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Marguerre shell type secant matrices for the postbuckling analysis of thin, shallow composite shells

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Abstract. The postbuckling behaviour of thin shells has fascinated researchers because the theoretical prediction and their experimental verification are often different. In reality, shell panels possess small imperfections and these can cause large reduction in static buckling strength. This is more relevant in thin laminated composite shells. To study the postbuckling behaviour of thin, imperfect laminated composite shells using finite elements, explicit incremental or secant matrices have been presented in this paper. These incremental matrices which are derived using Marguerre's shallow shell theory can be used in combination with any thin plate/shell finite element (Classical Laminated Plate Theory - CLPT) and can be easily extended to the First Order Shear deformation Theory (FOST). The advantage of the present formulation is that it involves no numerical approximation in forming total potential energy of the shell during large deformations as opposed to earlier approximate formulations published in the literature. The initial imperfection in shells could be modeled by simply adjusting the ordinate of the shell forms. The present formulation is very easy to implement in any existing finite element codes. The secant matrices presented in this paper are shown to be very accurate in tracing the postbuckling behaviour of thin isotropic and laminated composite shells with general initial imperfections.

Key words: Marguerre's shell theory; secant matrices; postbuckling; nonlinear finite element analysis; shallow shells.

1. Introduction

Thin shell structures are widely used in civil, aerospace and marine structures possibly due their high strength to weight ratio. Nevertheless, their thinness increases the problem of elastic buckling

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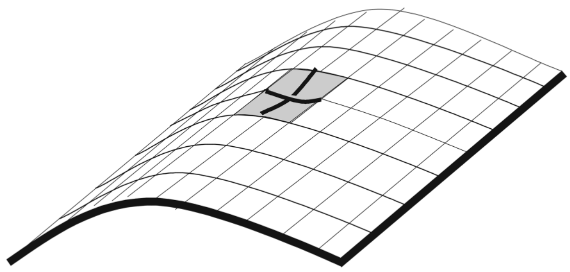

Fig. 1 Curved panel modeled as a plate with initial imperfection

under compressive loads and significant amount of research has been carried out to establish safe design criteria for practical shells. But this effort is still continuing as seen from the research work reported in the literature even in the 2000s.

A schematic diagram of a shell panel with local imperfection is shown in Fig. 1. The postbuckling behaviour of shells have several distinct features compared to the postbuckling behaviour of thin plates. Many shells under compression exhibit small displacement, linear pre-buckling behaviour. Normally, theoretical studies aim at determining this linear elastic critical load of the shell structure. Such analyses are basically linear eigenvalue analyses, which provides the buckling modes of the shell but gives no information on their subsequent stability. Only a study concerning the 'post buckled stability' concentrate on the nonlinear problem of determining the large displacement behaviour shells with plausible assumption of post buckled modes. The axial compression behaviour of shells is highly sensitive to initial imperfections and the imperfections are random in nature. Since imperfections are random in practice, correlating the buckling strength of imperfect shells to the statistical description of the initial imperfection may be a better approach. Nevertheless, such an approach will not be dealt in this paper.

2. Types of shell formulations for postbuckling analysis

The main consideration in the finite element analysis of shell structures is the modeling of geometry of the curved shell surfaces. There are several ways to model the shell curvature and their behaviour. A large number of papers have been published on finite element analysis of shells and the types of elements can be grouped largely into four categories such as

- Flat elements
- Curved elements based on shallow shell theory
- Curved elements based on deep shell theory
- Solid elements

The special finite elements for a particular geometrical shape such as axisymmetric shells and cylindrical shells falls into another category. In the present paper the curved elements based on shallow shell theory would be pursued further.

Many of the shell forms, which are encountered in practice, are shallow in nature. A review of the literature shows that the many of the general shell formulations presented are too rigorous to be implemented in existing finite element (FE) codes. In almost all the FE codes, routines would be available for the computation of displacement gradients that appear in strain-displacement relations.

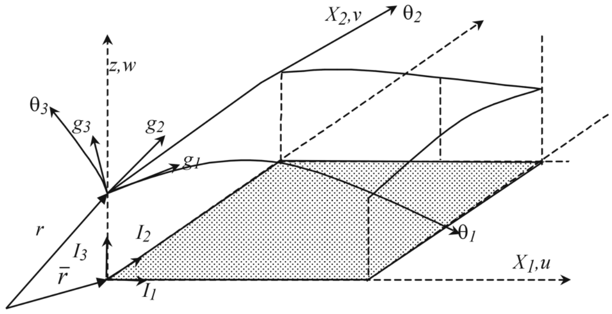

Fig. 2 Shell co-ordinates for Marguerre's shell theory

In the present formulation the secant matrices are presented as functions of these displacement gradients. Hence implementing the present formulation in existing FE codes is easy.

The problem of shallow shell was treated as a problem of a plate with initial curvature by Marguerre (1938). The principle of shallow shell theory lies in introducing the shallow shell terms to the governing strain-displacement relation based on certain assumptions. This theory is well established in the literature.

Representing the shell surface (Fig. 2) by a function $w_0(x_1,\ x_2)$ the membrane strains and curvatures for a shallow shell could be written in an expanded form as

$$\varepsilon_{x1} = \frac{\partial u}{\partial x_1} + \frac{\partial w_0}{\partial x_1}\frac{\partial w}{\partial x_1}$$

$$\varepsilon_{x2} = \frac{\partial v}{\partial x_2} + \frac{\partial w_0}{\partial x_1}\frac{\partial w}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial w_0}{\partial y}\frac{\partial w}{\partial x}$$

$$\chi_1 = \frac{\partial^2 w}{\partial x^2}$$

$$\chi_2 = \frac{\partial^2 w}{\partial y^2}$$

$$\tau = 2\frac{\partial^2 w}{\partial x \partial y} \tag{1}$$

where ε, γ are the membrane strains and χ, τ are the flexural strains. Eq. (1) can be written in the indicial notation form as

$$\varepsilon_{mn} = \frac{1}{2}(u_{m,\,n} + u_{n,\,m}) + \frac{1}{2}(w_{,m}w_{0,\,n} + w_{,n}w_{0,\,m}) \tag{2}$$

$$\varepsilon_{mn} = -zw_{,mn} \tag{3}$$

in which the comma denotes the differentiation of the preceding quantity with respect to the indexed co-ordinates. The strain measures are functions of co-ordinates $x_i = \{x_1, x_2\}$ and hence the range of indices m, n is two. When the geometry of a general shell is approximated by a patch of shallow shell elements as in the present case, with each single element behaves like a shallow shell with respect to the local element plane (Fig. 2). Nevertheless, the curvature in each element is constant. When the shallow shell elements are used to represent the geometry of a deep shell, the initial slope will be discontinuous across the inter-element boundary. Hence as emphasized in many texts, care must be exercised in using this shallow shell elements to deep shell problems.

