_3(1 + \alpha) - (\varepsilon + \eta)]\}}{2(1 + \alpha)(e_1 - e_2)(e_1 - e_3)}$$

$$\omega_1 = \frac{\alpha e_1(e_2 + e_3) + 2e_1^2(1 + \alpha) + e_2[2e_3(1 + \alpha) - \varepsilon - \eta] - e_3(\varepsilon + \eta)}{2(1 + \alpha)(e_1 - e_2)(e_1 - e_3)}$$

$$\omega_2 = \frac{-e_1(2 + 3\alpha) + (\varepsilon + \eta)}{2(1 + \alpha)(e_1 - e_2)(e_1 - e_3)} \tag{41}$$

and

$$\text{If } \boldsymbol{E} \in \Re_4: \quad \omega_0 = \frac{e_1 e_2[\alpha(e_1 + e_2) + e_3(2 + \alpha) - (\varepsilon + \eta)]}{(2 + \alpha)(e_1 - e_3)(e_2 - e_3)}$$

$$\omega_1 = \frac{(e_1 + e_2)[\alpha(e_1 + e_2) + e_3(2 + \alpha) - (\varepsilon + \eta)]}{(2 + \alpha)(e_1 - e_3)(e_2 - e_3)}$$

$$\omega_2 = \frac{\alpha(e_1 + e_2) + e_3(2 + \alpha) - (\varepsilon + \eta)}{(2 + \alpha)(e_1 - e_3)(e_2 - e_3)} \tag{42}$$

Furthermore:

$$\begin{aligned} \boldsymbol{C}\boldsymbol{E}^a \otimes \boldsymbol{C}\boldsymbol{E}^a &= \omega_0^2 \mathbf{1} \otimes \mathbf{1} + \omega_0\omega_1(\mathbf{1} \otimes \boldsymbol{E} + \boldsymbol{E} \otimes \mathbf{1}) + \omega_0\omega_2(\mathbf{1} \otimes \boldsymbol{E}^2 + \boldsymbol{E}^2 \otimes \mathbf{1}) + \\ &\quad \omega_1^2 \boldsymbol{E} \otimes \boldsymbol{E} + \omega_1\omega_2(\boldsymbol{E} \otimes \boldsymbol{E}^2 + \boldsymbol{E}^2 \otimes \boldsymbol{E}) + \omega_2^2 \boldsymbol{E}^2 \otimes \boldsymbol{E}^2 \\ \boldsymbol{C}\boldsymbol{E}^a \cdot \boldsymbol{E} &= 2\mu(\omega_0 I_1 + \omega_1 I_2 + \omega_2 I_3) + \lambda I_1(3\omega_0 + \omega_1 I_1 + \omega_2 I_2) \end{aligned} \tag{43}$$

4. The numerical examples

The proposed numerical method has been implemented on an existing structural FEM code named *Solver* that is distributed and commercialized in Italy by a software house.

In the numerical examples we use an eight-nodes finite element that is based on the formulation of Simo and Rifai (1990) extended to the 3-D case in the framework of the assumed strain method by using the incompatible displacement field of Wilson *et al.* (1973). The details on the formulation of the finite element will be illustrated in a forthcoming paper and, at this stage, one can refer to the paper of Pimpinelli (2003) for the case 2-D.

In order to illustrate the effectiveness of the numerical method developed in the preceding sections, we perform some numerical simulations. We begin to consider a rectangular block subjected to a trapezoidal load and to a thermal load. The second example considers a cantilever beam made of a bi-modular material subjected to a constant curvature. The same example is analyzed under a constant curvature and under a uniform thermal strain in the cases of no-tensile strength and bounded tensile strength, respectively. In order to show the effects of the thermal strains on the stress state of a masonry structure, firstly a double-clamped masonry wall under the actions of the self weight, a uniform load and a thermal load is analyzed and next it is performed an analysis on a clamped masonry arc subjected to the self weigh, a concentrated load and to a temperature variation.

4.1 The rectangular block

This is a rectangular block in plane stress state supported by a rigid plane. A trapezoidal load, as shown in Fig. 3, loads the block that is subject to a thermal strain too. The material is assumed with no-tensile strength.

The block is discretized first into fifty brick finite elements and next into two hundred elements. The calculation is performed by using the tangent method as provided by Lucchesi *et al.* (1995) with a relative error on the displacements $\varepsilon = 1.0 \times 10^{-5}$ and by using our secant method (see Eq. (34)) with a relative error on the displacements $\varepsilon = 1.0 \times 10^{-4}$. The analytic solution is shown in paper of Pimpinelli (2003) as well as the data used for the example.

