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Closed form solution for displacements of thick cylinders with varying thickness subjected to non-uniform internal pressure

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Abstract. In this paper a thick cylindrical shell with varying thickness which is subjected to static non-uniform internal pressure is analyzed. At first, equilibrium equations of the shell have been derived by the energy principle and by considering the first order theory of Mirsky-Herrmann which includes transverse shear deformation. Then the governing equations which are, a system of differential equations with varying coefficients have been solved analytically with the boundary layer technique of the perturbation theory. In spite of complexity of modeling the conditions near the boundaries, the method of this paper is very capable of providing a closed form solution even near the boundaries. Displacement predictions are in a good agreement with the calculated finite elements and other analytical results. The convergence of solution is very fast and the amount of calculations is less than the Frobenius method.

Key words: thick cylindrical shell; varying thickness; perturbation theory; finite elements method.

1. Introduction

Axisymmetric thick shells with varying thickness are very important in industries. In optimizing a shell with respect to weight or stress distribution, one method is to use shells with varying thickness.

Long and thick cylinders with constant thickness and subjected to axisymmetric internal pressure are analyzed with the Navier equations in cylindrical coordinates. Radial displacement of this shell is given by:

$$W = C_1 r + \frac{C_2}{r} \tag{1}$$

where
C_1, C_2 are constants and r is radius. With considering R as middle radius and z the distance from the middle surface, one can write $r = R + z$, if $z/R < 1$ and by the Taylor expansion:

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$$W = C_1(R+z) + \frac{C_2}{R(1+z/R)} = k_0 + k_1 z + k_2 z^2 + \dots \tag{2}$$

It means that the displacement can be written as a polynomial of z and k_0, k_1, … are coefficients. Different theories of thick shells have been obtained with an analogy of Eq. (2) and considering some finite terms of it.

The first order displacement field for thick shells was expressed by Mirsky-Herrmann (1958) which is the extension of the Mindlin (1951) plate theory and includes transverse shear deformation. Suzuki (1981, 1982, 1983) used the first order theory for vibration analysis of varying thickness vessels. He used a linear approximation for axial displacement while radial displacement is independent of z, also he assumed that the problem is in the state of plane stress and ignored the normal stress in the radial direction. Takahashi and Suzuki (1981) used the first order theory for vibrational analysis but for the stress analysis of conical shells, they (1986) used second order theory. Simkins (1994) used the first order theory for determining of displacements in a long and thick shell subjected to moving loads. Nzengwa (1999) derived a 2-Dimensional model of a thick elastic shell from 3-Dimensional theory by considering different orders of h/R in horizontal and vertical components. In recent years, static analysis of isotropic and homogeneous shells has not been examined frequently and the main investigations have been concentrated on composite shells with different orders for displacement field where the equations have been solved in *closed form* for *special cases* e.g. constant thickness or in general, *numerical methods* have been used. Reddy (1984) has collected some of these cases.

In this paper, the governing equations of ***thick cylinders with varying thickness*** are derived by the energy principle. Inner radius of the cylinder is constant but the outer radius and axisymmetric internal pressure varies linearly in vertical direction. The cylinder is homogeneous and isotropic and in an elastic state, stress-strain relations conform with Hooke's Law. The linear displacement field (Mirsky-Herrmann (1958) theory) which includes transverse shear deformation is used. The derived equations are a system of differential equations with variable coefficients. By solving this system in a special case, the *application of this theory* for different ratios of radius to thickness is obtained, then the equations are solved with the ***Boundary Layer Method (matched asymptotic method)*** of the perturbation theory.

2. Governing equations

In the axisymmetric case, the location of each point on the cross section of the shell is denoted by $A(r, x)$ where r is radius and x is the vertical coordinate in the symmetric axes direction. Consider $R(x)$ as middle radius, then $r = R(x) + z$ where z is the horizontal distance from the middle surface. Fig. 1 shows a schematic of the shell geometry.

Displacement of each point contains two components: vertical (x) and horizontal (z). Displacement field considering the transverse shear deformation (Mirsky-Herrmann (1958) theory) is:

$$\begin{aligned} \overline{U}_x &= u(x) + z\psi_x(x) \\ \overline{U}_z &= w(x) + z\psi_z(x) \end{aligned} \tag{3}$$

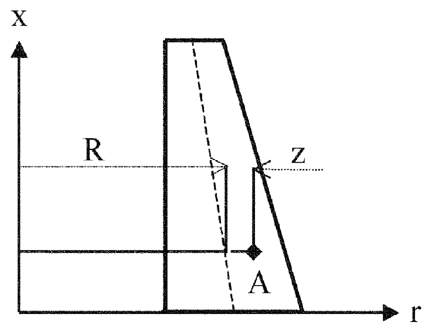

Fig. 1 Geometry of the shell

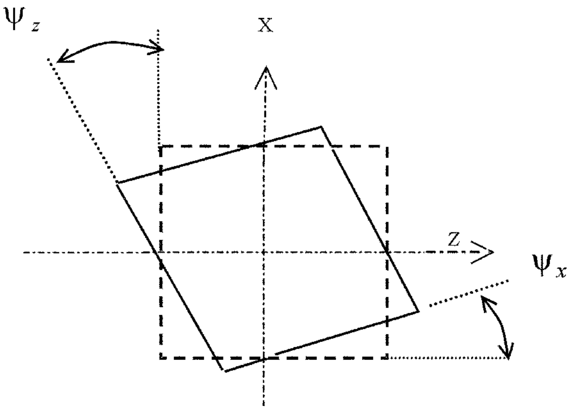

Fig. 2 Deformation of an element of the shell

where $\overline{U}_x$, $\overline{U}_z$ are approximate solutions in x and z directions; u and w displacements of the middle surface in x and z directions; ψ_x rotation in r-x plane and ψ_z is transverse normal strain. Fig. 2 shows the deformation of an element of length $2z$ with deformation described by relations (3) (u and w have not been displayed). The sides of this element remain straight but not perpendicular to the middle surface. Mirsky-Herrmann used relations (3) for dynamic analysis of shells. In their studies, the approximate solutions are the functions of the space and time. One can obtain relations (3) as the analogy of relation (2) by considering only the first order of expansion of series in which the coefficients are functions of x. Equilibrium equations can be derived by energy principle. Strain energy of an axisymmetric elastic body is:

