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On the eigenvalues of a uniform rectangular plate carrying any number of spring-damper-mass systems

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Abstract. The goal of this paper is to determine the eigenvalues of a uniform rectangular plate carrying any number of spring-damper-mass systems using an analytical-and-numerical-combined method (ANCM). To this end, a technique was presented to replace each "spring-damper-mass" system by a massless equivalent "spring-damper" system with the specified effective spring constant and effective damping coefficient. Then, the mode superposition approach was used to transform the partial differential equation of motion into the matrix equation, and the eigenvalues of the complete system were determined from the associated characteristic equation. To verify the reliability of the presented theory, all numerical results obtained from the ANCM were compared with those obtained from the conventional finite element method (FEM) and good agreement was achieved. Since the order of the property matrices for the equation of motion obtained from the ANCM is much lower than that obtained from the FEM, the CPU time required by the ANCM is much less than that by the FEM.

Key words: analytical-and-numerical-combined method (ANCM); eigenvalues; equivalent "spring-damper" system; finite element method (FEM).

1. Introduction

Several papers have been written on the free-vibration analysis of a uniform plate carrying a single "spring-mass" system (with the damping effect neglected). For example, Laura *et al.* (1977) determined the lowest two natural frequencies of a uniform beam and a uniform rectangular plate carrying a single sprung mass using the polynomial expansion and the Galerkin's method. Goldfracht and Rosenhouse (1984) determined the eigenvalues and the associated mode shapes of a uniform rectangular plate with beam-like stiffeners based on the Galerkin's method combined with the use of the special polynomial series. Then, Rosenhouse and Goldfracht (1984) used the last eigenvalues and mode shapes to determine the natural frequencies and the forced vibration responses of the stiffened plate carrying an elastically mounted vibrating machine by applying the Lagrange equations and multipliers. By using the normal mode with sinusoidal eigenfunction expansions, a closed-form solution for the natural frequencies of a simply-supported rectangular plate carrying a single "spring-mass" system was proposed by Avalos *et al.* (1993). Applying the Rayleigh-Ritz method and the polynomial coordinate functions, Avalos, Laredo and Laura (1994)

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studied the lowest six natural frequencies of a circular plate with transverse translation restrained at the edge. Bergman *et al*. (1993) presented the Levy series for the Green's functions of a rectangular plate with six kinds of supporting conditions and then determined the lowest ten natural frequencies of the rectangular plate with an intermediate rigid support and a sprung mass, respectively. Weaver applied the diagrammatic multiple-scattering theory to determine the mean responses (1997) and the mean-square responses (1998) of an infinite plate attached to a large number of randomly distributed undamped sprung masses. Because of complexity of the mathematical expressions, only the cases of a uniform plate carrying a single "spring-mass" system were illustrated in the foregoing literature except Weaver (1997, 1998). For this reason, Wu, Chou and Chen (2002) employed an analytical-and-numerical-combined method (ANCM) to determine the natural frequencies and mode shapes of a uniform rectangular plate carrying any number of elastically mounted masses.

For the vibration problem of a uniform plate carrying "spring-damper-mass" systems (with the damping effects considered), Das and Nazarene (1963) proposed a method to determine the natural frequencies of a rectangular plate carrying a single "spring-damper-mass" system. However, only the special case for a rectangular plate carrying a single dashpot was illustrated. Goyal and Sinha (1977) analyzed the vibration characteristics of a simply supported orthotropic square sandwich plate attached by a single spring-damper-mass system. Nicholson and Bergman (1986) derived the closed-form solutions for the eigenvalues and eigenfunctions of an undamped simply-supported rectangular plate attached by an undamped oscillator and a closed-form solution for the forced vibration responses of the foregoing constrained plate (or composite system) carrying a damper using the mode superposition method. To the authors knowledge, the literature relating to the vibration characteristics of a uniform rectangular plate carrying "any number of spring-damper-mass" systems is not found yet. Therefore, this paper aims at solving the last problem.