The results analysis are presented in Table 1 which shows that they are in agreement to the

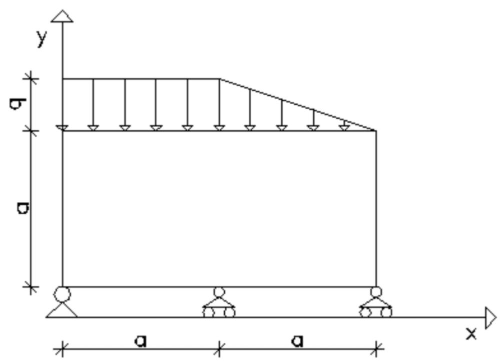

Fig. 3 The rectangular block

Table 1 The rectangular block

Displacements v (m) × 10^{-4} and stress σ (GPa) × 10^{-3}					
	50 elements		200 elements		Theoretical
Method	Tangent	Secant	Tangent	Secant	
$v(x = 0, y = a)$	–20.004	–19.918	–19.997	–19.986	–20.0
$v(x = a, y = a)$	–19.660	–19.525	–19.631	–19.564	–20.0
$v(x = 2a, y = a)$	–10.004	–9.8925	–9.9984	–9.9555	–10.0
$\sigma_Y(x = 0, y = a)$	–1.0014	–0.9994	–0.9975	–0.9999	–1.00
$\sigma_Y(x = a, y = a)$	–1.0007	–0.9923	–1.0023	–0.9962	–1.00
$\sigma_Y(x = 2a, y = a)$	–9.77E^{-4}	–4.90E^{-3}	–3.3128E^{-6}	–1.60E^{-3}	0
$\sigma_X(x = 0, y = a)$	–0.009988	–1.54E^{-7}	3.9954E^{-7}	–1.053E^{-6}	0
$\sigma_X(x = a, y = a)$	–0.1086	–0.00015	–0.0039	–0.00087	0
$\sigma_X(x = 2a, y = a)$	–0.0107	–0.0032	–0.000028	–0.00027	0
n. of. iterations	13	89	14	148	=

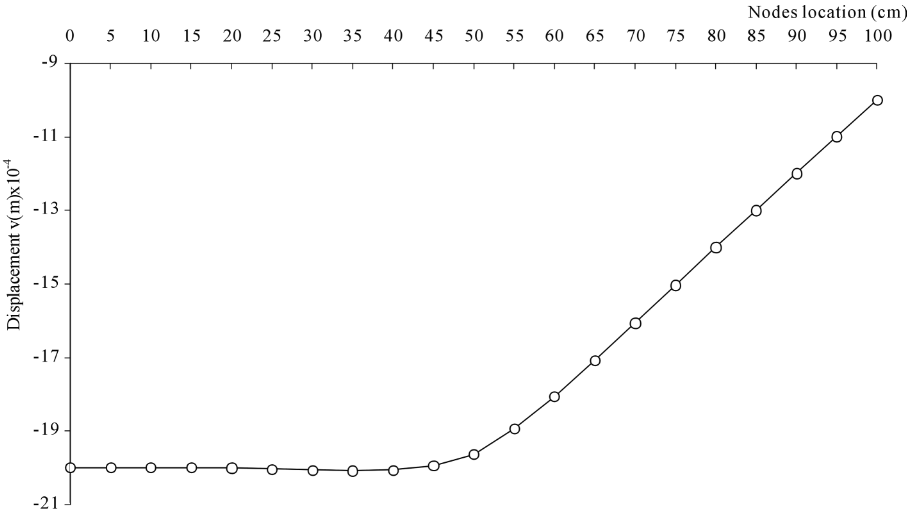

Fig. 4 The rectangular block. Top displacements

analytical values and that the refinement of the mesh has not a meaningful effect on the solution. The use of the coarse mesh is sufficient to fully describe the behavior of the loaded block. Moreover, the convergence ratio for the secant method is very slow and it is not possible to force the tolerance to small values. The comparison between the tangent method (approximated for the presence of the assumed small value of δ) and the secant method (virtually exact) does not show significative differences in terms of displacements and stress. Thus, the approximated tangent method is effective in the convergence ratio too.

In Fig. 4 we illustrate the top block displacement versus the nodal points location.

