Structural Engineering and Mechanics, Vol. 16, No. 5 (2003) 597-612
DOI: http://dx.doi.org/10.12989/sem.2003.16.5.597

Stability of multi-step flexural-shear plates with varying cross-section

J. Y. Xu†

Department of Civil Engineering, Wuhan University of Technology, Wuhan 430070, P.R. of China

Q. S. Li†

Department of Building and Construction, City University of Hong Kong
Tat Chee Avenue, Kowloon, Hong Kong

(Received February 4, 2003, Accepted July 18, 2003)

Abstract. In this paper, multi-story buildings with shear-wall structures and with narrow rectangular plane configuration are modeled as a multi-step flexural-shear plate with varying cross-section for buckling analysis. The governing differential equation of such a plate is established. Using appropriate transformations, the equation is reduced to analytically solvable equations by selecting suitable expressions of the distribution of stiffness. The exact solutions for buckling of such a one-step flexural-shear plate with variable stiffness are derived for several cases. A new exact approach that combines the transfer matrix method and closed from solution of one-step flexural-shear plate with continuously varying stiffness is presented for stability analysis of multi-step non-uniform flexural-shear plate. A numerical example shows that the present methods are easy to implement and efficient.

Key words: buckling; plates; tall buildings.

1. Introduction

Buckling is a primary consideration in the design of many structures, as it may reduce the load-carrying capacity. Buckling of structures depends on many factors and parameters, including those defining the structural deformation characteristics, the structural geometry, the material properties, the support and restraint conditions and the external load action. Thus, the appropriate selection of buckling analysis model should be made based on these factors and parameters mentioned above. Experimental results obtained in the field measurements of buildings (e.g., Wang 1978, Li *et al.* 1994, Jeary 1997) have shown that for a multi-story frame building with narrow rectangular plane configuration (narrow building), e.g., B/L < 1/4, where B and L are the width and length of the rectangular plane, respectively, shear deformation is usually dominant in the total deformation in its horizontal vibration. They reported that not only a relative motion among transverse frames is parallel, but also a parallel relative motion among floors is observed. The whole transverse

† Associate Professor

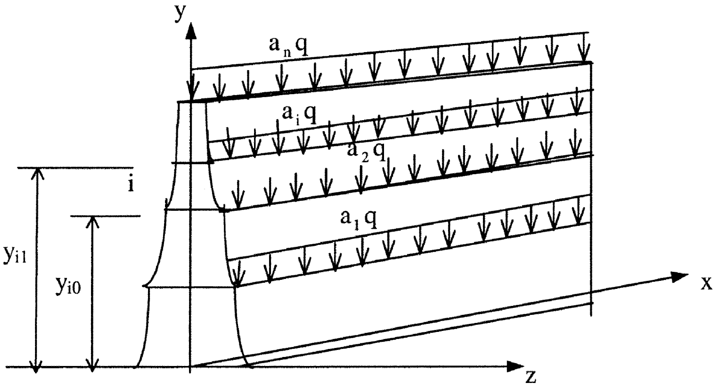


Fig. 1 A multi-step flexural-shear plate

deformation of a narrow frame building is similar to that of a cantilever shear plate (Li *et al.* 1998, Li 2000a). Thus, when analyzing free vibration and buckling of a narrow frame building, it is possible to regard such a structure as a cantilever shear plate. However, a narrow building with shear-wall structures may not be simplified as a shear plate for structural dynamic or buckling analysis. This is due to the fact that it has been recognized that the flexural deformation of shear-walls is dominant in the total lateral deformation of such a narrow building with shear-walls. It was reported (e.g. Li *et al.* 1996, Li 2000b) that in general a relative motion among transverse shear-walls is parallel, because the main deformation of each floor in-plane is shear deformation, and a relative motion among floors is not parallel, but rotation. Thus, the whole transverse deformation of a narrow building with shear-walls is similar to that of a cantilever flexural-shear plate. Hence, it is reasonable to treat a narrow building with shear-wall structures as a cantilever flexural-shear plate for vibration and buckling analysis, that is, the shear deformation is dominant in the longitudinal direction (the x-direction in Fig. 1) and the flexural deformation is dominant in the lateral direction. In general, the distribution of flexural stiffness of shear-walls is stepwise variation along the height of the building, thus, it is reasonable to treat a narrow building with shear-wall structures as a multi-step cantilever flexural-shear plate with variably distributed stiffness for buckling analysis.

The buckling of plates is a subject of considerable scientific and practical interest that has been studied extensively (e.g., Timoshenko and Gere 1961, Reddy 1998). However, there are very few equations for buckling of plates with variable cross-section where exact solutions can be obtained. These exact buckling solutions for shear plates or flexural plates are available only for certain plate shapes and boundary conditions. For example, Wittrick and Ellen (1962) studied the buckling of rectangular plates with two opposite edges simply supported and the other two simply supported or clamped. Linear and exponential thickness variations in one direction were considered in their study. Chehil and Dua (1973) investigated the buckling of simply supported rectangular plates with a linear thickness variation in one direction. Kobayashi and Sonoda (1989) presented an exact method to solve the buckling problem of uniaxially compressed rectangular plates with linearly tapered thickness analytically. Liew *et al.* (1996) derived analytical buckling solutions for Mindlin plates involving free edges. Recently, Xiang and Reddy (2001) presented exact solutions for free vibration and buckling of rectangular plates with intermediate line-supports using the Levy method. It should be mentioned that the concept and analytical model of the flexural-shear plates, which were recently proposed by Li (1999, 2000b, 2000c), are different from those of flexural plates or shear plates. The buckling of multi-step flexural-shear plate with varying cross-section has not previously been investigated and thus, the solution of this problem has not been proposed yet in the literature.