$$U = \int_V (\sigma_{xx}e_{xx} + \sigma_{\theta\theta}e_{\theta\theta} + \sigma_{zz}e_{zz} + \sigma_{xz}\gamma_{xz})dV \tag{4}$$

where V is the shell volume and $dV = rd\theta dxdz$, $0 \le \theta \le 2\pi$, $-h/2 \le z \le h/2$, $0 \le x \le L$, h is thickness and L is length of the shell. External work is due to internal pressure:

$$W = \int_S P\overline{U}_z dS = \int_S P\left(w - \frac{h}{2}\psi_z\right)dS \tag{5}$$

where S is the inner surface and $dS = 2\pi(R - h/2)dx$. Strain-Displacement relations are:

$$e_{xx} = \frac{\partial \overline{U}_x}{\partial x} = \frac{du}{dx} + z\frac{d\psi_x}{dx}$$

$$e_{zz} = \frac{\partial \overline{U}_z}{\partial z} = \psi_z$$

$$e_{\theta\theta} = \frac{\overline{U}_z}{r} = \frac{w + z\psi_z}{R + z}$$

$$\gamma_{xz} = \frac{\partial \overline{U}_x}{\partial z} + \frac{\partial \overline{U}_z}{\partial x} = \psi_x + \frac{dw}{dx} + z\frac{d\psi_z}{dx} \tag{6a}$$

According to the Hooke's Law, the stress-strain relations are:

$$\begin{aligned}\sigma_{xx} &= Ae_{xx} + \lambda(e_{zz} + e_{\theta\theta})\\ \sigma_{zz} &= Ae_{zz} + \lambda(e_{xx} + e_{\theta\theta})\\ \sigma_{\theta\theta} &= Ae_{\theta\theta} + \lambda(e_{zz} + e_{xx})\\ \sigma_{xz} &= \mu\gamma_{xz}\end{aligned} \tag{6b}$$

where μ and λ are Lame coefficients and $A = \lambda + 2\mu$.

The minimum energy principle states that: $\delta U = \delta W$ where:

$$\delta U = \int_V (\sigma_{xx}\delta e_{xx} + \sigma_{\theta\theta}\delta e_{\theta\theta} + \sigma_{zz}\delta e_{zz} + \sigma_{xz}\delta\gamma_{xz})dV$$

$$\delta W = \int_S P\left(\delta w - \frac{h}{2}\delta\psi_z\right)dS \tag{7}$$

By the substitution of relations (3), (6a) and (6b) in (7) and equating the coefficients of $\delta\psi_z$, $\delta\psi_x$, δw, δu to zero, one can find:

$$\frac{d}{dx}(RN_x) = 0$$

$$\frac{d}{dx}(RM_x) - RQ_x = 0$$

$$\frac{d}{dx}(RQ_x) - N_\theta + PR\left(1 - \frac{h}{2R}\right) = 0$$

$$\frac{d}{dx}(RM_{xz}) - M_\theta - RN_z - PR\frac{h}{2}\left(1 - \frac{h}{2R}\right) = 0 \tag{8a}$$

and boundary conditions are:

$$[R(N_x\delta u + M_x\delta\psi_x + Q_x\delta w + M_{xz}\delta\psi_z)]_0^L = 0 \tag{8b}$$

where the stress resultants are defined as:

$$N_x = \int_{-h/2}^{h/2} \sigma_{xx}(1 + z/R)dz = Ah\left(\frac{du}{dx} + \frac{h^2}{12R}\frac{d\psi_x}{dx}\right) + \lambda h\left(\frac{w}{R} + \psi_z\right) \tag{9a}$$

$$M_x = \int_{-h/2}^{h/2} \sigma_{xx}z(1 + z/R)dz = \frac{Ah^3}{12R}\left(\frac{du}{dx} + R\frac{d\psi_x}{dx}\right) + \frac{2\lambda h^3}{12R}\psi_z \tag{9b}$$

$$Q_x = \int_{-h/2}^{h/2} \sigma_{xz}(1 + z/R)dz = \kappa^2 \mu h\left(\frac{dw}{dx} + \psi_x + \frac{h^2}{12R}\frac{d\psi_z}{dx}\right) \tag{9c}$$

$$N_z = \int_{-h/2}^{h/2} \sigma_{zz}(1 + z/R)dz = Ah\psi_z + \lambda h\left(\frac{w}{R} + \frac{du}{dx} + \frac{h^2}{12R}\frac{d\psi_x}{dx}\right) \tag{9d}$$

$$M_{xz} = \int_{-h/2}^{h/2} \sigma_{xz} z(1 + z/R)dz = \kappa^2 \mu \frac{h^3}{12R}\left(\frac{dw}{dx} + \psi_x + R\frac{d\psi_z}{dx}\right) \tag{9e}$$

$$N_\theta = \int_{-h/2}^{h/2} \sigma_{\theta\theta} dz = A(\alpha w + \beta\psi_z) + \lambda h\left(\psi_z + \frac{du}{dx}\right) \tag{9f}$$

$$M_\theta = \int_{-h/2}^{h/2} \sigma_{\theta\theta} z dz = A(\beta w + \eta\psi_z) + \lambda\frac{h^3}{12}\frac{d\psi_x}{dx} \tag{9g}$$

$$\alpha = \int_{-h/2}^{h/2} \frac{dz}{R + z} = Ln\frac{2R + h}{2R - h}, \quad \beta = \int_{-h/2}^{h/2} \frac{zdz}{R + z} = h - R\alpha, \quad \eta = \int_{-h/2}^{h/2} \frac{z^2dz}{R + z} = \alpha R^2 - Rh \tag{9k}$$

$P(x)$ is the distribution of the internal pressure and κ is the shear correction factor that is embedded in shear stress term with an analogy of the Timoshenko beam theory (Mirsky-Herrmann 1958). If R is assumed constant, Eqs. (8a) result in Mirsky-Herrmann (1958) equations.