4.2 The cantilever beam

This is a cantilever beam that is subjected to a constant curvature and the geometry is shown in

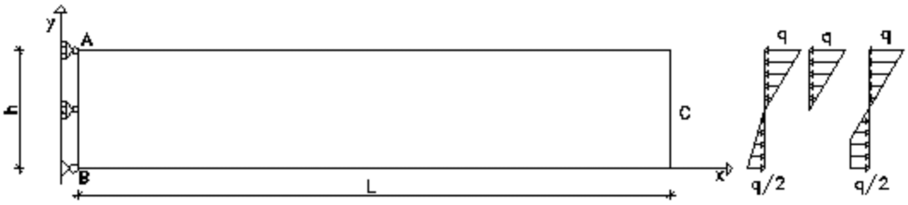

Fig. 5 The cantilever beam

Fig. 5. The proposed example is necessary to test the performance of the proposed numerical method in bending dominated situations. We assume the following geometrical data: $L = 10.0$ m, $h = 2.0$ m, $th = 1.0$ m.

We begin consider the first case for which we assume that the beam is constituted by a bi-modular material and we study it by considering the following data:

$$E_c = 1500 \text{ GPa},\ E_t = 750 \text{ GPa},\ \nu = 0.0,\ \delta = 0.50,\ \varepsilon = 10^{-5}$$

where E_c is the Young's modulus in compression and E_t is the Young's modulus in traction obtained by setting $\delta = 0.5$. The beam is discretized in $20 \times 4 \times 1$ brick elements.

Notice that for this kind of problem, two principal strains are ever equal to zero (different by zero for the double precision computer machine) and this fact has produced a numerical instability in the equation system during the iterative process. In order to avoid such a type of instability is was necessary to assume a small value of the Poisson's ratio. Furthermore the same example was examined by using our proposed tangent elasticity matrix and the convergence was assured. The results analysis are presented in Table 2 by which we note that our proposed numerical method provides exact results.

Then, we have considered the following data:

$$E = 1500 \text{ GPa},\ \nu = 0.25,\ \alpha_t = 0.00025(^{\circ}\mathrm{C})^{-1},\ \Delta\theta = 200^{\circ}\mathrm{C},\ \delta = 0.001,\ \varepsilon = 10^{-5}$$

and, by assuming $\sigma = 0$ and for a constant curvature $\chi = 0.0010$, we obtain $q = 1.50$ GPa. Furthermore we have considered the case $\sigma = 0.75$ GPa.

The theoretical vertical displacement of the cantilever end section at the point C is $v = \frac{\chi L^2}{2} + \frac{\alpha_t \Delta\theta h}{2}$ $= 0.10$ m and the deflection of the middle plane (x-z) is presented in Fig. 6.

Table 2 The bi-modular cantilever beam

Displacements u (m) and stress σ (GPa)		
	Computed	Theoretical
u_y(C)	0.050	0.050
u_x(C)	0	0
σ_x(B)	0.7500	0.75
σ_x(A)	−1.5000	−1.50