Apart from the several analytical methods for analyzing limited classes of plates, many approximate methods have been developed. For example, Liew and Wang (1992) conducted a buckling analysis of plates with straight/curved internal supports under uniform compression using the pb-2 Rayleigh-Ritz method. Wang *et al.* (1994) presented numerical buckling solutions for isotropic inplane loaded Mindin plates of regular polygonal, elliptical, semicircular and annular plates.

In this paper, multi-story buildings with shear-wall structures and with narrow rectangular plane configuration are modeled as a multi-step flexural-shear plate with varying cross-section for buckling analysis. An attempt is made here to establish and solve the governing differential equation for buckling of one-step and multi-step flexural-shear plates with varying stiffness. Exponential functions and power functions are adopted to describe the distribution of flexural stiffness and the exact solutions of the governing differential equation are given by means of Bessel functions and trigonometric functions. It is proved that a flexural-shear plate with free-free end conditions in the longitudinal direction, where the shear deformation is dominant, can be simplified by a flexural bar in buckling analysis. Numerical example shows that it is possible to simplify a multi-step flexural-shear plate with step varying distributions of stiffness as a one-step flexural-shear plate with continuously varying stiffness for buckling analysis.

The main purpose of this work is to present exact solutions and to propose an efficient analytical method for the buckling analysis of multi-step flexural-shear plates with variable stiffness. In the absence of the exact solutions, this problem may be solved using approximated methods (e.g., the Ritz method) or numerical methods (e.g., the finite element method). However, the present exact solutions could provide adequate insight into the physics of the problem and can be easily implemented. The availability of the exact solutions will help in examining the accuracy of the approximate or numerical solutions. Therefore, it is always desirable to obtain the exact solutions to such problems.

2. Theory

A multi-step flexural-shear plate is shown in Fig. 1. The axial force of the i-th step plate, N_i, is given by

$$N_i = \sum_{k=i}^{n} a_k q \tag{1}$$

where $a_k q$ is directly acted on the top of the k-th step plate.

In order to establish the governing differential equation for buckling of the i-th step plate, an infinitesimal element of the i-th step plate is taken, as shown in Figs. 2 and 3. Fig. 2 shows the element that is rotated through an angle of 90^{o}. A projection of the element shown in Fig. 2 on the y-z plane is presented in Fig. 3. The size of the element is $dx \times dy$. Considering the equilibrium condition in the z-axis for all the forces acting on the element leads to

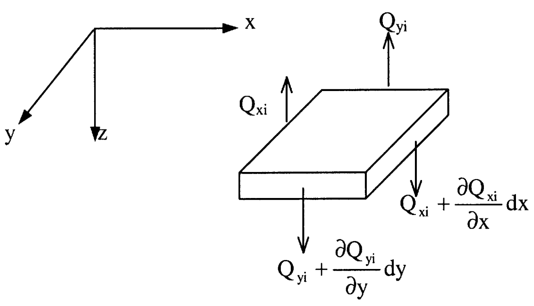


Fig. 2 A element of the plate

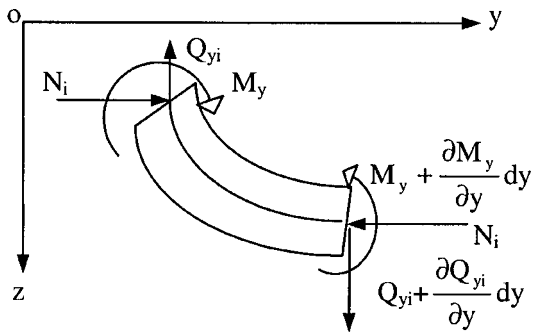


Fig. 3 A projection of the element on the *y*-*z* plane

$$\sum F_z = 0, \qquad \frac{\partial Q_{xi}}{\partial x} + \frac{\partial Q_{yi}}{\partial y} = 0 \tag{2}$$

where Q_{xi} and Q_{yi} are the shear forces acting on the element (see Fig. 2).

Using the equilibrium equation of moment of the forces acting on the element about the *z*-axis gives

$$\sum M_z = 0, \qquad Q_{yi} = \frac{\partial M_{yi}}{\partial y} - N_i \frac{\partial W_i}{\partial y} \tag{3}$$

where W_i is the displacement of the plate in the *z*-direction at the point (*x*, *y*), M_{yi} is the bending moment about the *x*-axis.

As is well known, the bending-curvature relation is given by (see Fig. 3)

$$M_{yi} = -K_{yi} \frac{\partial^2 W_i}{\partial y^2} \tag{4}$$

where K_{yi} is the transverse flexural stiffness in the *y*-direction.

As discussed previously, the deformation in the *x*-direction is shear deformation only, one yields

$$Q_{xi} = K_{xi} \frac{\partial W_i}{\partial x} \tag{5}$$

where K_{xi} is the transverse shear stiffness in the *x*-direction.