3. Review of literature

A thorough enumeration of all the works on finite element shell instability problems would not be attempted in this paper as it is a formidable task. Only some major developments are enumerated. For an extensive review of shell formulation the reader may see Crisfield (1991) and Yang (2000).

The first and significant step in developing a theory for post buckling phenomenon on shells was due to Koiter (1945). Koiter's work was on the immediate vicinity of the branching point called the 'initial postbuckling theory'. This theory relies on perturbation methods wherein the difficulty of solving nonlinear equilibrium equation is transformed into solving a sequence of linear problem. Koiter (1945) presented his original work for small initial imperfections and later he came out with procedures for shells with finite imperfections.

A thorough work on finite elements applied to shell problems was carried by Horrigmoe (1977), Horrigmoe and Bergan (1978). They used the co-rotational approach to develop flat shell elements. One of the earliest works on the nonlinear analysis of laminated composite shell was due to Reddy (1981). Sheinman and Simites (1983) presented thin shell formulations for the post buckling of imperfect cylindrical shells under axial compression. Oliver and Onate (1984) presented finite element formulations based on total Lagrangian format for the post buckling analysis of plates and shells. Zhang and Mathews (1985) presented a formulation for the large deflection behaviour laminated composite shell panels using modified Galerkhin's method. Elements based on Marguerre shell theory was also used by Jetteur and Frey (1986) and Stolarski *et al.* (1984). Fu and Chia (1989) used the Marguerre shell theory to study the post buckling of composite panels. The shear deformation effects were included. Saigal *et al.* (1986) presented a curved shell element for the analysis of imperfect laminated shell. Chia (1987) presented shallow shell formulation for the behaviour laminated composite shell panels resting on elastic foundation. Yang *et al.* (1989) presented 48 d.o.f. thin element based on a Love-Kirchoff thin shell theory for the post buckling analysis of shells without membrane locking problems.

Sheinman and Frostig (1990) modified Galerkhin procedure for the post buckling analysis of stiffened laminated curved panels. The computational models for shell problems presented in the literature before 1990 was reviewed by Noor and Burton (1990). Librescu and Chang (1992) presented analytical formulation for the post buckling analysis of multilayered composite curved panels. Palazotto *et al.* (1992) presented formulations for the stability of laminated composite shells using secant matrix formulation. Regarding the degenerated shell elements Boisse *et al.* (1994) and Parish (1995) presented solid element formulation for the nonlinear analysis of shells. Onate *et al.* (1994) proposed a very simple triangular shear locking free shell element. The element has nine degrees of freedom, three displacements at the corners and six rotations at the mid-sides. The element produces no spurious modes and applicable to both thin and thick shell elements. Allman (1995) proposed a shallow curved shell element constructed on a triangular flat facet approximation to general thin elastic shell. The novelty of this element is the quadratic middle surface geometry to allow for accurate modeling of arbitrary curvatures. Madenci and Barut (1994) presented thin shell element for the analysis of thin composite shells. Sabir and Djoudi (1995) presented a shallow shell element for geometrically nonlinear analysis of shells. Regarding the large deflection analysis of composite shells, Pai and Palazotto (1995) presented a total Lagrangian formulation of laminated composite shells undergoing large displacements. Ganapathy and Varadhan (1995) used the secant matrix technique for the nonlinear vibration of laminated composite shells. A comprehensive reference on finite element analysis for the prediction of post buckling strength of composite shells is given by Atluri (1997). Kim *et al.* (1998) presented finite element formulations in updated Lagrangian formulation for the post buckling analysis of composite shells with initial imperfections. A review and evaluation of geometrically nonlinear multilayered shell models was by Erasmo Carrera and Parisch (1998). Chaudri and Raymond (1998) presented a serendipity type thick shell element in the total Lagrangian format in curvilinear co-ordinates. Kheyrkhahan and Ralf Peek (1999) developed a general shell element based on Lyapunov-Schimidt-Koiter (LSK) and postbuckling studies on cylindrical and spherical shells are carried out. Ferreira and Barbosa (2000) presented a layered formulation for the buckling analysis of shells using Marguerre's shell theory. The results were not good compared to the results of Horrigmoe (1977). Teng and Song (2001) presented issues involved in modelling imperfection in shells and the influence of scaled down models in postbuckling analysis. A review of the literature above indicate that in many of the formulations, the curvature modeling is the pivotal issue and simplification of modeling this is done at the cost of accuracy. The present paper attempts to present a simple yet an accurate formulation for the postbuckling analysis of thin shells.

4. Shell formulation based on Green's strain and second Piola Kirchoff stress

The strain energy density U_0 of a Hookean material in a total Lagrangian (TL) format could be written in terms of Second Piola - Kirchoff Stress tensor S_{mn} and Green's strain tensor ε_{mn} as

$$U_0 = \int S_{mn} d\varepsilon_{mn} \tag{4}$$

The deformation of the shell element is described in the total Lagrangian formulation in a fixed right handed rectangular Cartesian frame of reference, with the X_1, X_2 plane coinciding with the middle surface of the plate in its undeformed state and the Z axis normal to it as shown in Fig. 3. The components of the in-plane and out-of plane displacements of a particle of the plate originally at ($x1$, $x2$, z) are denoted by $u_i = \{u_1, u_2\}$ and 'w' respectively. The thickness of the shell is assumed to be very thin and obeys the classical laminated plate theory (CLPT) and the shear deformation is neglected. Since Marguerre's assumption is valid, the strain terms dependent on the square of the membrane displacements u_1, u_2 are neglected.