So, the equilibrium equations of the shell can be rewritten in the abbreviated form:

$$\frac{d}{dx}\left(B_1\frac{dy_1}{dx}\right) + \frac{d}{dx}(B_2 y_1) + B_3\frac{dy_1}{dx} + B_4 y_1 + F_1 = 0 \tag{10a}$$

The vectors of forces and displacements and the matrices of coefficient are defined as:

$$y_1 = \{u(x), \psi_x(x), w(x), \psi_z(x)\}^T, \quad F_1 = \frac{PR}{A}\left(1 - \frac{h}{2R}\right)\left\{0, 0, 1, -\frac{h}{2}\right\}^T \tag{10b}$$

$$B_1 = \begin{bmatrix} Rh & \frac{h^3}{12} & 0 & 0 \\ \frac{h^3}{12} & \frac{Rh^3}{12} & 0 & 0 \\ 0 & 0 & \theta_2 Rh & \theta_2\frac{h^3}{12} \\ 0 & 0 & \theta_2\frac{h^3}{12} & \theta_2\frac{Rh^3}{12} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & \theta_1 h & \theta_1 Rh \\ 0 & 0 & 0 & 2\theta_1\frac{h^3}{12} \\ 0 & \theta_2 Rh & 0 & 0 \\ 0 & \theta_2\frac{h^3}{12} & 0 & 0 \end{bmatrix} \tag{10c}$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta_2 Rh & -\theta_2 \frac{h^3}{12} \\ -\theta_1 h & 0 & 0 & 0 \\ -\theta_1 Rh & -\theta_1 \frac{h^3}{12} & 0 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\theta_2 Rh & 0 & 0 \\ 0 & 0 & -\alpha & -(\beta + \theta_1 h) \\ 0 & 0 & -(\beta + \theta_1 h) & -\alpha R^2 \end{bmatrix} \tag{10d}$$

$$\theta_1 = \frac{\lambda}{A}, \qquad \theta_2 = \frac{\kappa^2 \mu}{A}$$

and boundary conditions have to satisfy Eq. (8b).

If R and h are assumed constant, these equations simplify to Simkins (1994) equations.

The coefficients in the above equations are the geometrical and mechanical properties of the shell. In the elastic state, for a homogeneous and isotropic shell, mechanical properties are constant but geometrical properties are functions of x. Note that in calculating of α one needs to consider $2R > h$, i.e. the non-solid shell.

Integrating the first equation of (8a) directly:

$$RN_x = C_0 \tag{11}$$

where C_0 is a constant and C_0/R for a cylinder with constant thickness is axial prestress (Simkins 1994). From Eq. (11), one can obtain:

$$-h\frac{du}{dx} = \frac{h^3}{12R}\frac{d\psi_x}{dx} + \theta_1 \frac{h}{R} w + \theta_1 h \psi_z - \frac{C_0}{AR} \tag{12}$$

By substituting Eq. (12) in Eqs. (10), equilibrium equations are derived with respect to displacements:

$$\frac{d}{dx}\left(A_1 \frac{dy}{dx}\right) + \frac{d}{dx}(A_2 y) + A_3 \frac{dy}{dx} + A_4 y + F = 0 \tag{13a}$$

$$y = \{\psi_x, w, \psi_z\}^T, \qquad F = \begin{Bmatrix} \frac{d}{dx}\left(\frac{h^2}{12R}\right)\frac{C_0}{A} \\ -\theta_1 \frac{C_0}{AR} + \frac{PR}{A}\left(1 - \frac{h}{2R}\right) \\ -\theta_1 \frac{C_0}{A} - \frac{PRh}{2A}\left(1 - \frac{h}{2R}\right) \end{Bmatrix} \tag{13b}$$

$$A_1 = \begin{bmatrix} \frac{Rh^3}{12}\left(1 - \frac{h^2}{12R^2}\right) & 0 & 0 \\ 0 & \theta_2 Rh & \theta_2 \frac{h^3}{12} \\ 0 & \theta_2 \frac{h^3}{12} & \theta_2 \frac{Rh^3}{12} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -\theta_1 \frac{h^3}{12R} & \theta_1 \frac{h^3}{12} \\ \theta_2 Rh & 0 & 0 \\ \theta_2 \frac{h^3}{12} & 0 & 0 \end{bmatrix} \tag{13c}$$

$$A_3 = \begin{bmatrix} 0 & -\theta_2 Rh & -\theta_2 \frac{h^3}{12} \\ \theta_1 \frac{h^3}{12R} & 0 & 0 \\ -\theta_1 \frac{h^3}{12} & 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -\theta_2 Rh & 0 & 0 \\ 0 & \theta_1^2 \frac{h}{R} - \alpha & \theta_1^2 h - (\beta + \theta_1 h) \\ 0 & \theta_1^2 h - (\beta + \theta_1 h) & \theta_1^2 Rh - \alpha R^2 \end{bmatrix} \quad (13d)$$

and the boundary conditions for Eqs. (12) and (13) is Eq. (8b).