Substituting Eqs. (3), (4) and (5) into Eq. (2) gives

$$\frac{\partial}{\partial x}\left(K_{xi} \frac{\partial W_i}{\partial x}\right) - \frac{\partial^2}{\partial y^2}\left(K_{yi} \frac{\partial^2 W_i}{\partial y^2}\right) - N_i \frac{\partial^2 W_i}{\partial y^2} = 0 \tag{6}$$

It is assumed that K_x is only a function of *y* as follows

$$K_{xi} = K_i \varphi_i(y) \tag{7}$$

Because K_x is mainly dependent on the size and material properties of building floors, and the stiffness distribution of each floor is usually approximately uniform along the x-direction, thus, this assumption is reasonable for most narrow buildings.

Using the method of separation of variables gives

$$W_i(x, y) = X_i(x)Y_i(y) \tag{8}$$

Substituting Eq. (8) into Eq. (6) leads to

$$\frac{K_i \dfrac{d^2 X_i}{dx^2}}{X_i} = \frac{\dfrac{d^2}{dy^2}\left(K_{yi}\dfrac{d^2 Y_i}{dy^2}\right) + N_i \dfrac{d^2 Y_i}{dy^2}}{Y_i \varphi_i(y)} \tag{9}$$

The left hand side of the above equation is a function of x and the right hand side is a function of y. Thus, both sides should be equal to a constant. It is assumed that the constant is $(-\alpha_i^2)$, then, two ordinary differential equations are obtained from Eq. (9) as follows

$$K_i \frac{d^2 X}{dx^2} + \alpha_i^2 X = 0 \tag{10}$$

$$\frac{d^2}{dy^2}\left(K_{yi}\frac{d^2 Y_i}{dy^2}\right) + N_i \frac{d^2 Y_i}{dy^2} + \alpha_i^2 \varphi_i(y) Y = 0 \tag{11}$$

In general, Eq. (10) should be solved first. It is easy to find the general solution of Eq. (10) as follows

$$X_i(x) = C_{1i}\sin a_i x + C_{2i}\cos a_i x \tag{12}$$

where

$$a_i^2 = \frac{\alpha_i^2}{K_i} \tag{13}$$

A narrow building treated as a flexural-shear plate has free-free end conditions in the x-direction, i.e. the shear forces are zero at the free-free end conditions

$$\frac{dX}{dx} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L \tag{14}$$

where L is length in the longitudinal direction of the plate.

Using Eqs. (12) and (14) gives the eigenvalue equation as

$$\sin a_i L = 0 \tag{15}$$

or

$$a_{ij} = \frac{(j-1)\pi}{L} \tag{16}$$

in which a_{ij} represent the value of a_i corresponding to the j-th mode of buckling.

The minimum a_{ij} is equal to zero when $j = 1$, the corresponding mode of buckling is

$$X_{1i}(x) = \text{constant} \tag{17}$$

This constant can be taken as any value, usually taken as 1.

It is evident that a flexural-shear plate with free-free end conditions in the x-direction can be simplified as a flexural bar in the y-direction for buckling analysis. In this case, $\alpha = 0$, Eq. (11) becomes

$$\frac{d^2}{dy^2}\left(K_{yi}\frac{d^2Y_i}{dy^2}\right) + N_i\frac{d^2Y_i}{dy^2} = 0 \tag{18}$$

This equation can be rewritten in terms of Eq. (4) as follows

$$\frac{d^2M_{yi}}{dy^2} + \frac{N_i}{K_{yi}}M_{yi} = 0 \tag{19}$$

The general solution of this equation can be expressed as

$$M_{yi} = D_{1i}S_{1i}(y) + D_{2i}S_{2i}(y) \tag{20}$$

where $S_{1i}(y)$ and $S_{2i}(y)$ are linearly independent solutions of Eq. (19) and D_{1i}, D_{2i} are constants of integration. It is obvious that $S_{1i}(y)$ and $S_{2i}(y)$ are dependent on the expression of K_y. Several cases are considered and discussed as follows:

Case 1. $K_{yi} = K_{1i} = \text{constant}$ (21)

If the flexural stiffness of the i-th step plate is constant, then

$$S_{1i}(y) = \sin\sqrt{\frac{N_i}{K_{1i}}}y, \qquad S_{2i}(y) = \cos\sqrt{\frac{N_i}{K_{1i}}}y \tag{22}$$

Case 2. $K_{yi} = \alpha_i e^{-\beta_i\frac{y}{H}}$ (23)

where α_i and β_i are parameters that can be determined by the values of the flexural stiffness at the critical sections of the i-th step plate; H is the height of the plate.

Substituting Eq. (22) into Eq. (19) and setting

$$\eta = e^{\frac{\beta_i y}{2H}} \tag{24}$$

one obtains a Bessel equation, the two linearly independent solutions are as

$$S_{1i}(y) = J_0\left(\lambda e^{\frac{\beta y}{2H}}\right), \quad S_{2i}(y) = Y_0\left(\lambda e^{\frac{\beta y}{2H}}\right), \quad \lambda^2 = \frac{4N_iH^2}{\alpha_i\beta_i^2} \tag{25}$$

The exact solutions for other six types of distributions of K_{yi} are also derived and are presented in the Appendix of this paper.