The Green's strain tensor, based on Marguerre's shell theory can be written combining the membrane and bending strains in terms of additional deflection 'w' as

$$\varepsilon_{mn} = \frac{1}{2}(u_{m,n} + u_{n,m}) + \frac{1}{2}w_{,m}w_{,n} + \frac{1}{2}(w_{,m}w_{0,n} + w_{,n}w_{0,m}) - zw_{,mn} \tag{5}$$

where w_0 is the ordinate of the shell form. Since Green's strain is assumed for the description of motion, it's work conjugate, the second Piola - Kirchoff stress is used as the stress measure. In the

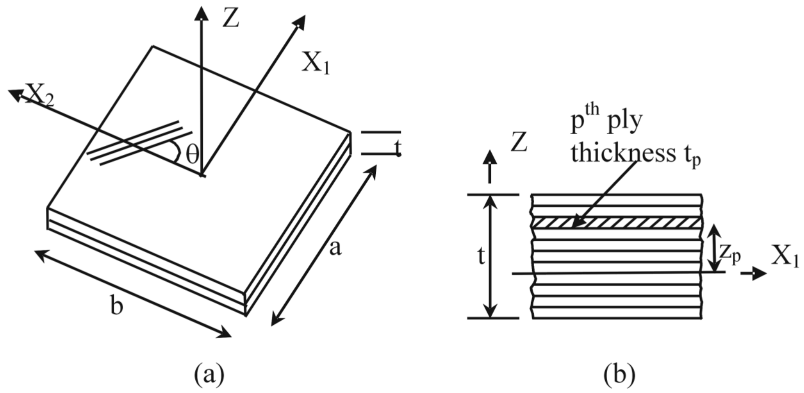

Fig. 3 Co-ordinates of the plate/shell element and ply orientations

case of isotropic material the second Piola - Kirchoff stress tensor can be related to the Green's strain using a fourth order elastic compliance tensor as

$$S_{mn} = C_{mnrs}\varepsilon_{rs} \tag{6}$$

where

$$C_{mnrs} = \lambda\delta_{mn}\delta_{rs} + G(\delta_{mr}\delta_{ns} + \delta_{ms}\delta_{nr}) \tag{7}$$

and λ and G are the usual Lame's constants and δ is the Kronecker delta and the range of indices m, n, r, s, is two. Alternatively, the displacement gradients in the strain terms in Eq. (5) can be collected into a column vector and then the second Piola - Kirchoff stress can be related to the Green's Strain using a second order material compliance tensor as,

$$\tilde{S}_i = \tilde{C}_{ij}\tilde{\varepsilon}_j \tag{8}$$

For a layered composite plate (Fig. 3b) based on the classical laminated plate theory (CLPT), the material compliance relations can be written in terms of the usual stretching, bending and coupling rigidities (Datoo 1991) as

$$\tilde{C}_{ij} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{B} & \tilde{D} \end{bmatrix} \tag{9}$$

Using the parameters defined in Fig. 3, the rigidities can be expressed as Dattoo (1991)

$$\tilde{A}_{ij} = \sum_{p=1}^{N} t_p(\overline{Q}_{ij})_p$$

$$\tilde{B}_{ij} = \sum_{p=1}^{N} -t_p\bar{z}_p(\overline{Q}_{ij})_p$$

$$\tilde{D}_{ij} = \sum_{p=1}^{N} \left(t_p\bar{z}_p^2 + \frac{t_p^3}{12}\right)(\overline{Q}_{ij})_p \tag{10}$$

where $(\overline{Q}_{ij})_p$ is the transformed reduced stiffness term of the p^{th} ply. The strain energy for the domain of the plate 'Ω' in terms of the displacement gradients can be written as

$$U_0 = \int_\Omega \tilde{S}:\tilde{\varepsilon} d\Omega = \int_\Omega \tilde{\varepsilon}^T \tilde{C} \tilde{\varepsilon} d\Omega \tag{11}$$

where the integrand in the first term represents a double dot product or scalar tensor product. Applying Eq. (5) into Eq. (8) and then into Eq. (11) the elastic strain energy could be obtained as a function of displacement gradients. By examining the order of dependence of Green's strain tensor, the product terms in Eq. (11) could be split as strain energy components which are functions of second, third and fourth order of displacement gradients and accordingly they could be written symbolically as

$$U_0 = U^2 + U^3 + U^4 \tag{12}$$

The superscripts of the terms on the right hand side of Eq. (12) denote the order of dependence of energy terms with respect to displacement gradients. To get the secant matrices in quadratic form, a vector $g_\alpha^T = \{u_{,x} \ u_{,y} \ v_{,x} \ v_{,y} \ w_{,x} \ w_{,y} \ w_{,xx} \ w_{,yy} \ w_{,xy}\}$ is assumed which is the set of all displacement gradients of additional displacements found in the Green's strain tensor in Eq. (5). The range of the Greek indices varies as the number of elements in the non-empty set g_α. The shell ordinates 'w_0' being constants, they are excluded from the gradient vector g_α. It should be emphasized that the basis of the secant matrix technique (Mallet and Marcal 1968, Rajasekaran and Murray 1973 and Oñate 1995) is in expressing Eq. (11) in a quadratic form as

$$U_0 = g_\alpha^T U g_\alpha \qquad \forall g_\alpha \tag{13}$$

which could be rewritten using Mallet and Marcal (1968) symbolism as

$$U_0 = g_\alpha^T \left[\frac{1}{2}K_{\alpha\beta} + \frac{1}{2}N0_{\alpha\beta} + \frac{1}{6}N1_{\alpha\beta} + \frac{1}{12}N2_{\alpha\beta}\right] g_\alpha \tag{14}$$

It is seen from Eq. (14) that there are indeed two terms which are independent of displacement gradients. The first one contributes the linear stiffness matrix and is purely a function of material compliance relations and the second term arises from the third term of the Green's strain tensor in Eq. (5) namely the shell ordinate term. The third and fourth terms are dependent on first and second order of displacement gradient g_α. The energy terms given in Eq. (14) after carrying out the double differentiation defined in Eq. (13) would give rise to four incremental or secant matrices. After lengthy calculations, the complete set of displacement gradients for the secant matrices for the postbuckling analysis of thin laminated composite shells have been presented in the Appendix. To the author's knowledge such explicit secant matrices for the nonlinear analysis of laminated composite shells based on Marguerre's shell theory have not been published in the literature. The incremental matrices presented in this study could be extended to the postbuckling analysis of shear deformable shells by adding the linear shear flexibility matrix. This technique was demonstrated by Ganapathy and Varadhan (1995). For the finite element implementation, the potential of the laminated composite plate as given in Eq. (11) can be represented as sum of potentials contributed by individual plate/shell finite elements. Let $p_i \in E^n(\Omega)$ be n- tuples of generalised displacements of any plate/shell finite element in the domain Ω of the plate/shell. The element displacements p_e

can be interpolated using a linear combination of the generalised nodal displacements p_i and an approximating shape function φ_i as

$$p_e = \varphi_i \, p_i \tag{15}$$

φ_i can be assumed for any plate bending element. The displacement gradient can be related to the generalised nodal displacements of a typical plate/shell element as

$$g_\alpha = R_{\alpha i} \, p_i \tag{16}$$

Using Eq. (14) the strain energy equation can be written symbolically as

$$U_n = p^T (R^T U R) p \tag{17}$$

Combining Eq. (14) and Eq. (15) the total potential can be written in terms of generalised nodal displacements as

$$\Pi_e = \frac{1}{2} K_{ij} p_i p_j + \frac{1}{2} {}^0 n_{ij} p_i p_j + \frac{1}{6} {}^1 n_{ij} p_i p_j + \frac{1}{12} {}^2 n_{ij} p_i p_j - p_i q_i \tag{18}$$

As defined earlier, the first term represents the linear stiffness matrix, the second term the shell ordinate matrix and the third and fourth terms are the first and second order secant matrices respectively. Eq. (18) is the total potential energy (TPE) which is a function of incremental matrices which are in turn functions of displacement vector p_i.