Wu and Luo (1997a, 1997b, 1997c) have shown that the ANCM was available for the free vibration analysis of a rectangular plate carrying any number of concentrated elements (such as point mass, translational springs, etc.). Hence, this paper tries to apply the ANCM to determine the eigenvalues of a rectangular plate carrying any number of spring-damper-mass systems. To this end, a technique was presented to replace each "spring-damper-mass" system by an equivalent massless "spring-damper " system. Since the degree of freedom for the lumped mass in each spring-damper-mass system was eliminated by the last equivalent system, one may use the ANCM (Wu and Luo 1997a) to solve the characteristic problem regarding a uniform plate carrying any number of "spring-damper-mass" systems.

Because of the existence of damping, the equation of motion derived from the ANCM is in "complex form" composed of a real part and an imaginary part. From either part, a set of simultaneous equations can be obtained. Since the simultaneous equations are in "real forms", the eigenvalues of the "constrained" plate were determined by the half-interval method (Faires and Burden 1993). To confirm the reliability of the introduced technique, all the numerical results obtained from the ANCM were compared with those obtained from the conventional finite element method (FEM).

2. Equation of motion for a uniform plate with a single spring-damper-mass system

For convenience, a plate without carrying any concentrated elements is called the "unconstrained" plate and the one with any concentrated elements attached is called the "constrained" plate in this

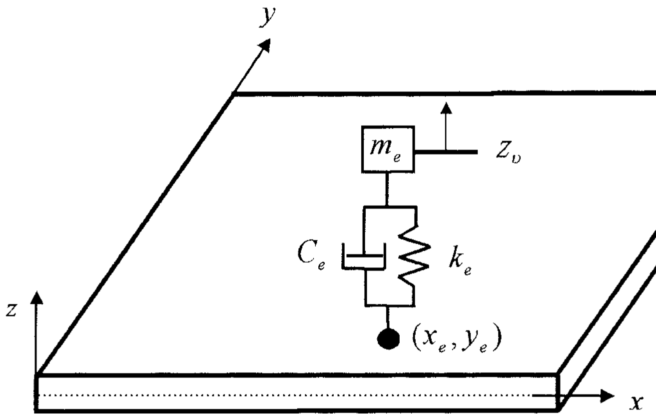

Fig. 1 A uniform rectangular plate carrying a spring-damper-mass system

paper. If the effects of shear deformation and rotatory inertia are neglected, then the equation of motion for a uniform plate carrying a single spring-damper-mass system is given by

$$D_E \nabla^4 w(x, y, t) + \bar{m} \frac{\partial^2 w(x, y, t)}{\partial t^2} = F_e(t) \cdot \delta(x - x_e)\delta(y - y_e) \tag{1}$$

where $D_E = Eh^3/[12(1 - \nu^2)]$ is the flexural rigidity of the plate, E is the Young's modulus, h is the thickness of the plate, ν is the Poisson's ratio, $\bar{m}$ is the mass per unit area of the plate, ∇^4 is the biharmonic differential operator, $w(x, y, t)$ is the transverse deflection of plate at position (x, y) and time t, $F_e(t)$ is the interaction force between the spring-damper-mass system and the plate at time t, (x_e, y_e) is the coordinate of the attaching point and $\delta(\cdot)$ is the Dirac delta function.

The equation of motion for the spring-damper-mass system is given by (see Fig. 1)

$$F_e(t) = -m_e \ddot{z}_\upsilon(t) = C_e[\dot{z}_\upsilon(t) - \dot{w}_e(t)] + k_e[z_\upsilon(t) - w_e(t)] \tag{2}$$

or

$$m_e \ddot{z}_\upsilon(t) + C_e \dot{z}_\upsilon(t) + k_e z_\upsilon(t) = C_e \dot{w}_e(t) + k_e w_e(t) \tag{3}$$

where k_e, C_e and m_e are the spring constant, damping coefficient and lumped mass of the spring-damper-mass system, respectively; $\ddot{z}_\upsilon(t), \dot{z}_\upsilon(t)$ and $z_\upsilon(t)$ are the acceleration, velocity and displacement of the lumped mass m_e, respectively; while $\dot{w}_e(t)$ and $w_e(t)$ are the transverse velocity and displacement of the plate at the attaching point (x_e, y_e).