After $S_{1i}(y)$ and $S_{2i}(y)$ are found, differentiating Eq. (20) obtains

$$\left.\begin{aligned} M'_{yi}(y) &= D_{1i}S'_{1i}(y) + D_{2i}S'_{2i}(y) \\ \theta_{yi}(y) &= D_{1i}\frac{S'_{1i}(y)}{N_i} + D_{2i}\frac{S'_{2i}(y)}{N_i} + \frac{D_{oi}}{N_i} \end{aligned}\right\} \tag{26}$$

Eqs. (20) and (26) can be expressed as a matrix equation as follows

$$\begin{bmatrix} \theta_{yi}(y) \\ M_{yi}(y) \\ M'_{yi}(y) \end{bmatrix} = [W_i(y)] \begin{bmatrix} D_{1i} \\ D_{2i} \\ D_{0i} \end{bmatrix} \tag{27}$$

in which

$$[W_i(y)] = \begin{bmatrix} \dfrac{S'_{1i}(y)}{N_i} & \dfrac{S'_{2i}(y)}{N_i} & \dfrac{1}{N_i} \\ S_{1i}(y) & S_{2i}(y) & 0 \\ S'_{1i}(y) & S'_{2i}(y) & 0 \end{bmatrix} \tag{28}$$

The relationship between the parameters introduced above at the two ends of the i-th step can be expressed as

$$\begin{bmatrix} \theta_{yi}(y_{i1}) \\ M_{yi}(y_{i1}) \\ M'_{yi}(y_{i1}) \end{bmatrix} = [T_i] \begin{bmatrix} \theta_{yi}(y_{i0}) \\ M_{yi}(y_{i0}) \\ M'_{yi}(y_{i0}) \end{bmatrix} \tag{29}$$

in which

$$[T_i] = [W_i(y_{i1})][W_i(y_{i0})]^{-1} \tag{30}$$

$[T_i]$ is called the transfer matrix because it transfers the parameters at the end y_{i0} to those at the end y_{i1} of the i-th step.

The relationships of the parameters between the i-th step and the $(i + 1)$ step plate at the connection section are as

$$\left.\begin{aligned} \theta_{yi}(y_{i0}) &= \theta_{y(i-1)}(y_{(i-1)1}) \\ M_{yi}(y_{i0}) &= M_{y(i-1)}(y_{(i-1)1}) \\ M'_{yi}(y_{i0}) &= M'_{y(i-1)} + (N_i - N_{i-1})\,\theta_{y(i-1)}(y_{(i-1)1}) \end{aligned}\right\} \tag{31}$$

Applying Eq. (31) to the end of the $(i + 1)$-th step and that of the i-th step leads to

$$\begin{bmatrix} \theta_{yi}(y_{i1}) \\ M_{yi}(y_{i1}) \\ M'_{yi}(y_{i1}) \end{bmatrix} = [T_{iN}] \begin{bmatrix} \theta_{y(i-1)}(y_{(i-1)1}) \\ M_{y(i-1)}(y_{(i-1)1}) \\ M'_{y(i-1)}(y_{(i-1)1}) \end{bmatrix} \tag{32}$$

in which

$$[T_{iN}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ N_i - N_{i-1} & 0 & 1 \end{bmatrix} [T_i] \tag{33}$$

The relationship between the parameters of the n-th step and those of the first step can be established by using Eqs. (32) and (29) repeatedly as follows

$$\begin{bmatrix} \theta_{yn}(y_{n1}) \\ M_{yn}(y_{n1}) \\ M'_{yn}(y_{n1}) \end{bmatrix} = [T] \begin{bmatrix} \theta_{y1}(y_{10}) \\ M_{y1}(y_{10}) \\ M'_{y1}(y_{10}) \end{bmatrix} \tag{34}$$

where

$$[T] = [T_{nN}][T_{(n-1)N}]\ldots[T_{2N}][T_1] \tag{35}$$

and $[T]$ has the form as

$$[T] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \tag{36}$$

The elements T_{ij} (i, j = 1, 2, 3) can be found from Eq. (35)

The boundary conditions in the y-direction are as follows

$$\theta_{y1}(y_{10}) = 0, \quad M'_{y1}(y_{10}) = 0, \quad M'_{yn}(y_{n1}) = 0 \tag{37}$$

Substituting Eqs. (37) into Eq. (34) gives

$$\begin{bmatrix} \theta_{yn}(y_{n1}) \\ 0 \\ M_{yn}(y_{n1}) \end{bmatrix} = [T] \begin{bmatrix} 0 \\ M_{y1}(y_{10}) \\ 0 \end{bmatrix} \tag{38}$$

From the above equation, one yields

$$T_{22} M_{y1}(y_{10}) = 0 \tag{39}$$

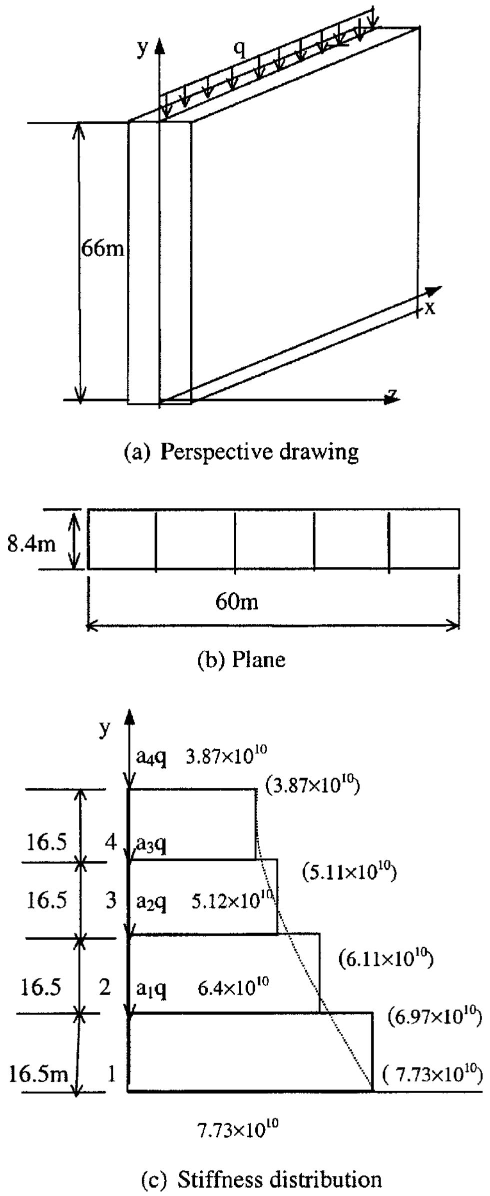
Because $M_{y1}(y_{10}) \neq 0$, we have,