5. Equilibrium equations and method of analysis

The equilibrium equation can be obtained by applying the condition of equilibrium to Eq. (18) as Π^e, i (comma represents differentiation with respect to succeeding indices) it becomes,

$$q_i = K_{ij} p_j + {}^0 n_{ij} p_j + \frac{1}{2} {}^1 n_{ij} p_j + \frac{1}{3} {}^2 n_{ij} p_j \tag{19}$$

It is seen that the secant matrices in Eq. (19) is solvable by pure iterative procedure. However with further differentiation of Eq. (19) with respect to p_i the tangent stiffness matrix relation could be obtained as

$$\Delta q_i = K_{ij} \Delta p_j + {}^0 n_{ij} \Delta p_j + {}^1 n_{ij} \Delta p_j + {}^2 n_{ij} \Delta p_j \tag{20}$$

where Δ represents the incremental operator. The matrices required to form these equations are presented in the Appendix. In this study, cubic Hermitian polynomials are assumed for in-plane $(u_i \in C^2)$ and out-of-plane displacements $(w \in C^2)$ as proposed by Bogner *et al.* (1966). In spite of numerous developments that had taken place in the evolution of the new plate bending elements, Bogner *et al.* (1966) element has been used because it yields very good results in the case of postbuckling problems. The equilibrium equation is solved incrementally by first computing the Euler predictor $\Delta \boldsymbol{p}_i$ corresponding to a load increment $\Delta \boldsymbol{q}_i$ and then correcting with Newton corrector iterations by computing the unbalanced forces from the equilibrium equations. For problems involving snap through behaviour, the 'arc length' procedure (Crisfield 1991) and the 'minimum residual desplacement' method (Chan 1988) has been implemented in the present study.

6. Numerical studies

Two types of shell post buckling problems are considered in this study to validate the derived shallow shell formulation. The first category is of shells that are loaded out-of-plane and the second category is of shells axially loaded.

6.1 Shallow shells transversely loaded

In this category first two problems are the standard cylindrical shell panels analysed by Sabir and Lock (1973). The first problem is the cylindrical shell loaded by a central concentrated load as shown in Fig. 4. The thickness of the shell is taken as 6.35 mm. The resulting behaviour is a snap through and a snap back behaviour at points A and B respectively. The results of the present study is compared with the results of displacement of other researchers at point B. It is seen that the present formulation predicts the limit load very accurately. Many researchers (Ramm 1982) have observed that the snap back portion of the load-displacement curve presented by Sabir and Lock (1973) is a little erroneous.

Regarding the present formulation the comparison of pre-buckling behaviour is good and shows little flexibility in deep post buckling ranges. Since Surana (1983) used the displacement control algorithm for path tracing the snap back behaviour is completely missing. Regarding the small deviation in the snap back range, it may be pointed out that such variations do exist in the results presented by many authors and there appears to be no fixed bench mark problem which could be treated as a standard problem.

In the second problem the thickness is increased form 6.35 to 12.7 mm and since the stiffness of the shell during postbuckling gets increased, the upward movement of the shell is reduced. The problem presents a snap-through behaviour at both ends A and B as shown in Fig. 5. The results are compared with Surana (1983) and the finite strip results of Madasamy (1994).

The agreement is very good except for a slight deviation in the deep post buckling ranges. The

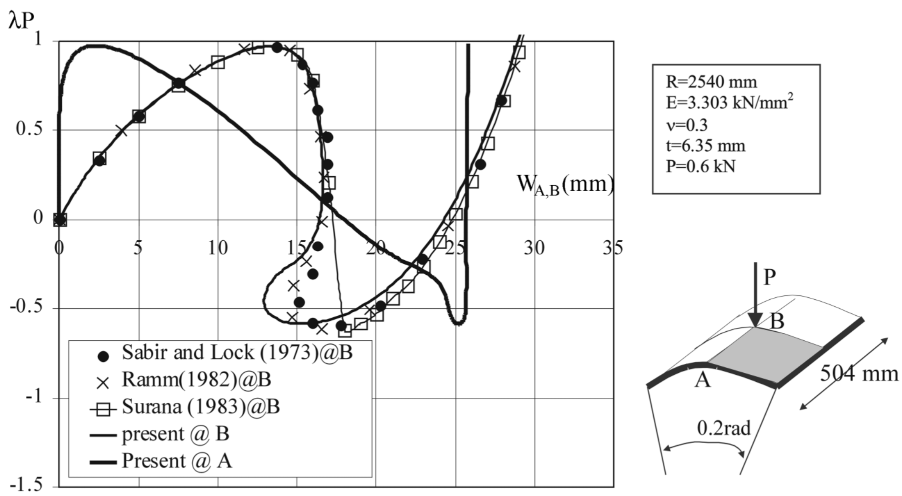

Fig. 4 Postbuckling behaviour of cylindrical shell subjected to point load (t = 6.35 mm)