3. Restriction of using the first order theory

Mirsky-Herrmann (1958) used the first order theory for studying wave motion in an axisymmetric cylinder. They state that the error of using this theory in the calculation of the phase velocity is not more than 15-20 percent even for a solid shell. Bhaskar (1991) found that the first order theory is accurate in determining the general characteristics of a shell e.g. deflection, natural frequencies and buckling loads but it is not suitable in calculation of stresses, mode shapes and high natural frequencies.

For difference estimation of the first order theory in calculating the deflection of a thick shell, Eqs. (13) have been solved for a long and thick cylinder with constant thickness and pressure. In this case, displacements do not depend on *x* and the terms that contain *d/dx*, are removed. Assuming $C_0 = 0$ then $\psi_x = 0$ and ψ_z and w are determined and the radial displacement on the inner wall (w_1) is calculated. On the other hand, this radial displacement can be calculated from the relation (1) (w_2). If one assumes R as the average radius and h as the thickness, the difference percentage i.e. *Diff*(%) $= \left|\frac{w_2 - w_1}{w_2}\right| \times 100$ with respect to $m = R/h$ is plotted in Fig. 3. This figure shows that the first order theory is very accurate for small thickness and by increasing the thickness, its accuracy is reduced.

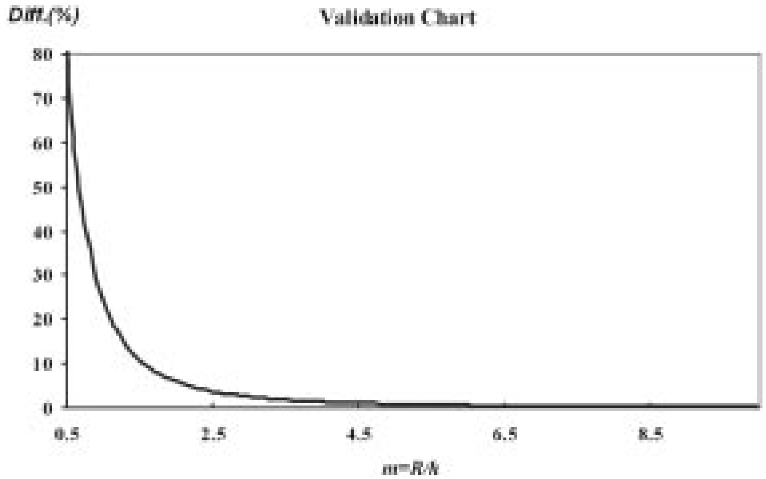

Fig. 3 Difference percentage with respect to $m = R/h$

4. Solution of equations

For a cylinder with constant thickness the coefficients of Eqs. (13) are constant and by the elementary theory of differential equations, one can solve these equations. Simkins (1994) solved these equations for a long thick cylinder with constant thickness which is subjected to moving pressure by this method. The general method for solving differential equations with variable coefficients is the Frobenius method. This method that is used in analyzing the vessels with varying thickness by Suzuki (1981, 1982, 1983) and Takahashi and Suzuki (1981, 1986), requires a large amount of calculations. Takahashi and Suzuki (1981) used 100 terms of series for desired convergence. Before using this method, the *inner* and *outer profiles of the shell* have to be defined.

In this article, *Boundary Layer Method (matched asymptotic method)* of the perturbation theory is used for solving these equations. This method does not require the knowledge of the inner and outer profiles of shell before formulation. In addition, the convergence is fast and the solution is in *closed form*. The beauty of this method is in its compatibility with the *physics of the shell*. It can explain the behavior of the shell successfully even near the boundaries. Solving the equations with varying thickness give rise to solving a system of algebraic equations and two systems of differential equations with constant coefficients.

To make the equations dimensionless, one can use the nondimensional parameters:

$$x^* = x/L,\ \ h^* = h/h_0,\ \ R^* = R/h_0,\ \ w^* = w/h_0,\ \ u^* = u/h_0 \tag{14}$$

the dimensionless form of Eqs. (13) is as below:

$$\varepsilon^2\frac{d}{dx^*}\left(A_1^*\frac{dy^*}{dx^*}\right) + \varepsilon\left[\frac{d}{dx^*}(A_2^*y^*) + A_3^*\frac{dy^*}{dx^*}\right] + A_4^*y^* + F^* = 0 \tag{15a}$$

where L is length of the shell, h_0 is characteristic thickness and $\varepsilon = h_0/L$ is assumed a *small parameter*. Matrices of $A_1^*, A_2^*, A_3^*, A_4^*$ are formed by substitution of R^* with R and h^* with h in the matrices of A_1, A_2, A_3, A_4 and $y^* = \{\psi_x, w^*, \psi_z\}^T$

$$F^* = F_1^* + \varepsilon F_2^*, \qquad F_2^* = \left\{-\frac{d}{dx^*}\left(\frac{h^{*2}}{12R^*}\right)C_0^*, 0, 0\right\}^T$$

$$F_1^* = \begin{Bmatrix} 0 \\ \theta_1\dfrac{C_0^*}{R^*} + \dfrac{PR^*}{A}\left(1 - \dfrac{h^*}{2R^*}\right) \\ \theta_1 C_0^* - \dfrac{PR^*h^*}{2A}\left(1 - \dfrac{h^*}{2R^*}\right) \end{Bmatrix}, \qquad C_0^* = -\frac{C_0}{Ah_0^2} \tag{15b}$$

Eqs. (15a) are singular and have two boundary layers at two ends of the shell. So, solution of the problem contains three parts: a solution away from the boundaries that is called *outer expansion* and two solutions near two boundaries which are called *inner expansions* (Nayfeh 1981).