$$T_{22} = 0 \tag{40}$$

This is the eigenvalue equation of a multi-step cantilever flexural-shear plate with free-free end conditions in the x-direction, setting n = 1 gives that of a one-step plate. Obviously, the eigenvalue equation is a transcendental one. The minimum eigenvalue root of buckling can be found by using a trial method.

3. Numerical example

A residence building with narrow rectangular plane configuration located in Beijing has 24 stories. Fig. 4 shows a sketch of this building. There are six shear-walls in each storey (Fig. 4b). Although this building looks like a one-step plate, the distribution of stiffness of which is stepwise variation



[Note: The dotted line and the values in parentheses are the evaluated results by using Eq. (44)]

Fig. 4 A narrow building

(Fig. 4c) due to the shear-walls with varying thickness along the building height. The procedure for determining the critical buckling force of the narrow building is as follows:

3.1 Calculation model

From the distribution of the stiffness (Fig. 4c), it is assumed that this building can be treated as a four-step cantilever flexural-shear plate with free-free end conditions in the x-direction for buckling analysis and four constant linear distributed axial forces are acted on the top of each step, respectively. N_i is the equivalent axial force of the i-th step plate, and $N_4 = q$, $N_3 = N_4 + q$, $N_2 = N_3 + q$, $N_1 = N_2 + q$.

In order to apply the method proposed in this paper to the buckling analysis of this building, the step varying distribution of flexural stiffness is approximated by a continuously varying one (see Fig. 4c).

3.2 Determination of the flexural stiffness K_y

The total flexural stiffness of the six shear-walls, EI_1, of the first step (from the first storey to the sixth storey) is found as

$$EI_1 = 4.64 \times 10^{12} \text{ N·m}^2$$

The total flexural stiffness of the second step (from the seventh storey to the twelfth storey) is found as

$$EI_2 = 3.84 \times 10^{12} \text{ N·m}^2$$

The total flexural stiffness of the third step (from the thirteenth storey to the eighteenth storey) and the fourth step (from the nineteenth to twenty-fourth storey) are found as

$$EI_3 = 3.07 \times 10^{12} \text{ N·m}^2$$
$$EI_4 = 2.32 \times 10^{12} \text{ N·m}^2$$

The stiffness of the first, second, third and fourth step, K_{11}, K_{12}, K_{13}, K_{14} are the values of EI_1, EI_2, EI_3, EI_4 divided by the length of the plate (i.e., the length of this building) as follows

$$K_{11} = \frac{EI_1}{L} = 7.73 \times 10^{10} \text{ N·m}, \qquad K_{12} = 6.4 \times 10^{10} \text{ N·m},$$
$$K_{13} = 5.12 \times 10^{10} \text{ N·m}, \quad K_{14} = 3.87 \times 10^{10} \text{ N·m}$$

3.3 Determination of the shear stiffness, K_{xi}, in the x-direction

The shear stiffness in the x-direction of the i-th step, K_{xi}, is the value of the shear stiffness of the i-th floor, GF, divided by the storey height.

Because the stiffness, GF, for each floor of this building is a constant, i.e., $\varphi(y)$ in Eq. (7) is equal to 1, GF is found as 7.8×10^8 N, we have

$$K_{xi} = K_i = \frac{7.8\times10^8}{3} = 2.8\times10^8 \text{ N/m}$$

In fact, it is not necessary to determine K_{2i}, because this building is treated as a four-step cantilever flexural-shear plate with free-free end conditions in the x-direction, as discussed previously, such a kind of plate can be replaced by a four-step cantilever bar in the y-direction for buckling analysis.

3.4 Determination of the transfer matrix

The transfer matrix for this example is found as

$$[T] = [T_{4N}][T_{3N}][T_{2N}][T_1] \tag{41}$$

where

$[T_{iN}]$ is given by Eq. (33), i.e.,

$$[T_{iN}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ N_i - N_{i-1} & 0 & 1 \end{bmatrix} [W_i(y_{i1})][W_i(y_{i0})]^{-1} \tag{42}$$

$$y_{10} = 0, \quad y_{11} = y_{20} = 16.5, \quad y_{21} = y_{30} = 33, \quad y_{31} = y_{40} = 49.5, \quad y_{41} = 66$$

Because $K_{yi} = K_{1i}$ (i = 1, 2, 3, 4), the special solutions are given by Eq. (22), and the matrix $[W_i(y)]$ is as

$$[W_i(y)] = \begin{bmatrix} \sqrt{\frac{1}{K_{1i}N_i}}\cos\sqrt{\frac{N_i}{K_{1i}}}y & -\sqrt{\frac{1}{K_{1i}N_i}}\sin\sqrt{\frac{N_i}{K_{1i}}}y & \frac{1}{N_i} \\ \sin\sqrt{\frac{N_i}{K_{1i}}}y & \cos\sqrt{\frac{N_i}{K_{1i}}}y & 0 \\ \sqrt{\frac{N_i}{K_{1i}}}\cos\sqrt{\frac{N_i}{K_{1i}}}y & -\sqrt{\frac{N_i}{K_{1i}}}\cos\sqrt{\frac{N_i}{K_{1i}}}y & 0 \end{bmatrix} \tag{43}$$

3.5 Determination of the eigenvalue equation

Using Eqs. (41), (42) and (43) obtains $[T]$, the eigenvalue equation is Eq. (40). It is evident that the unknown variable is only q.