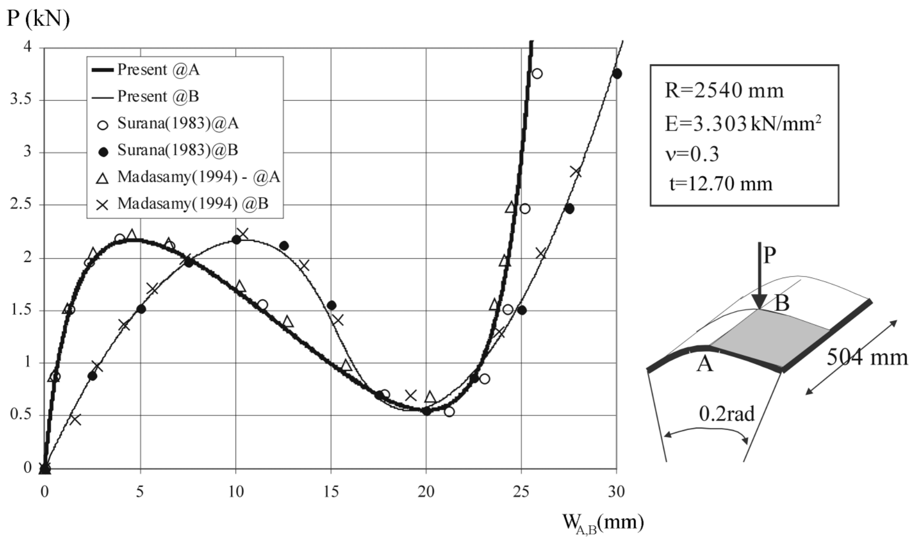

Fig. 5 Postbuckling behaviour of cylindrical shell subjected to point load (*t* = 12.70 mm)

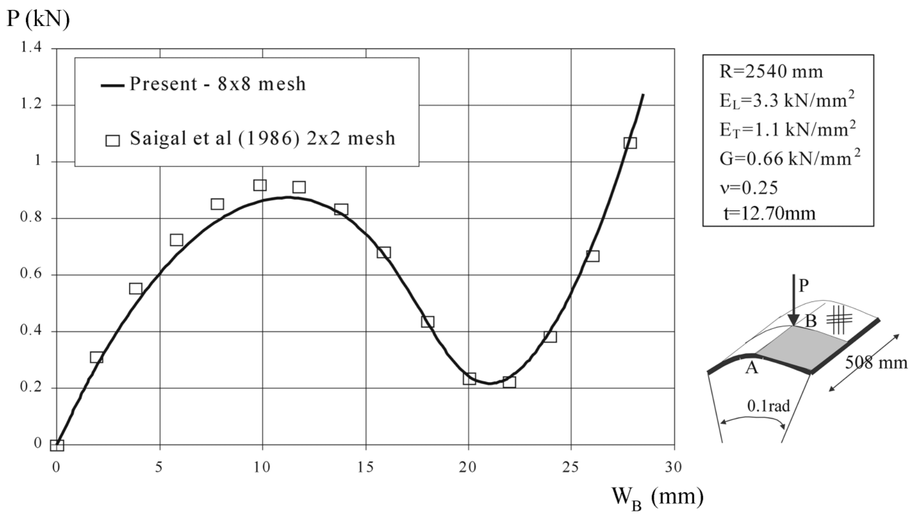

Fig. 6 Postbuckling behaviour of cylindrical composite shell subjected to point load

next problem involves a similar cylindrical shell shown in Fig. 7, but with a thickness of 12.6 mm. The fibre composite shell contains two layers (±45) construction. The ply characteristics are shown in Fig. 6.

From the result shown in Fig. 6, the results of the present study compare very well, given the fact that Saigal *et al.* (1983) used only 2×2 mesh per quarter and the present study is presented for an 8×8 mesh per quarter. The next problem is that of a clamped shell. The shell ordinates are specified as negative imperfections for a clamped plate. Various magnitudes of imperfection have been specified and the results are presented in Fig. 7. For the case with zero imperfection the problem reduces to that of a clamped plate. The results of the present study agree very well with the

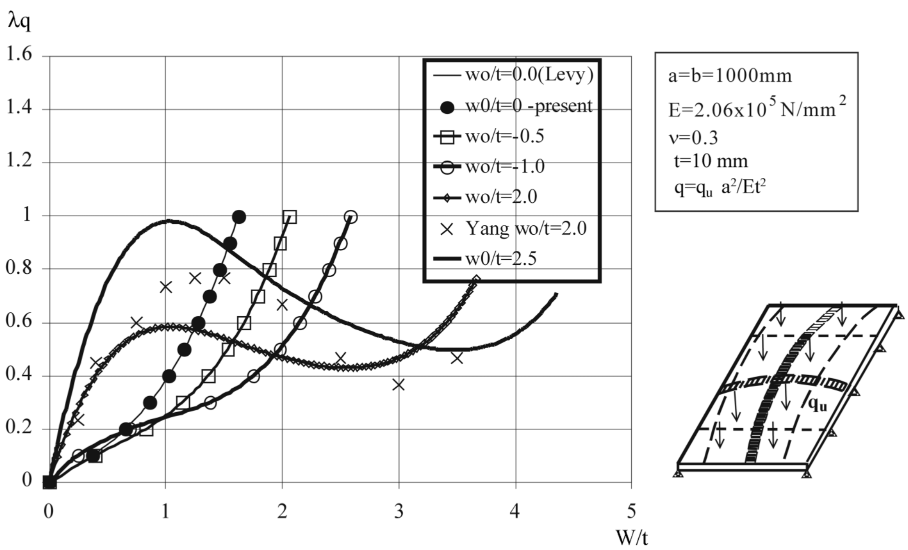

Fig. 7 Postbuckling behaviour of a clamped shell under UDL

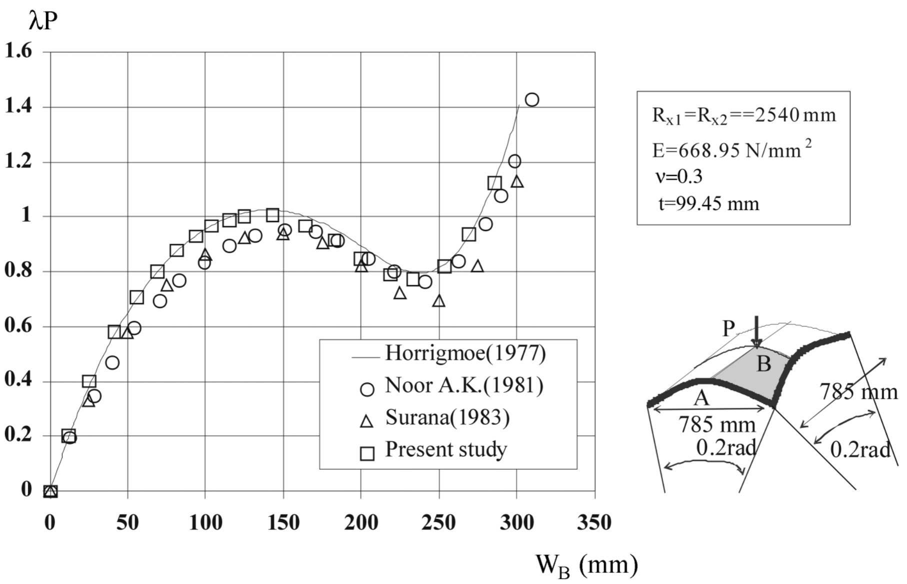

Fig. 8 Postbuckling behaviour of spherical shell subjected to point load

analytical solution of Levy (1948). As the imperfection shape is increased the behavior gradually changes, from a stable behaviour to limit point behaviour. On increasing the shell ordinates by increasing w_0/t, the shell gets bending stiffness because of the arch action and hence we see from Fig. 7 that as the negative initial imperfection shape is increased, initial stiffness is increased. However once this initial stiffness is overcome, the shell form unloads at limit point and snap to the other side.