4.1 Outer expansion

This solution is considered as a uniform series of ε:

$$y^*_{out} = y_{00} + \varepsilon y_{11} + \dots \tag{16a}$$

With substituting of this solution in Eqs. (15a) and considering the terms with the same order of ε, results:

$$\varepsilon^0 : A^*_4 y_{00} + F^*_1 = 0$$
$$\varepsilon^1 : A^*_4 y_{11} + \frac{d}{dx^*}(A^*_2 y_{00}) + A^*_3 \frac{dy_{00}}{dx^*} + F^*_2 = 0 \tag{16b}$$

Eqs. (16b) are systems of algebraic equations and by solving them, solution of the shell away from the boundaries is obtained.

4.2 Inner expansion at $x^* = 0$

The fast variable $\eta = \dfrac{x^*}{\varepsilon^\nu}$, $\nu > 0$ is considered as new variable for this region. With the Taylor expansion:

$$A^*_i(x^*) = A^*_i(0) + x^* \left[\frac{dA^*_i}{dx^*}\right]_{x^*=0} + \dots, \qquad i = 1\dots4 \tag{17a}$$

$$F^*_j(x^*) = F^*_j(0) + x^* \left\{\frac{dF^*_j}{dx^*}\right\}_{x^*=0} + \dots, \qquad j = 1, 2 \tag{17b}$$

By using relations (17) and applying the fast variable, Eqs. (15a) is formed as:

$$\varepsilon^{2-2\nu} \frac{d}{d\eta}\left[\left(A^*_1(0) + \varepsilon^\nu \eta \left[\frac{dA^*_1}{dx^*}\right]_{x^*=0} + \dots\right)\frac{dy^*}{d\eta}\right] + \varepsilon^{1-\nu}\frac{d}{d\eta}\left[\left(A^*_2(0) + \varepsilon^\nu \eta \left[\frac{dA^*_2}{dx^*}\right]_{x^*=0} + \dots\right)y^*\right] +$$
$$\varepsilon^{1-\nu}\left[\left(A^*_3(0) + \varepsilon^\nu \eta \left[\frac{dA^*_3}{dx^*}\right]_{x^*=0} + \dots\right)\frac{dy^*}{d\eta}\right] + \varepsilon\left[F^*_2(0) + \varepsilon^\nu \eta \left\{\frac{dF^*_2}{dx^*}\right\}_{x^*=0} + \dots\right] +$$
$$\left(A^*_4(0) + \varepsilon^\nu \eta \left[\frac{dA^*_4}{dx^*}\right]_{x^*=0} + \dots\right)y^* + \left[F^*_1(0) + \varepsilon^\nu \eta \left\{\frac{dF^*_1}{dx^*}\right\}_{x^*=0} + \dots\right] = 0 \tag{18}$$

In above equations, ν is derived from balancing of the dominant terms when $\varepsilon \to 0$. In this case $\nu = 1$ and the fast variable at $x^* = 0$ is $\eta = x^*/\varepsilon$. By considering the series solution as:

$$y^*_{in}(\eta) = u_0(\eta) + \varepsilon u_1(\eta) + \dots \tag{19a}$$

and substituting in Eq. (18), the terms with the same order of ε are:

$$\varepsilon^0:\ L(u_0, \eta, 0) + F_1^*(0) = 0$$

$$\varepsilon^1:\ L(u_1, \eta, 0) + M(u_0, a_1, b_1, c_1, d_1, \eta) + \eta e_1 + F_2^*(0) = 0 \tag{19b}$$

where:

$$L(y, x, i) = A_1^*(i)\frac{d^2y}{dx^2} + (A_2^*(i) + A_3^*(i))\frac{dy}{dx} + A_4^*(i)y \tag{20a}$$

$$M(y, a_1, b_1, c_1, d_1, \eta) = a_1\frac{d}{d\eta}\left(\eta\frac{dy}{d\eta}\right) + b_1\frac{d}{d\eta}(\eta y) + c_1\eta\frac{dy}{d\eta} + d_1\eta y \tag{20b}$$

a_1, b_1, c_1, d_1 are the derivatives of $A_1^*, A_2^*, A_3^*, A_4^*$ with respect to x^* at $x^* = 0$ and e_1 is the derivative of F_1^* with respect to x^* at $x^* = 0$. Eqs. (19b) are systems of ordinary differential equations (ODE) with constant coefficients which are solved using the elementary differential equations theory (Wylie 1979).

Determination of the solution constants is by applying the boundary conditions of the shell at $\eta = 0$ and the physics of the structure e.g. the deflections have to be restricted as $\eta \to \infty$.

4.3 Inner expansion at $x^* = 1$

Assuming the fast variable as $\zeta = (x^* - 1)/\varepsilon^\nu$ and substituting in Eqs. (15a), one gets $\nu = 1$ from balancing of dominant terms. By the uniform solution as:

$$y_{IN}^* = V_0 + \varepsilon V_1 + \ldots \tag{21a}$$

the equations with the orders of zero and one are:

$$\varepsilon^0:\ L(V_0, \zeta, 1) + F_1^*(1) = 0$$

$$\varepsilon^1:\ L(V_1, \zeta, 1) + M(V_0, a_2, b_2, c_2, d_2, \zeta) + \zeta e_2 + F_2^*(1) = 0 \tag{21b}$$

a_2, b_2, c_2, d_2 are the derivatives of $A_1^*, A_2^*, A_3^*, A_4^*$ with respect to x^* at $x^* = 1$ and e_2 is the derivative of F_1^* with respect to x^* at $x^* = 1$. Eqs. (21b) are also systems of ordinary differential equations with constant coefficients which are solved using the elementary differential equations theory (Wylie 1979).