According to the expansion theorem or the mode superposition method (Meirovitch 1967, Clough and Penzien 1975), the transverse deflection of the uniform plate can be obtained from

$$w(x, y, t) = \sum_{j=1}^{n'} \bar{W}_j(x, y) q_j(t) \tag{4}$$

where $\bar{W}_j(x, y)$ is the j-th normal mode shape of the "unconstrained" uniform plate, $q_j(t)$ is the associated j-th generalized coordinate and n' is the total number of modes considered. Hence, the transverse deflection of the unconstrained plate at the attaching point (x_e, y_e) is given by

$$w_e(t) = \sum_{j=1}^{n'} \overline{W}_j(x_e, y_e) q_j(t) \tag{5}$$

From Eqs. (3) and (5), one has the particular solution of $z_\upsilon(t)$ to be

$$z_\upsilon(t) = \bar{z}_\upsilon \sum_{j=1}^{n'} q_j(t) \tag{6}$$

where $\bar{z}_\upsilon(t)$ is the amplitude of $z_\upsilon(t)$.

When the "constrained" plate performs harmonic free vibration, the generalized coordinate $q_j(t)$ can be expressed by

$$q_j(t) = \sum_{j=1}^{n'} \bar{q}_j e^{(\overline{\omega}_R + \bar{i}\,\overline{\omega}_I)t} \tag{7}$$

where $\bar{q}_j$ is the amplitude of $q_j(t)$, $\overline{\omega}_R$ and $\overline{\omega}_I$ are the real part and imaginary part of the eigenvalue, t is time and $\bar{I} = \sqrt{-1}$.

To substitute Eq. (7) into Eq. (5) gives

$$w_e(t) = \sum_{j=1}^{n'} \overline{W}_j(x_e, y_e) \cdot \bar{q}_j e^{(\overline{\omega}_R + \bar{i}\,\overline{\omega}_I)t} \tag{8}$$

From Eqs. (6) and (7), one obtains

$$z_\upsilon(t) = \bar{z}_\upsilon \sum_{j=1}^{n'} \bar{q}_j e^{(\overline{\omega}_R + \bar{i}\,\overline{\omega}_I)t} \tag{9}$$

The derivatives of Eqs. (9) and (8) with respect to time t give

$$\begin{aligned} \dot{z}_\upsilon(t) &= \bar{z}_\upsilon(\overline{\omega}_R + \bar{i}\,\overline{\omega}_I) \sum_{j=1}^{n'} \bar{q}_j e^{(\overline{\omega}_R + \bar{i}\,\overline{\omega}_I)t} \\ &= (\overline{\omega}_R + \bar{i}\,\overline{\omega}_I) z_\upsilon(t) \end{aligned} \tag{10}$$

$$\begin{aligned} \ddot{z}_\upsilon(t) &= \bar{z}_\upsilon(\overline{\omega}_R + \bar{i}\,\overline{\omega}_I)^2 \sum_{j=1}^{n'} \bar{q}_j e^{(\overline{\omega}_R + \bar{i}\,\overline{\omega}_I)t} \\ &= [(\overline{\omega}_R^2 - \overline{\omega}_I^2) + \bar{i} \cdot 2\overline{\omega}_R \cdot \overline{\omega}_I] z_\upsilon(t) \end{aligned} \tag{11}$$

$$\dot{w}_e(t) = (\overline{\omega}_R + \bar{i}\,\overline{\omega}_I) \sum_{j=1}^{n'} \overline{W}_j(x_e, y_e) \bar{q}_j e^{(\overline{\omega}_R + \bar{i}\,\overline{\omega}_I)t} = (\overline{\omega}_R + \bar{i}\,\overline{\omega}_I) w_e(t) \tag{12}$$

From Eq. (12) one obtains

$$\bar{i} \cdot w_e(t) = \frac{1}{\overline{\omega}_I} \dot{w}_e(t) - \left(\frac{\overline{\omega}_R}{\overline{\omega}_I}\right) w_e(t) \tag{13}$$