It is seen from Fig. 7, that the results of Yang's formulation based on total displacements $w + w_0$

(Yang 1972) is for w_0/t = 2.0 much away from the results of the present study for w_0/t = 2.0. Yang (1972) formulated the shell formulation using ($w + w_0$), the total displacement as the displacement variable. Such an approach has been shown to be erroneous by Arul Jayachandran *et al.* (2003). The next problem is that of a spherical shell subjected to a central concentrated load. The details of the spherical shell are shown in Fig. 8. The results of the present study are compared with Horrigmoe (1977), Noor and Anderson (1981) and Surana (1983). The results of Horrigmoe (1977) who used a co-rotational procedure for shell element formulation is considered as a standard example by many researchers in literature. It is seen that the present study agrees very well with the results of Horrigmoe (1977).

6.2 Axially loaded shallow shells

In the case of axially loaded shells, the first example chosen is the post buckling of shell panel presented by Riks *et al.* (1990) that considers the fact that the unloaded edges are immovable.

Because of this condition, while the shell panel is axially loaded, the development of uniform axial stress is avoided and hence the shell panel do not show any bifurcation behaviour. The results of the present study shown in Fig. 9 agree well with the results of Riks *et al.* (1990) except for a flexible behaviour in the deep post buckling ranges.

The next problem is that of laminated composite carbon epoxy panel subjected to axial compression. For a 90/0 lay up postbuckling solutions were presented by Kim *et al.* (1998). The problem is presented in two parts. In the first case, the shell panel is taken to be perfectly flat and axially loaded.

But because of the axial flexural coupling of the lay-up sequence, the plate starts deflecting out-of-plane right from the onset of loading. For such a case the results of the present study compare

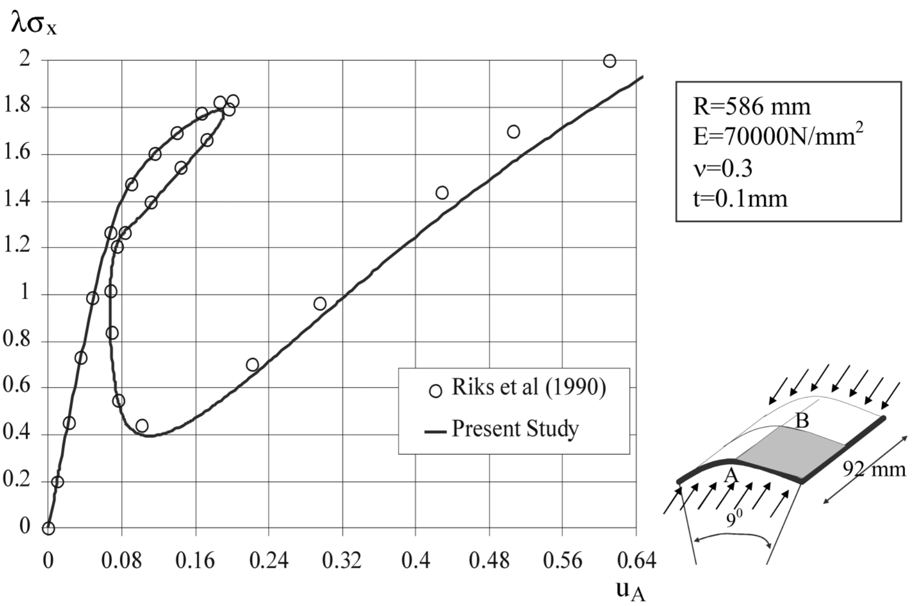

Fig. 9 Postbuckling behaviour of cylindrical shell subjected to axial compression

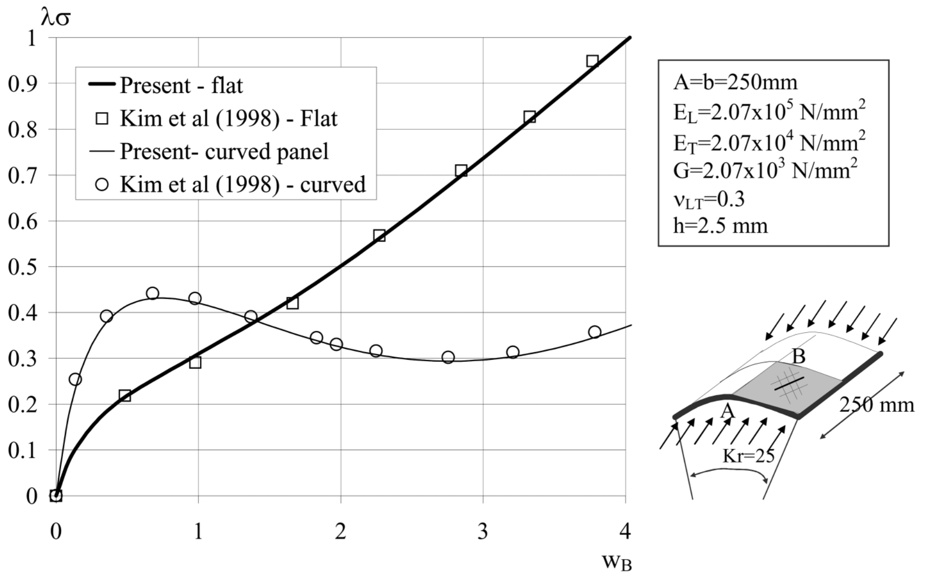

Fig. 10 Postbuckling behaviour flat and curved panels subjected to axial compression