Determination of the solution constants is by applying the boundary conditions of the shell at $\zeta = 0$ and the physics of the structure e.g. the deflection have to be restricted as $\zeta \to -\infty$.

4.4 Composite solution

In the boundary layer method, a composite solution is the summation of these three calculated solutions minus the overlapped parts of them. Outer solution at $x \to 0$ and inner solution at $\eta \to \infty$

are overlapped and this common part, have to be removed from the composite solution. This is true for outer solution at $x \to L$ and inner solution at $\zeta \to -\infty$. Therefore the composite solution is:

$$y^* = y^*_{in} + y^*_{IN} + y^*_{out} - (J_1 + J_0) \tag{22}$$

where J_1 and J_0 are common parts of inner and outer solutions at two ends of the shell. These common parts can be determined by definition of the intermediate variable or Van-Dyke algorithm (Nayfeh 1981).

5. Analytical results

In this section, two problems have been studied. At first, a thick shell with constant thickness and pressure have been examined and then, a thick shell with varying thickness and pressure have been analyzed.

5.1 A thick cylinder with constant thickness and subjected to constant internal pressure with clamped-free boundary conditions

Geometrical and mechanical properties of the shell are listed in Table 1. In this problem, $\varepsilon = h_0/L \cong 1/10$ is a small parameter. This problem is solved by three methods:

Table 1 Characteristics of the shell with constant thickness

Length of the shell	L = 0.4 m
Inner diameter	80 mm
Thickness	40 mm
Internal pressure	100 MPa
Young’s Modulus	210 GPa
Poisson’s ratio	0.3
Shear correction factor (κ^2)	5/6

5.1.1 Perturbation method (Boundary layer method)

By derived formulation, solving algebraic Eqs. (16b) gives rise to outer expansion. Inner expansion at $x^* = 0$ relates to solution of three coupled differential equations with six constants. Three constants are zero because of the limitation of deflection as $\eta \to \infty$. Three remaining constants are determined from applying the fixed boundary condition at $x^* = 0$, i.e. $w^* = \psi_x = \psi_z = 0$. Inner expansion at end of the shell is obtained by solving Eqs. (21b). This solution also contains six constants. Three constants are zero because of the limitation of deflection as $\zeta \to -\infty$. Other constants are determined by considering free edge condition. In the free edge, the deflection is the same as the deflection away from the boundary or solution does not depend on ζ and three remaining constants are zero too. So, the inner solution at end of the shell contains only the particular solution of (21b). This solution satisfies the free boundary conditions i.e. $N_x = M_x = Q_x = M_{xz} = 0$(dimensionless). Then with Eqs. (22), the composite solution is calculated. For determining u, it is enough to integrate Eq. (12). The coefficients of this equation are constant and this integration is performed directly. The constant of integrating from Eq. (12) and C_0 are calculated from the equation $N_x u = 0$ at two boundaries. The calculations are performed by a program in MAPLE 5 environment and the radial displacement on the inner wall, away from the clamped boundary is estimated (w_1).

5.1.2 Navier equations

By using the relation (1), one can calculate the radial displacement w_2 on the inner wall of this cylinder.

5.1.3 Ordinary differential equations theory (ODE)

In this problem, the coefficients of Eqs. (13a) are constant and its solution by elementary differential equations theory is possible. The equations have general and particular solutions. By unknown coefficients method, one can determine the particular solution. The general solution is in the form of $y = e^{i\alpha}\{v_1 \;\; v_2 \;\; v_3\}^T$ and by substitution in homogenous equations, one can calculate eigenvalues (α), eigenvectors (y), and then get general solution. Total solution is the summation of general and particular solutions. The constants of solution are estimated by applying free and clamped boundary conditions. The displacement on the inner wall and away from the boundaries is w_3. Table 2 compares radial displacement on the inner surface by these three methods.

Table 2 Radial displacement

Radial Displacement	(mm)
w_1 (Maple)	0.03343
w_2 (Navier)	0.03746
w_3(ODE)	0.03341

5.2 A thick cylinder with varying thickness and subjected to linear internal pressure with clamped-free boundary conditions

The shell characteristics are defined in Fig. 4 and Table 3. The matched asymptotic method is used for solving Eqs. (15a). Derived equations are also solved by a program on MAPLE 5 like section 5.1.1 for this problem. The only difference is in calculating of u. In this case, one can not

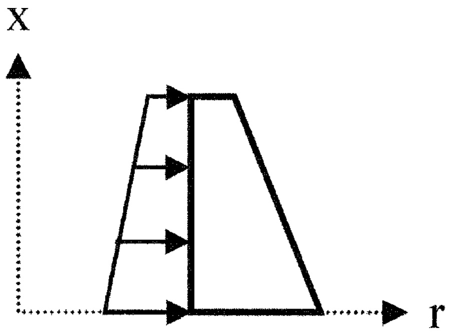

Fig. 4 Schematic of the shell loading

Table 3 Characteristics of the shell with varying thickness

Length of the shell	L = 0.4 m
Inner diameter	80 mm
Thickness at $x = 0$	40 mm
Thickness at $x = L$	10 mm
Internal pressure at $x = 0$	100 MPa
Internal pressure at $x = L$	25 MPa
Young's Modulus	210 GPa
Poisson's ratio	0.3
Shear correction factor (κ^2)	5/6

integrate all terms of the right hand side of Eq. (12). If $f(x)$ contains these unintegrable terms, the Taylor expansion of $f(x)$ is calculated and then, the integration of $f(x)$ has been determined. Radial and axial displacements of inner wall are presented later.