Solving the eigenvalue equation obtains the critical distributed axial force

$$q_{cr} = 1.93\times10^7 \text{ N·m}$$

Because q_{cr} is a constant linear distributed axial force, the critical buckling force is

$$q_{cr}\cdot L = 1.158\times10^9 \text{ N}$$

If the step varying distribution of flexural stiffness is approximated by a continuously varying one described by

$$K_y = \alpha(1 + \beta y)^b \tag{44}$$

and it is assumed that all distributed axial forces are acted on the top of this building.

Then, α, β, b can be determined by K_{11} and K_{14} as follows

$$\alpha = K_{11} = 7.73 \times 10^{10}\ \mathrm{N \cdot m}, \quad \beta = \frac{3}{4}, \quad b = \frac{1}{2}, \quad v = \frac{2}{3}$$

The eigenvalue equation is also Eq. (40), but $n = 1$ in which. T_{22} for this case can be determined from Eq. (20) and Eq. (A-5) in the appendix as well as the boundary conditions in the y-direction as follows

$$J_{-\frac{1}{3}}\left(\frac{4n}{3}\right) J_{-\frac{2}{3}}\left(\frac{4n}{3} \cdot 4^{-\frac{3}{4}}\right) = -J_{\frac{2}{3}}\left(\frac{4n}{3} \cdot 4^{-\frac{3}{4}}\right) J_{\frac{1}{3}}\left(\frac{4n}{3}\right) \tag{45}$$

where

$$n = \frac{q}{\alpha\beta^2}$$

Solving Eq. (45) obtains

$$n = 3.6366$$

The critical value of q, which is only acted on the top of the building (Fig. 4a), is found as

$$q'_{cr} = 3.63 \times 10^7\ \mathrm{N/m}$$

The critical buckling force is

$$q'_{cr} \cdot L = 2.178 \times 10^9\ \mathrm{N}$$

If $N_1 = N_2 = N_3 = N_4$, i.e., only a linear distributed axial force, q, is acted on the top of the building (Fig. 4a), then, using the calculation model of the four-step flexural-shear plate gives

$$q''_{cr} = 3.62 \times 10^7\ \mathrm{N/m}$$

It can be seen from the above results that the value of q'_{cr} is closed to that of q''_{cr}. This implies that a multi-step flexural-shear plate with step varying stiffness can be treated as a one-step flexural-shear plate with continuously varying stiffness for buckling analysis.

4. Conclusions

In this paper, narrow buildings with multi-step shear-walls are treated as a multi-step flexural-shear plate for buckling analysis. The governing differential equation of such a kind of plate is established and is reduced to a Bessel's equation or an ordinary differential equation with constant coefficients by selecting suitable expressions of the distribution of stiffness. The exact buckling solutions for a one-step flexural-shear plate with variable stiffness are derived for several important cases. A new exact approach that combines the transfer matrix method and the derived closed from solutions of one-step flexural-shear plate with continuously varying stiffness is presented. It is shown that a flexural-shear plate with free-free end conditions in the longitudinal direction, where the shear deformation is dominant, can be simplified as a flexural bar for buckling analysis, the boundary conditions of the bar are the same as those of the plate. The numerical example shows that: (1) the present methods are easy to implement and efficient for analyzing the buckling of multi-step flexural-shear plates, (2) it is possible to regard a multi-step flexural-shear plate with step varying cross-section as a one-step flexural-shear plate with continuously varying cross-section for buckling analysis.

Acknowledgements

The work described in this paper was fully supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China [Project Ref. No: CityU 1093/ 02E].

References

Chehil, D.S. and Dua, S.S. (1973), "Buckling of rectangular plates with general variation in thickness", *J. Appl. Mech.*, ASME, **40**(3), 745-751.

Jeary, A.P. (1997), *Designer's Guide to the Dynamic Response of Structures*, E & FN Span, London, U. K.

Kobayashi, H. and Sonoda, K. (1989), "Buckling of rectangular plates with tapered thickness", *J. Struct. Eng.*, ASCE, **116**(5), 1278-1289.

Li, Q.S. (1999), "Free vibration analysis of flexural-shear plates under the action of axial forces", *Advance in Structural Engineering, An Int. J.*, **2**(4), 305-320.

Li, Q.S. (2000a), "Exact solutions for free vibration of multi-step orthotropic shear plates", *Struct. Eng. Mech., An Int. J.*, **9**(3), 269-288.

Li, Q.S. (2000b), "A computational approach for free vibration of non-uniform flexural-shear plates", *Comput. Method Appl. Mech. Eng.*, **190**, 3-23.