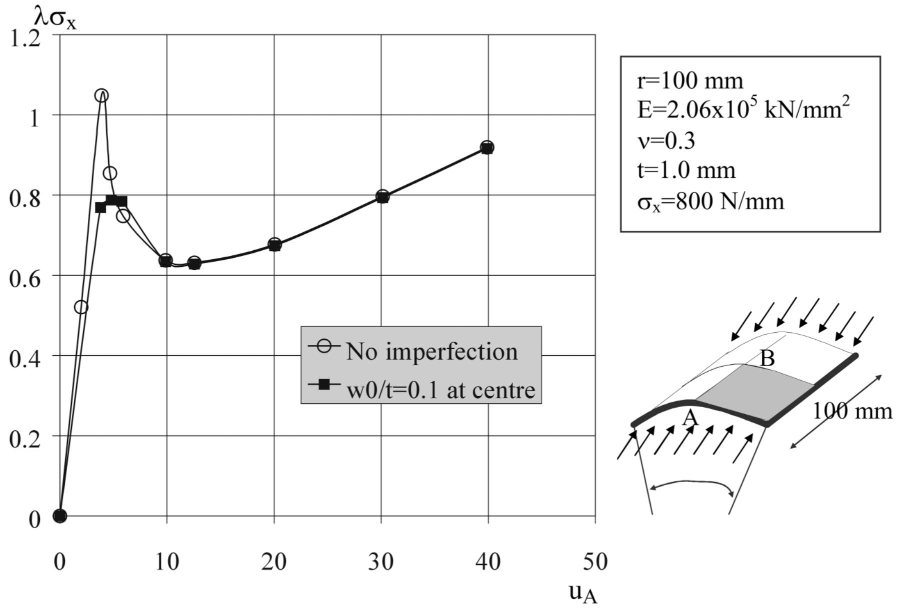

Fig. 11 Postbuckling behaviour of curved panels subjected to axial compression

very well with the results presented by Kim *et al.* (1998) as seen from Fig. 10. The next part is the curved panel with curvature parameter Kr = 25 defined as $Kr = b^2/rt$, where 'b' is the breadth, 'r' radius and 't' thickness of the shell panel. The panel exhibits limit point behaviour and also in this case, the results of the present study agree very well with results presented by Kim *et al.* (1998) as evident from Fig. 10.

The next problem taken for comparison is the problem presented by Kroplin (1982). The

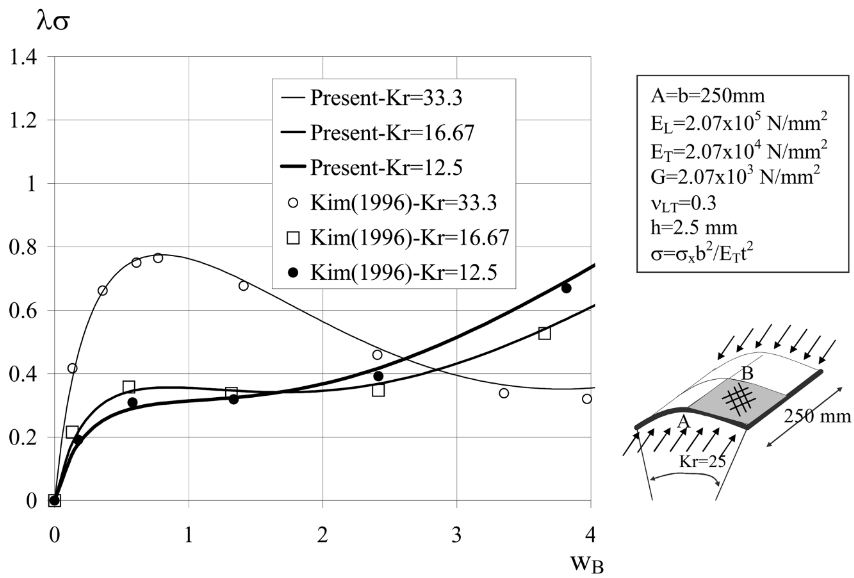

Fig. 12 Postbuckling behaviour of curved panels with various curvatures

cylindrical shell panel is under compression is shown in Fig. 11. Two analyses were considered: one with a local imperfection of 0.1 mm at B and the other without imperfection.

The results are shown in Fig. 11. It is seen from Fig. 11, that the results of the present study are quite efficient in modelling the post buckling behaviour of cylindrical shells. This example also exemplifies the capability of the present formulation in modelling localised imperfections. In order to check the present formulation in modelling shell panels of different curvatures, the results presented by and Kim (1996) are chosen. Fig. 12 shows the results of the present study compared with the results of Kim (1996) for various shell curvatures.

7. Conclusions

In this paper explicit secant matrices are presented for the postbuckling analysis of shallow laminated composite shells. These secant matrices can be used in combination with any plate/shell element. These incremental matrices are derived using Marguerre shallow shell theory. Numerical studies, which comprises of isotropic and anisotropic laminated composite shell panels, subjected to both transverse and axial loads are presented in this paper. From the results presented in this paper, the derived shallow shell formulations have been shown to be very efficient in the analysis of composite panels subjected to both out-of-plane and axial compressive loads.

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Appendix

The displacement gradient vector 'g_α' can be written as

$$g_\alpha^T = \{u_{,x}\ \ u_{,y}\ \ v_{,x}\ \ v_{,y}\ \ w_{,x}\ \ w_{,y}\ \ w_{,xx}\ \ w_{,yy}\ \ w_{,xy}\}_{9x1}$$

and the comma represents differentiation of the preceding quantity by the succeeding indices. The nonzero coefficients matrices $K_{\alpha\beta}$, $N0_{\alpha\beta}$, $N1_{\alpha\beta}$, $N2_{\alpha\beta}$ as in Eq. (14) is -presented below. The coefficients of the matrices A, B and D could be obtained from Eq. (10).