6. Numerical analysis

The coefficients of Eqs. (13a) are variable and solving them by elementary theory of differential equations is not possible. So, for comparison purpose, one can use numerical methods for analyzing a thick shell with varying thickness. The finite element method (FEM) is a powerful numerical method in structural analysis and can be used in our studies. For a thick cylindrical shell with varying thickness, due to geometrical and loading symmetric, the shell is studied in the field of plane elasticity. In this field, it suffices to model only the shell section which reduces three dimensional analysis to a problem in two dimensions. Characteristics of the shell are listed in Table 3. For analysis, *Ansys5.4* package has been used. The element of *PLANE82* which is an axisymmetric element, have been used for meshing. It is a quadrilateral element with eight nodes, the degrees of freedom are two translations in the radial and axial directions for each nodes and the stiffness matrix is $16 * 16$. Fig. 5 shows this element. The pressure is imposed linearly. The boundary conditions are clamped-free and the shell is assumed isotropic and elastic (Table 3). By investigating

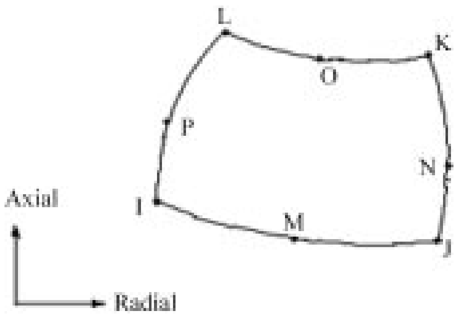

Fig. 5 PLANE82 Element (Ansys Element Manual 1997)

of several mesh configurations, a *refined* mesh pattern which include 272 elements was chosen. In Fig. 6, the mesh pattern, loading and boundary conditions of the shell, have been shown. Figs. 7 and 8 present the radial and axial displacements of the shell on inner surface that is calculated by FEM and perturbation method. Displacements are dimensionless with respect to h_0 and distance from the edge $x = 0$ with respect to L. Deformation of the shell is shown in Fig. 9.

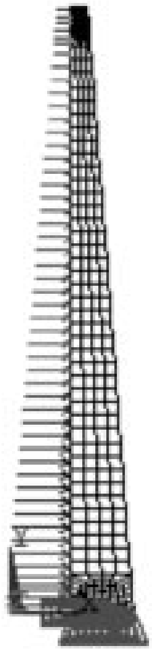

Fig. 6 Finite elements model

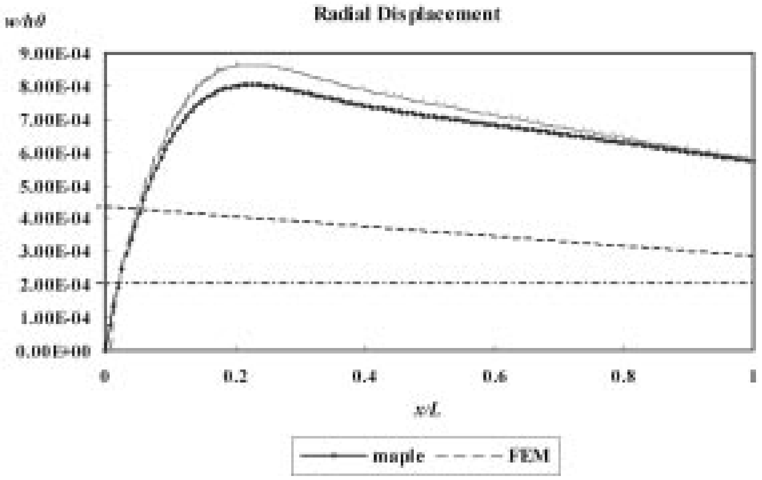

Fig. 7 Radial displacement with respect to distance (dimensionless)

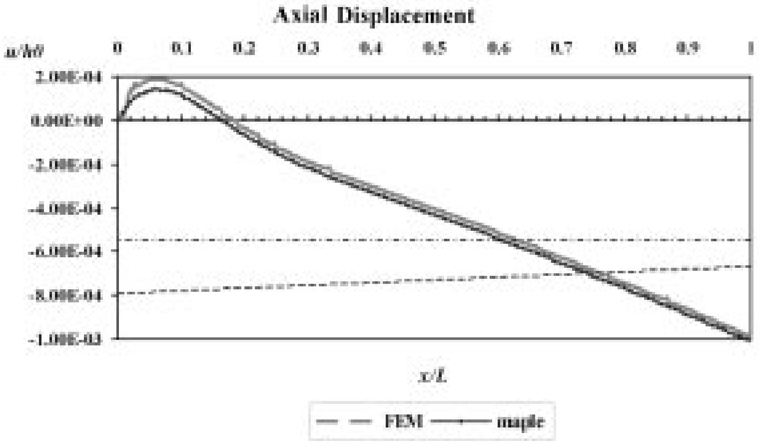

Fig. 8 Axial displacement with respect to distance (dimensionless)

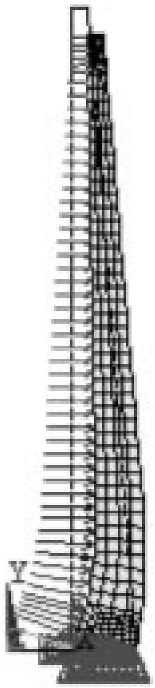

Fig. 9 Deformation of the shell (FEM)

7. Discussion of results

Figs. 7 and 8 comprise analytical and numerical solutions of radial and axial displacements on inner wall of the varying thickness shell from the edge. Non-uniform radial displacement caused axial reaction and results the axial deflection. The maximum axial deflection is at the free edge of the shell. These axial and radial displacements are physically reasonable as shown in Fig. 9. The