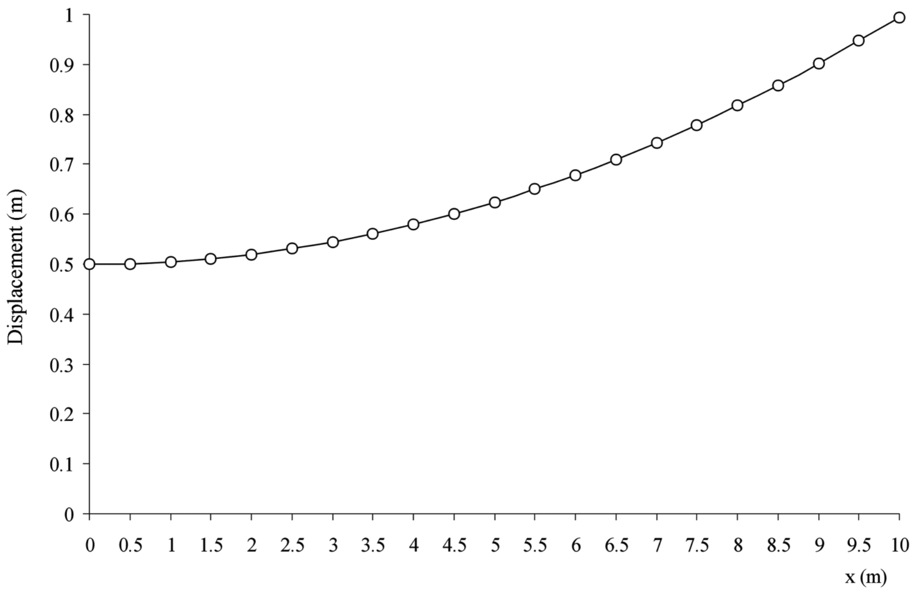

Fig. 6 The deflection of the cantilever beam

Table 3 The cantilever beam under the action of the thermal load

Displacements u (m) and stress σ (GPa)						
	$\sigma = 0$			$\sigma = 0.75$		
	Tangent	Secant	Theoretical	Tangent	Secant	Theoretical
u_y(C)	0.09952	0.099795	0.10	0.9987	0.9986	0.10
u_x(C)	0.4999	0.4999	0.50	0.500	0.5000	0.50
σ_x(B)	0.00179	$-2.8482E^{-6}$	0	0.7508	0.7499	0.75
σ_x(A)	–1.5029	–1.5023	–1.50	–1.5034	–1.5032	–1.50
n. of iterations	8	68	=	4	17	

In these cases too, the method proposed by Lucchesi *et al.* (1995) fails whereas our proposed tangent method reaches the convergence in few iterations. Indeed, also having adopted the corrective factor δ, we have found a negative term on the diagonal of the tangent elastic matrix. This is not attributed to the method of Lucchesi *et al.* (1995) but only to a numerical instability related to the computer precision. Moreover the examples were examined by using the secant method (virtually no approximated) and the results analysis are illustrated in Table 3. As we see, there is not a significative difference between the two methods, but the tangent method is very fast.

4.3 The masonry clamped wall

This is a masonry wall which is clamped at the left and right edges: the geometry is shown in Fig. 7. We assume the following data:

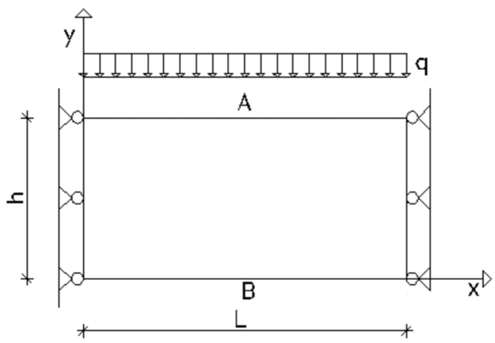

Fig. 7 The clamped wall

Table 4 The clamped wall

Displacements u (m) and stress σ (KPa)		
	$\Delta\theta = 0$	$\Delta\theta = -15^{o}C$
$u_y(A)$	$-1.26E^{-4}$	$-2.81E^{-4}$
$\sigma_x(A)$	-42.73	-58.67
$\sigma_x(B)$	42.72	100.400

$$L = 4.0 \text{ m}, \quad h = 2.0 \text{ m}, \quad th = 0.20 \text{ m}, \quad \gamma = 18000 \text{ N/m}^3, \quad E = 1.0 \text{ GPa}, \quad q = 1.0 \text{ KPa}$$
$$\sigma = 100.0 \text{ KPa}, \quad \nu = 0.25, \quad \alpha_t = 0.00001(^{o}C)^{-1}, \quad \Delta\theta = -15^{o}C, \quad \delta = 0.005$$

The block is discretized into $20 \times 10 \times 1$ brick finite elements and it is analyzed firstly on the action of the self weight and the uniform load q and then in presence of the thermal load too. The results of the analysis are presented in Table 4.

This example is instructive to understand the meaningful modifications that the thermal strains can produce on the stress state in the masonry walls. Indeed, by Table 4, we observe that the effect of the thermal load increases, in absolute value, both the value of the stress σ_x at the top and the bottom of the clamped wall. The effects of the thermal strains on the static of the masonry structures will be more evident in the following example

4.4 The masonry arc

The masonry arc, which geometry is illustrated in Fig. 8, is clamped at the edges and it is subjected to the self weigh and to a concentrated load P. We assume that the temperature at the act of the construction is $\theta = 20^{o}C$ and we want analyze the effect of the temperature decrement up to zero. We assume the following data:

$$R_i = 600 \text{ cm}, \quad R_e = 680 \text{ cm}, \quad th = 30 \text{ cm}, \quad \gamma = 1800 \text{ daN/m}^3, \quad E = 30000 \text{ daN/cm}^2$$
$$P = 800 \text{ daN}, \quad \sigma = 0.10 \text{ daN/cm}^2, \quad \nu = 0.25, \quad \alpha_t = 0.00001(^{o}C)^{-1}, \quad \delta = 0.0005$$