To substitute the values of $\dot{z}_\upsilon(t)$, $\ddot{z}_\upsilon(t)$, $w_e(t)$ and $\dot{w}_e(t)$ defined by Eqs. (8)-(12) into Eq. (3), one has

$$z_\upsilon(t) = \frac{(C_e\bar{\omega}_R + k_e) + \bar{i}\,C_e\bar{\omega}_I}{[m_e(\bar{\omega}_R^2 - \bar{\omega}_I^2) + C_e\bar{\omega}_R + k_e] + \bar{i}\,[2\bar{\omega}_R\bar{\omega}_I m_e + C_e\bar{\omega}_I]} w_e(t) \tag{14}$$

From Eqs. (2), (11) and (14), one obtains the interactive force between the spring-damper-mass system and the plate

$$\begin{aligned} F_e(t) &= \left\{\frac{-m_e[(\bar{\omega}_R^2 - \bar{\omega}_I^2) + \bar{i}\cdot 2\bar{\omega}_R\bar{\omega}_I]\cdot[(C_e\bar{\omega}_R + k_e) + \bar{i}\cdot C_e\bar{\omega}_I]}{[m_e(\bar{\omega}_R^2 - \bar{\omega}_I^2) + C_e\bar{\omega}_R + k_e] + \bar{i}\,[2\bar{\omega}_R\bar{\omega}_I m_e + C_e\bar{\omega}_I]}\right\} w_e(t) \\ &= -\left[\frac{E_1 + \bar{i}\cdot F_1}{G_1 + \bar{i}\cdot H_1}\right] w_e(t) = -\left[\frac{(E_1G_1 + F_1H_1) + \bar{i}\,(F_1G_1 - E_1H_1)}{G_1^2 + H_1^2}\right] w_e(t) \end{aligned} \tag{15}$$

Substituting the value of $\bar{i}\cdot w_e(t)$ defined by Eq. (13) into Eq. (15) gives

$$\begin{aligned} F_e(t) &= -\frac{(E_1G_1 + F_1H_1)}{G_1^2 + H_1^2} w_e(t) - \frac{(F_1G_1 - E_1H_1)}{G_1^2 + H_1^2}\left[\frac{1}{\bar{\omega}_I}\dot{w}_e(t) - \left(\frac{\bar{\omega}_R}{\bar{\omega}_I}\right) w_e(t)\right] \\ &= \left[-\frac{(E_1G_1 + F_1H_1)}{G_1^2 + H_1^2} + \frac{(F_1G_1 - E_1H_1)}{G_1^2 + H_1^2}\cdot\left(\frac{\bar{\omega}_R}{\bar{\omega}_I}\right)\right] w_e(t) - \frac{(F_1G_1 - E_1H_1)}{G_1^2 + H_1^2}\left(\frac{1}{\bar{\omega}_I}\right)\dot{w}_e(t) \\ &= k_{eff} w_e(t) + C_{eff}\dot{w}_e(t) \end{aligned} \tag{16}$$

where

$$k_{eff} = \left[-\frac{E_1G_1 + F_1H_1}{G_1^2 + H_1^2} + \frac{F_1G_1 - E_1H_1}{G_1^2 + H_1^2}\left(\frac{\bar{\omega}_R}{\bar{\omega}_I}\right)\right] = \text{effective spring constant} \tag{17a}$$

$$C_{eff} = \left[-\frac{F_1G_1 - E_1H_1}{G_1^2 + H_1^2}\left(\frac{1}{\bar{\omega}_I}\right)\right] = \text{effective damping coefficient} \tag{17b}$$

$$E_1 = m_e[(\bar{\omega}_R^2 - \bar{\omega}_I^2)\cdot(C_e\bar{\omega}_R + k_e) - 2C_e\bar{\omega}_R\bar{\omega}_I^2]$$

$$F_1 = m_e[2\bar{\omega}_R\bar{\omega}_I(C_e\bar{\omega}_R + k_e) + C_e\bar{\omega}_i(\bar{\omega}_R^2 - \bar{\omega}_I^2)]$$

$$G_1 = [m_e(\bar{\omega}_R^2 - \bar{\omega}_I^2) + C_e\bar{\omega}_R + k_e]$$

$$H_1 = [2\bar{\omega}_R\bar{\omega}_I m_e + C_e\bar{\omega}_I]$$

Eq. (16) reveals that the effect of a “spring-damper-mass” system on the attached plate can be replaced by the effect of a massless equivalent “spring-damper” system with effective spring constant k_{eff} defined by Eq. (17a) and effective damping coefficient C_{eff} defined by Eq. (17b), as shown in Fig. 2.