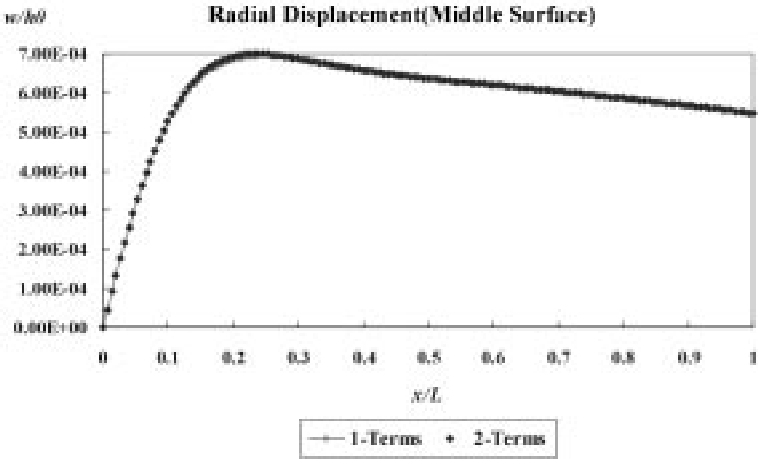

Fig. 10 Displacement for one and two terms of series

prediction of displacements of the cylinder with constant thickness (Table 2) show the percentage of difference between Navier and ODE solutions divided to Navier solution is about 10.8% for this shell which is adapt with Fig. 3 for m = 6/4. It is possible to approximate this difference for varying thickness shell. For this case, m is a function of x and $6/4 \leq m \leq 9/2$, so the difference is 1 to 11 percent (Fig. 3). In solving the problem by the boundary layer technique, only two terms of the series is considered (to order ε). These two terms suffice as a good approximation and one reason is that the perturbation parameter is small. Fig. 10 shows the radial displacement of the shell with respect to the distance from clamped edge (dimensionless) for one and two terms of series. It is seen that even the order one term doesn't have significant effect on the order zero term. So, in spite of the Freobenius series which requires a large number of terms for the desired convergence, this method requires few numbers of terms in series. Also the exact solution of Eqs. (13a) for the thick shell with constant thickness by ODE and solution of (15a) for this shell by the perturbation method, *do not* have any significant difference. So, *the perturbation theory could present an exact approximation for the solution and the differences of the solutions are due to the first order theory and not the boundary layer method.* For the varying thickness shell, calculated displacements are in a good agreement with FEM results and away from the boundaries, the difference is about 8%.

8. Conclusions

The presented method is successful in determining the displacements of shells with varying thickness and pressure. This method relates the solution of a system of differential equations with variable coefficients to solutions of a system of algebraic equations and two systems of differential equations with constant coefficients. In contrast with the Frobenous and Galerkin weighted residual methods, the formulation of problem does not depend to the form of pressure and outer radius distributions, i.e. this formulation is correct not only for linear functionality of the pressure and the outer radius but also for all functions which could describe these parameters. In comparison with the Frobenous and Galerkin methods, this method has *less computations, fast rate of convergence,* *closed form solution* for displacements and *short running time* of computations. Boundary layer method could predict the deflection of shells well, even near the boundaries and compatible with the *physics of the problem*. This method can be used in *verifying the finite elements results* and vice versa. Also one can use this method in optimizing the shell thickness with trial and error, provided that displacements are considered as the optimization criteria because various models can be defined and analyzed by a few variations in the program and it *does not require meshing*. This formulation can be extended to axisymmetric shells subjected to various kinds of axisymmetric loads (static or dynamics).

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Notation

k_0, k_1, k_2, C_0, c_1, c_2, ν	: Constants
r	: Horizontal coordinate (radius)
x	: Vertical coordinate
z	: Distance from middle surface
h, R	: Thickness and mean radius
h_0	: Characteristic thickness
u, w	: Axial and radial displacement of middle surface
ψ_x, ψ_z	: Rotation components of approximate solution
σ_{xx}, ε_{xx}	: Axial stress and strain
σ_{zz}, ε_{zz}	: Radial stress and strain
$\sigma_{\theta\theta}$, $\varepsilon_{\theta\theta}$	: Hoop stress and strain
σ_{xz}, γ_{xz}	: Shear stress and strain

δ	: Variation operator
δU, δW	: Variation of strain energy and external work
w_1, w_2, w_3	: Radial Displacements on the inner surface
y, y_1	: Vectors of displacements
ε	: Perturbation parameter
y^*_{out}	: Outer expansion
y_{00}, y_{11}	: Components of y^*_{out}
y^*_{in}	: Inner expansion at $x^* = 0$
u_0, u_1	: Components of y^*_{in}
y^*_{IN}	: Inner expansion at $x^* = 1$
a_1, b_1, c_1, d_1	: Derivatives of the matrices of $A^*_1, A^*_2, A^*_3, A^*_4$ at $x^* = 0$
a_2, b_2, c_2, d_2	: Derivatives of the matrices of $A^*_1, A^*_2, A^*_3, A^*_4$ at $x^* = 1$
η	: Fast variable at $x^* = 0$
ζ	: Fast variable at $x^* = 1$
λ and μ	: Lame coefficients
L	: Length of the shell
V	: Volume
$\overline{U}_x$, $\overline{U}_z$	: Approximate solutions for axial and radial displacements
N_x, M_x, Q_x, N_z, M_{xz}, N_θ, M_θ	: Stress resultants
A_i, B_j	: Matrices of coefficients (i = 1..3, j = 1..4)
F, F_1	: Vectors of force
$L(.)$, $M(.)$	: Defined operators
V_0, V_1	: Components of y^*_{IN}
J_0, J_1	: Common parts of inner and outer solutions

Parameters that are designated with * and do not enter in above list, are dimensionless e.g. R^* is dimension–less form of R