For symmetry reasons, only half arc is analyzed discretizing it into $8 \times 4 \times 1$ brick elements and in

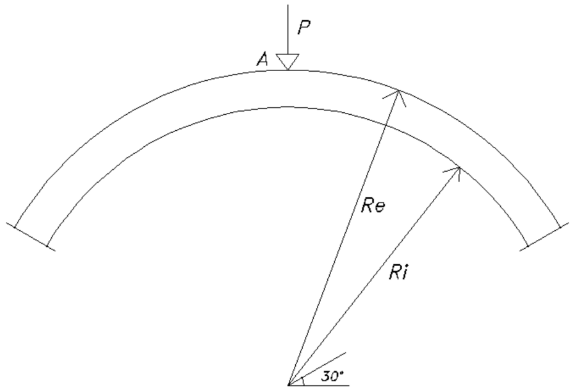

Fig. 8 The masonry arc

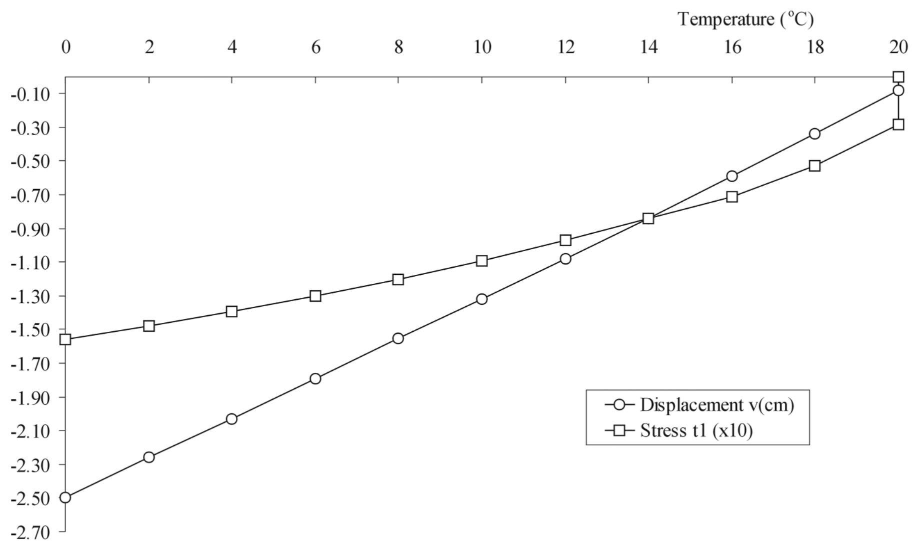

Fig. 9 The masonry arc. The displacement and the principal stress at the point A

Fig. 9 are shown the curves relating the vertical displacement and the minimum principal stress t_1(daN/cm^2) at the point A of the structure versus the temperature.

By Fig. 9 we see that the vertical displacement v at the point A is very small compared to the total displacement when the temperature goes to zero. Furthermore, the principal stress t_1 change its nominal value from 2.86 daN/cm^2 under the loads at $\theta = 20^{o}$C to 15.63 daN/cm^2 at $\theta = 0^{o}$C and the ratio is approximately 5.46. Thus, the temperature variations can compromise the equilibrium of the structure if the compressive principal stresses exceed the compressive strength of the material. In order to evaluate these aspects, a more sophisticated constitutive 3-D model can be formulated taking into account the compressive strength of the material (see Lucchesi *et al.* (1996)) and using the results obtained by the present paper.

5. Conclusions

We have presented constitutive equations and numerical methods to study structural problems concerning bodies made of masonry-like materials with bounded tensile strength in presence of thermal strains. By setting $\delta > 0$, we have shown that the tangent elasticity tensor is positive definite and thus the Newton-Raphson proposed numerical method is stable and convergent. Moreover, the numerical examples show that the method is effective too. The use of the eight-node finite element based on the model by Simo and Rifai (1990) contributes to reduce the equations system number and to obtain a considerable accuracy respect to the use of a twenty-four nodes element.

Furthermore, the convergence rate is fast respect to the secant method and the approximation due to the assumed values of the parameter δ is negligible. Finally, the illustrative examples show that the analysis results obtained by the proposed numerical methods, are in agreement with the theoretical solution.

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