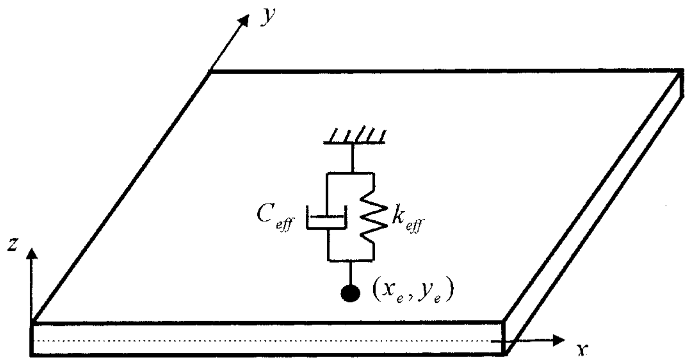

Fig. 2 The "spring-damper-mass" system shown in Fig. 1 may be replaced by a massless equivalent "spring-damper" system with effective spring constant k_{eff} and effective damping coefficient C_{eff}

To substitute Eqs. (4) and (16) into Eq. (1), to premultiply both sides of the resulting expression by $\overline{W}_k(x, y)$, and then to integrate the final equation over the area of the entire plate, A, one has

$$\int_A \sum_{j=1}^{n'} \overline{W}_k(x, y) D_E \nabla^4 \overline{W}_j(x, y) q_j(t) dA + \int_A \sum_{j=1}^{n'} \overline{W}_k(x, y) \overline{m} \overline{W}_j(x, y) \ddot{q}_j(t) dA$$

$$= \int_A F_e(t) \overline{W}_k(x, y) dA, \qquad k = 1, 2, \ldots, n' \tag{18}$$

The orthogonalitity of the normal mode shapes can be referred in Wu *et al.* (1997a, 1997b) and leads to

$$\int_A \overline{W}_k(x, y) D \nabla^4 \overline{W}_j(x, y) dA = 0, \qquad k \neq j \tag{19a}$$

$$\int_A \overline{W}_k(x, y) \overline{m} \overline{W}_j(x, y) dA = 0, \qquad k \neq j \tag{19b}$$

Hence, Eq. (18) reduces to

$$\overline{M}_{jj} \ddot{q}_j(t) + \overline{K}_{jj} q_j(t) = \overline{N}_{jj}(t), \qquad j = 1, 2, \ldots, n' \tag{20}$$

where

$$\overline{M}_{jj} = \int_A \overline{W}_j \overline{m} \overline{W}_j dA \tag{21a}$$

$$\overline{K}_{jj} = \int_A \overline{W}_j D_E \nabla^4 \overline{W}_j dA \tag{21b}$$

$$N_{jj} = [(k_{eff} + \overline{\omega}_R \cdot C_{eff}) + \bar{i} \cdot (\overline{\omega}_I \cdot C_{eff})] \cdot \sum_{k=1}^{n'} \overline{W}_k(x_e, y_e) \overline{W}_k(x_e, y_e) q_j(t) \tag{21c}$$

Since $\overline{M}_{jj} = 1.0$, Eq. (20) can further be reduced to

$$\ddot{q}_j(t) + \omega_j^2 q_j(t) = N_{jj}(t), \qquad j = 1, 2, \ldots, n' \tag{22}$$

where $\omega_j = \sqrt{K_{jj}/M_{jj}} = \sqrt{K_{jj}}$ is the j-th natural frequency of the "unconstrained" plate.