Li, Q.S. (2000c), "Vibration analysis of flexural-shear plates with varying cross-section", *Int. J. Solids Struct.*, **37**(9), 1339-1360.

Li, Q.S., Cao, H. and Li, G.Q. (1994), "Analysis of free vibrations of tall buildings", *J. Eng. Mech.*, ASCE, **120**, 1861-1876.

Li, Q.S., Cao, H. and Li, G.Q. (1996), "Static and dynamic analysis of straight bars with variable cross-section", *Comput. Struct.*, **59**, 1185-1191.

Li, Q.S., Fang, J.Q. and Jeary, A.P. (1998), "Free vibration analysis of multi-story buildings with narrow rectangular plane configuration", *Eng. Struct.*, **21**(6), 507-518.

Liew, K.M. and Wang, C.M. (1992), "Elastic buckling of rectangular plates with curved internal supports", *J. Struct. Eng.*, ASCE, **118**(6), 1480-1493.

Liew, K.M., Xiang, Y. and Wang, C.M. (1996), "Analytical buckling solutions for Mindlin plates involving free edges", *Int. J. Mech. Sci.*, **38**(10), 1127-1138.

Reddy, J.N. (1998), *Theory and Analysis of Elastic Plates*, Taylor and Francis, Philadelphia, USA.

Timoshenko, S.P. and Gere, G.M. (1961), *Theory of Elastic Stability*, McGraw-Hill, New York.
Wang, C.M. and Liew, K.M. (1994), "Buckling solutions for Mindlin plates of various shapes", *Eng. Struct.*, **16**(2), 119-127.
Wang, G.Y. (1978), *Vibration of Buildings and Structures*, Science Press, Beijing.
Wittrick, W.H. and Ellen, C.H. (1962), "Buckling of tapered rectangular plates in compression", *Aeronaut. Q.*, **13**(4), 308-326.
Xiang, Y. and Reddy, J.N. (2001), "Buckling and vibration of stepped symmetric cross-ply laminated rectangular plates", *Int. J. Structural Stability and Dynamics,* **1**(3), 385-408.

Appendix: The exact solutions of Eq. (19) for six cases

Obviously, the closed form solutions of Eq. (19) are dependent on the distributions of flexural stiffness. For the following six types of distributions of K_{yi}, the exact solutions are found as

Case 3. $K_{yi} = \left(\alpha_i e^{\beta_i \frac{y}{H}} + b_i\right)^{-1}$ (A-1)

Using the same approach adopted in Case 2, one obtains

$$\left.\begin{aligned} &S_{1i}(y) = J_{v_i}\left(a_i e^{\frac{\beta_i y}{2H}}\right), \quad a_i^2 = \frac{4\alpha_i N_i H^2}{\beta_i^2}, \quad v_i^2 = \frac{4(-b_i)N_i H^2}{\beta_i^2} \\ &S_{2i}(y) = J_{-v_i}\left(a_i e^{\frac{\beta_i y}{2H}}\right) \qquad v_i = a \text{ non-integer} \\ &\text{or} \\ &S_{2i}(y) = Y_{v_i}\left(a_i e^{\frac{\beta_i y}{2H}}\right) \qquad v_i = \text{an integer} \end{aligned}\right\} \tag{A-2}$$

Case 4. $K_{yi} = \alpha_i(1+\beta_i y)^{-b_i}$ (A-3)

in which α_i, β_i, b_i are parameters that can be determined by the values of the flexural stiffness at $y = 0$, $H/2$ and H, H is the total height of the plate.

Substituting Eq. (A-3) into Eq. (19) and letting

$$\eta = (1+\beta_i y), \qquad M_{yi} = \eta^{\frac{1}{b_i + z}} Z \tag{A-4}$$

one obtains a Bessel equation, the solutions are

$$\left.\begin{aligned} &S_{1i}(y) = (1+\beta_i y)^{\frac{1}{2}} J_{v_i}\left[\frac{n_i}{k_i}(1+\beta_i y)^{k_i}\right] \quad k_i = \frac{2+b_i}{2}, \quad v_i = \frac{1}{2+b_i}, \quad n_i^2 = \frac{N_i}{\alpha_i \beta_i^2} \\ &S_{2i}(y) = (1+\beta_i y)^{\frac{1}{2}} J_{-v_i}\left[\frac{n_i}{k_i}(1+\beta_i y)^{k_i}\right], \quad v_i = \text{a non-integer} \\ &\text{or} \\ &S_{2i}(y) = (1+\beta_i y)^{\frac{1}{2}} Y_{v_i}\left[\frac{n_i}{k_i}(1+\beta_i y)^{k_i}\right], \quad v_i = \text{an integer} \end{aligned}\right\} \tag{A-5}$$

where $J_v(\cdot)$ is the Bessel function of the first kind, of order v; $Y_v(\cdot)$ is the Bessel function of the second kind, of order v.