The non-zero coefficients of the $[K]_{9\times 9}$ matrix are given below

$K(1,1) = A_{11}$ $\quad K(1,2) = \dfrac{A_{13}+A_{31}}{2}$ $\quad K(1,3) = K(1,2)$

$K(1,4) = \dfrac{A_{12}+A_{21}}{2}$ $\quad K(1,7) = B_{11}$ $\quad K(1,8) = B_{12}$

$K(1,9) = 2B_{13}$ $\quad K(2,2) = A_{33}$ $\quad K(2,3) = K(2,2)$

$K(2,4) = \dfrac{A_{23}+A_{32}}{2}$ $\quad K(2,7) = B_{31}$ $\quad K(2,8) = B_{12}$

$K(2,9) = 2B_{33}$ $\quad K(3,3) = K(2,2)$ $\quad K(3,4) = K(2,4)$

$K(3,7) = K(2,7)$ $\quad K(3,8) = K(2,8)$ $\quad K(3,9) = K(2,9)$

$K(4,4) = A_{22}$ $\quad K(4,7) = B_{21}$ $\quad K(4,8) = B_{22}$

$K(4,9) = 2B_{23}$ $\quad K(7,7) = D_{11}$ $\quad K(7,8) = \dfrac{D_{12}+D_{21}}{2}$

$K(7,9) = D_{13} + D_{31}$ $\quad K(8,8) = D_{22}$ $\quad K(8,9) = D_{23} + D_{32}$

$K(9,9) = 4D_{33}$ $\quad K_{ij} = K_{ji}$

The non-zero components of initial imperfection/ shell ordinate matrix $N0_{ij} = N0_{ji}(9 \times 9)$ is given by

$N0(1,5) = A_{11}w_{0,x} + A_{31}w_{0,y}$ $\quad N0(1,6) = A_{21}w_{0,y} + A_{31}w_{0,x}$

$N0(2,5) = A_{13}w_{0,x} + A_{33}w_{0,y}$ $\quad N0(2,6) = A_{23}w_{0,y} + A_{33}w_{0,x}$

$N0(3,5) = N0(2,5)$ $\quad N0(3,6) = N0(2,6)$

$N0(4,5) = A_{12}w_{0,x} + A_{32}w_{0,y}$ $\quad N0(4,6) = A_{22}w_{0,y} + A_{32}w_{0,x}$

$N0(5,5) = A_{11}w_{0,x}^{\ 2} + (A_{31} + A_{13})w_0 w_{0,y} + A_{33}w_{0,y}^{\ 2}$

$$\text{N0}(5,6) = \frac{(A_{21} + A_{12} + 2A_{33})}{2} w_{0,x} w_{0,y} + \frac{(A_{13} + A_{31})}{2} w_{0,x}{}^2 + \frac{(A_{23} + A_{32})}{2} w_{0,y}{}^2$$

$\text{N0}(5,7) = B_{11} w_{0,x} + B_{31} w_{0,y}$ $\text{N0}(5,8) = B_{12} w_{0,x} + B_{32} w_{0,y}$

$\text{N0}(5,9) = 2B_{13} w_{0,x} + 2B_{33} w_{0,y}$ $\text{N0}(6,6) = A_{22} w_{0,y}{}^2 + (A_{32} + A_{23}) w_{0,x} w_{0,y} + A_{33} w_{0,x}{}^2$

$\text{N0}(6,7) = B_{21} w_{0,y} + B_{31} w_{0,x}$ $\text{N0}(6,8) = B_{22} w_{0,y} + B_{32} w_{0,x}$

$\text{N0}(6,9) = 2B_{23} w_{0,y} + 2B_{33} w_{0,x}$

The non-zero coefficients of first order incremental matrix $N1_{ij} = N1_{ji}(9 \times 9)$ is given by

$\text{N1}(1,5) = A_{11} w_{,x} + A_{13} w_{,y}$ $\text{N1}(1,6) = A_{12} w_y + A_{13} w_x$

$\text{N1}(2,5) = A_{13} w_{,x} + A_{33} w_{,y}$ $\text{N1}(2,6) = A_{32} w_y + A_{33} w_x$

$\text{N1}(3,5) = A_{31} w_{,x} + A_{33} w_{,y}$ $\text{N1}(3,6) = A_{32} w_y + A_{33} w_x$

$\text{N1}(4,5) = A_{21} w_{,x} + A_{23} w_{,y}$ $\text{N1}(4,6) = A_{22} w_y + A_{23} w_x$

$\text{N1}(5,5) = A_{11} u_{,x} + A_{23} v_{,y} + A_{31} u_{,y} + A_{31} v_{,x} + 6c_1 w_x + 2c_3 w_y - B_{11} w_{xx} - B_{12} w_{yy} - 2B_{13} w_{xy}$

$\text{N1}(5,6) = A_{13} u_{,x} + A_{23} v_{,y} + A_{31} u_{,y} + A_{33} v_{,x} + 2c_3 w_x + 2c_4 w_y - B_{31} w_{xx} - B_{32} w_{yy} - 2B_{33} w_{xy}$

$\text{N1}(6,6) = A_{12} u_{,x} + A_{22} v_{,y} + A_{32} u_{,y} + A_{32} v_{,x} + 6c_2 w_y + 2c_4 w_x - B_{21} w_{xx} - B_{22} w_{yy} - 2B_{23} w_{xy}$

$\text{N1}(5,7) = -B_{11} w_{,x} - B_{31} w_{,y}$ $\text{N1}(5,8) = -B_{12} w_{,x} - B_{32} w_{,y}$

$\text{N1}(5,9) = -2B_{13} w_{,x} - 2B_{33} w_{,y}$ $\text{N1}(6,7) = -B_{21} w_{,y} - B_{31} w_{,x}$

$\text{N1}(6,8) = -B_{22} w_{,y} - B_{32} w_{,x}$ $\text{N1}(6,9) = -2B_{23} w_{,y} - 2B_{33} w_{,x}$

$C_1 = 0.5A_{11} w0_{,x} + 0.5A_{13} w0_{,y}$ $C_2 = 0.5A_{22} w0_{,y} + 0.5A_{23} w0_{,x}$

$C_3 = A_{31} w0_{,x} + 0.5A_{12} w0_{,y} + 0.5A_{13} w0_{,x} + A_{33} w0_{,y}$

$C_3 = A_{21} w0_{,x} + 0.5A_{32} w0_{,y} + 0.5A_{23} w0_{,x} + A_{33} w0_{,x}$

The non-zero coefficients of $N2_{ij} = N2_{ji}$ (9 × 9) is given by

$$\text{N2}(5,5) = \frac{3}{2} A_{11} w_{,x}{}^2 + \frac{3}{2}(A_{13} + A_{31}) w_{,x} w_{,y} + \frac{(A_{12} + A_{21} + 4A_{33})}{4} w_{,y}{}^2$$

$$\text{N2}(5,6) = \frac{3}{4}(A_{13} + A_{31}) w_{,x}{}^2 + \frac{3}{4}(A_{23} + A_{32}) w_{,y}{}^2 + \frac{(A_{12} + A_{21} + 4A_{33})}{2} w_{,x} w_{,y}$$

$$\text{N2}(6,6) = \frac{3}{2} A_{22} w_{,y}{}^2 + \frac{3}{2}(A_{23} + A_{32}) w_{,x} w_{,y} + \frac{(A_{12} + A_{21} + 4A_{33})}{4} w_{,x}{}^2$$