To place the values of $q_j(t)$ and $\bar{N}_{jj}(t)$ defined by Eqs. (7) and (21c) into Eq. (22) gives the equation of motion for a uniform plate carrying a spring-damper-mass system

$$(\bar{\omega}_R + \bar{i} \cdot \bar{\omega}_I)^2 \sum_{j=1}^{n'} \bar{q}_j(t) e^{(\bar{\omega}_R + \bar{i}\,\bar{\omega}_I)} + \omega_j^2 \sum_{j=1}^{n'} \bar{q}_j(t) e^{(\bar{\omega}_R + \bar{i}\,\bar{\omega}_I)}$$
$$= [(k_{eff} + \bar{\omega}_R \cdot C_{eff}) + \bar{i} \cdot (\bar{\omega}_I \cdot C_{eff})] \cdot \sum_{k=1}^{n'} \bar{W}_k(x_e, y_e) \bar{W}_k(x_e, y_e) \bar{q}_j(t) e^{(\bar{\omega}_R + \bar{i}\,\bar{\omega}_I)} \tag{23a}$$

or

$$\omega_j^2 \bar{q}_j(t) - (k_{eff} + \bar{\omega}_R \cdot C_{eff}) \cdot \sum_{k=1}^{n'} \bar{W}_k(x_e, y_e) \bar{W}_k(x_e, y_e) \bar{q}_j(t)$$
$$- \bar{i} \cdot (\bar{\omega}_I \cdot C_{eff}) \cdot \sum_{k=1}^{n'} \bar{W}_k(x_e, y_e) \bar{W}_k(x_e, y_e) \bar{q}_j(t)$$
$$= -(\bar{\omega}_R^2 - \bar{\omega}_I^2) \bar{q}_j(t) - \bar{i} \cdot 2 \bar{\omega}_R \bar{\omega}_I \bar{q}_j(t), \qquad j = 1, 2, \ldots, n' \tag{23b}$$

To separate the real parts and imaginary parts on both sides of the last equation, one has

$$\omega_j^2 \bar{q}_j(t) - (k_{eff} + \bar{\omega}_R \cdot C_{eff}) \cdot \sum_{k=1}^{n'} \bar{W}_k(x_e, y_e) \bar{W}_k(x_e, y_e) \bar{q}_j(t)$$
$$= -(\bar{\omega}_R^2 - \bar{\omega}_I^2) \bar{q}_j(t), \qquad j = 1, 2, \ldots, n' \text{ (from real parts)} \tag{24a}$$

and

$$(\bar{\omega}_I \cdot C_{eff}) \cdot \sum_{k=1}^{n'} \bar{W}_k(x_e, y_e) \bar{W}_k(x_e, y_e) \bar{q}_j(t)$$
$$= 2 \bar{\omega}_R \bar{\omega}_I \bar{q}_j(t), \qquad j = 1, 2, \ldots, n' \text{ (from imaginary parts)} \tag{24b}$$

The eigenvalues of the "constrained" plate can be obtained either from Eq. (24a) or Eq. (24b). To avoid confusing, continuous derivation of the equation of motion for the constrained plate from Eq. (24b) is placed in Appendix A at the end of this paper.

The matrix form of Eq. (24a) is given by

$$[A]\{\bar{q}_j\} = (\bar{\omega}_I^2 - \bar{\omega}_R^2)[B]\{\bar{q}_j\} \tag{25}$$

where

$$[A]_{n' \times n'} = [\ulcorner \omega^2 \lrcorner]_{n' \times n'} + [A']_{n' \times n'} \tag{26a}$$

$$[B]_{n' \times n'} = [\ulcorner I \lrcorner]_{n' \times n'} = \ulcorner 11 \cdots\cdots 11 \lrcorner_{n' \times n'} \tag{26b}$$

$$[A']_{n' \times n'} = -(k_{eff} + \bar{\omega}_R \cdot C_{eff}) \cdot [\bar{W}_j(x_e, y_e)]_{n' \times n'} \tag{26c}$$

$$[\overline{W}_j(x_e, y_e)]_{n' \times n'} = \{\overline{W}_j(x_e, y_e)\}_{n' \times 1} \cdot \{\overline{W}_j(x_e, y_e)\}^T_{n' \times 1} \tag{26d}$$