If $b_i = -2$, then

$$\left.\begin{aligned} S_{1i}(y) &= (1+\beta_i y)^{\frac{1}{2}+\sqrt{A_i}} \quad A_i^2 = n_i^2 - \frac{1}{4}, \quad n_i^2 = \frac{N_i}{\alpha_i \beta_i^2} \\ S_{2i}(y) &= (1+\beta_i y)^{\frac{1}{2}-\sqrt{A_i}} \end{aligned}\right\} \quad \text{for} \quad n_i^2 < \frac{1}{4} \tag{A-6}$$

or

$$\left.\begin{aligned} S_{1i}(y) &= (1+\beta_i y)^{\frac{1}{2}}\cos[\sqrt{A_{1i}}\ln(1+\beta_i y)] \\ S_{2i}(y) &= (1+\beta_i y)^{\frac{1}{2}}\sin[\sqrt{A_{1i}}\ln(1+\beta_i y)] \end{aligned}\right\} \quad \text{for} \quad n_i^2 > \frac{1}{4} \tag{A-7}$$

or

$$\left.\begin{aligned} S_{1i}(y) &= (1+\beta_i y)^{\frac{1}{2}} \\ S_{2i}(y) &= (1+\beta_i y)^{\frac{1}{2}}\ln(1+\beta_i y) \end{aligned}\right\} \quad \text{for} \quad n_i^2 = \frac{1}{4} \tag{A-8}$$

Case 5. $K_{yi} = (a_i + \beta_i y)^{-c_i}$ (A-9)

This is an alteration of Case 4. The solutions are

$$\left.\begin{aligned} S_{1i}(y) &= (a_i + b_i y)^{\frac{1}{2}} J_{\nu_i}[\tilde{\alpha}_i (a_i + b_i y)]^{\frac{1}{2\nu_i}} \\ S_{2i}(y) &= (a_i + b_i y)^{\frac{1}{2}} J_{-\nu_i}[\tilde{\alpha}_i (a_i + b_i y)]^{\frac{1}{2\nu_i}} \\ &\text{or} \\ S_{2i}(y) &= (a_i + b_i y)^{\frac{1}{2}} Y_{\nu_i}[\tilde{\alpha}_i (a_i + b_i y)]^{\frac{1}{2\nu_i}} \end{aligned}\right\} \tag{A-10}$$

where

$$\tilde{\alpha}_i^2 = \frac{4\nu_i^2 N_i}{b^2}, \qquad \nu_i = \frac{1}{c_i + 2} \tag{A-11}$$

If $c_i = -2$, then

$$\left.\begin{aligned} S_{1i}(y) &= (a_i + b_i y)^{\frac{1}{2}+B_i} \\ S_{2i}(y) &= (a_i + b_i y)^{\frac{1}{2}-B_i} \\ B_i^2 &= \frac{1}{4} - \frac{N_i}{b_i^2} \end{aligned}\right\} \quad \text{for} \quad \frac{N_i}{b_i^2} < \frac{1}{4} \tag{A-12}$$

or

$$\left.\begin{aligned} S_{1i}(y) &= (a_i + b_i y)\sin[B_i \ln(a_i + b_i y)] \\ S_{2i}(y) &= (a_i + b_i y)\cos[B_i \ln(a_i + b_i y)] \end{aligned}\right\} \quad \text{for} \quad \frac{N_i}{b_i^2} > \frac{1}{4} \tag{A-13}$$

or

$$\left.\begin{aligned} S_{1i}(y) &= (a_i + b_i y) \\ S_{2i}(y) &= (a_i + b_i y)\ln(a_i + b_i y) \end{aligned}\right\} \quad \text{for} \quad \frac{N_i}{b_i^2} = \frac{1}{4} \tag{A-14}$$

Case 6. $K_{yi} = a_i(y_i^2 + b_i)^2, \quad a_i > 0, \quad b_i > 0$ (A-15)

The solutions for this case are given by

$$\left.\begin{aligned} S_{1i}(y) &= (y_i^2 + b_i)^{\frac{1}{2}}\sin\xi \\ S_{2i}(y) &= (y_i^2 + b_i)^{\frac{1}{2}}\cos\xi \end{aligned}\right\} \tag{A-16}$$

where

$$\xi = \left(\frac{N_i + a_i b_i}{a_i b_i}\right)^{\frac{1}{2}} \arctan\frac{y_i}{b_i^{1/2}} \tag{A-17}$$

Case 7. $K_{yi} = a_i(y_i^2 - b_i)^2, \quad a_i > 0, \quad b_i > 0$ (A-18)

The solutions for this case are as

$$\left.\begin{aligned} S_{1i}(y) &= (y_i^2 - b_i)^{\frac{1}{2}}\sin\xi \\ S_{2i}(y) &= (y_i^2 - b_i)^{\frac{1}{2}}\cos\xi \end{aligned}\right\} \tag{A-19}$$

where

$$\xi = \frac{1}{2}\left(\frac{N_i - a_i b_i}{a_i b_i}\right)\ln\frac{b_i^{1/2} + y}{b_i^{1/2} - y}, \qquad |y| < b_i^{1/2} \tag{A-20}$$

Case 8. $K_{yi} = (a_i y_i^{C_i - 1} - b_i^2 y^{2C_i - 2})^{-1}$

The solutions for this case are as

$$S_{1i}(y) = y^{\frac{1 - C_i}{2}}\Phi\left(\frac{a_i N_i^{1/2}}{2b_i C_i}, \frac{1}{2C_i}; \frac{2b_i N_i^{1/2}}{C_i} y^{C_i}\right) \tag{A-21}$$

$$S_{2i}(y) = \left(\frac{2b_i N_i^{1/2}}{C_i} y^{C_i}\right)\Phi\left(\frac{a_i N_i^{1/2}}{2b_i C_i} - \frac{1}{2C_i} + 1, 2 - \frac{1}{2C_i}; \frac{2b_i N_i^{1/2}}{C_i} y^{C_i}\right) \tag{A-22}$$

in which $\Phi(x, x; x)$ represents the Φ-function