$$\{\overline{W}_j(x_e, y_e)\}_{n' \times 1} = \{\overline{W}_1(x_e, y_e)\overline{W}_2(x_e, y_e)\cdots\cdots\overline{W}_{n'}(x_e, y_e)\}_{n' \times 1} \tag{26e}$$

$$\{\bar{q}_j\}_{n' \times 1} = \{\bar{q}_1\bar{q}_2\cdots\cdots\bar{q}_{n'}\}_{n' \times 1} \tag{26f}$$

$$[\diagdown\omega^2\diagdown]_{n' \times n'} = \ulcorner\omega_1^2\ \omega_2^2 \cdots \omega_{n'}^2\lrcorner \tag{26g}$$

In the above equations, the symbols { }, [ ] and ⌜ ⌟ represent the column matrix, square matrix and diagonal matrix, respectively.

Since k_{eff} and C_{eff} are function of the unknown $\overline{\omega}_R$ and $\overline{\omega}_I$ as shown in Eqs. (17a)-(17b), the eigenvalues $\overline{\omega}_R \pm \bar{i}\,\overline{\omega}_I$ can not be obtained from Eq. (25) by means of the Jacobi method (Meirovitch 1967). However, Eq. (25) can be rewritten as

$$([A] - (\overline{\omega}_I^2 - \overline{\omega}_R^2)[B])\{\bar{q}_j\} = 0 \tag{27}$$

The nontrivial solution of Eq. (27) requires that

$$\left|[A] - (\overline{\omega}_I^2 - \overline{\omega}_R^2)[B]\right| = 0 \tag{28}$$

which is the frequency equation and its roots define the eigenvalues of the constrained plate, $\overline{\omega}_R \pm \bar{i}\,\overline{\omega}_I$. Therefore, the half-interval method (Faires and Burden 1993) may be used to solve Eq. (28). From Eqs. (28), (16) and (17a)-(17b), one sees that the frequency equation is a function of two unknown $\overline{\omega}_R$ and $\overline{\omega}_I$, hence two trial values for $\overline{\omega}_R$ and $\overline{\omega}_I$ are required when cut and trial procedures were performed. It is evident that simultaneous guessing two trial values for $\overline{\omega}_R$ and $\overline{\omega}_I$ is very difficult. To overcome this difficulty, a relationship between $\overline{\omega}_R$ and $\overline{\omega}_I$ is derived by

$$\overline{\omega}_{jR} = -\frac{\zeta_j}{\sqrt{1-\zeta_j^2}}\overline{\omega}_{jI}, \qquad j = 1, 2, \ldots \tag{29}$$

Eq. (29) was obtained from the free vibration curves and the relationship between the damped natural frequency and the undamped one for a single-degree-of-freedom damped system (Librescu and Na 1997).

In Eq. (29), ζ_j is the damping ratio associated with the j-th mode shape of the “unconstrained” plate and is defined by

$$\zeta_j = C_j^* / (2 \cdot m_j^* \omega_j) \tag{30}$$

In Eq. (30), C_j^* and m_j^* are the generalized damping coefficient and generalized mass given by

$$\begin{aligned} C_j^* &= \int_A \overline{W}_j(x, y)\left[\sum_{\upsilon=1}^{r} C_{e,\upsilon} \cdot \delta(x - x_e)\delta(y - y_e)\right]\overline{W}_j(x, y)dA \\ &= \sum_{\upsilon=1}^{r} C_{e,\upsilon} \cdot \overline{W}_j^2(x_{e,\upsilon}, y_{e,\upsilon}) \end{aligned} \tag{31}$$

$$m_j^* = \int_A \bar{m} \cdot \overline{W}_j(x, y) \cdot \overline{W}_j(x, y) dA = 1 \tag{32}$$

and ω_j is the j-th natural frequency of the "unconstrained" plate.

Therefore, one only requires to guess the value of $\overline{\omega}_I$ and then to calculate the associated value of $\overline{\omega}_R$ from Eq. (29). If this pair of values for $\overline{\omega}_R$ and $\overline{\omega}_I$ satisfy Eq. (28), then they represent one of the eigenvalues of the constrained plate, otherwise, iteration with a new pair of values for $\overline{\omega}_R$ and $\overline{\omega}_I$ is required.

3. Equation of motion for a uniform plate carrying any number of spring-damper-mass systems

For the uniform plate carrying r spring-damper-mass systems as shown in Fig. 3, from Eq. (23) one may infer the equation of motion for the constrained plate to be

$$\begin{aligned} &\omega_j^2 \bar{q}_j(t) - \sum_{\upsilon=1}^{r} (k_{eff,\upsilon} + \overline{\omega}_R \cdot C_{eff,\upsilon}) \sum_{k=1}^{n'} \overline{W}_k(x_{e,\upsilon}, y_{e,\upsilon}) \overline{W}_k(x_{e,\upsilon}, y_{e,\upsilon}) \bar{q}_j(t) \\ &- \bar{i} \cdot \sum_{\upsilon=1}^{r} (\overline{\omega}_I \cdot C_{eff,\upsilon}) \sum_{k=1}^{n'} \overline{W}_k(x_{e,\upsilon}, y_{e,\upsilon}) \overline{W}_k(x_{e,\upsilon}, y_{e,\upsilon}) \bar{q}_j(t) \\ &= -(\overline{\omega}_R^2 - \overline{\omega}_I^2) \bar{q}_j(t) - \bar{i} \cdot 2 \overline{\omega}_R \overline{\omega}_I \bar{q}_j(t), \qquad j = 1, 2, \ldots, n' \end{aligned} \tag{33}$$

Equating the real parts on the both sides of the last equation yields

$$\begin{aligned} &\omega_j^2 \bar{q}_j(t) - \sum_{\upsilon=1}^{r} (k_{eff,\upsilon} + \overline{\omega}_R \cdot C_{eff,\upsilon}) \sum_{k=1}^{n'} \overline{W}_k(x_{e,\upsilon}, y_{e,\upsilon}) \overline{W}_k(x_{e,\upsilon}, y_{e,\upsilon}) \bar{q}_j(t) \\ &= -(\overline{\omega}_R^2 - \overline{\omega}_I^2) \bar{q}_j(t), \qquad j = 1, 2, \ldots, n' \end{aligned} \tag{34a}$$

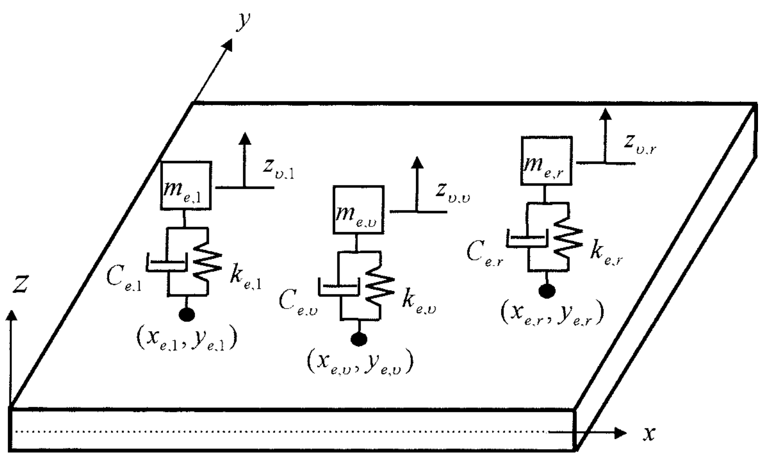

Fig. 3 A uniform rectangular plate carrying r spring-damper-mass systems

Similarly, considering the equality of the imaginary parts of Eq. (33), one obtains

$$\sum_{\upsilon=1}^{r}(\overline{\omega}_I \cdot C_{eff,\,\upsilon})\sum_{k=1}^{n'}\overline{W}_k(x_{e,\,\upsilon}, y_{e,\,\upsilon})\overline{W}_k(x_{e,\,\upsilon}, y_{e,\,\upsilon})\overline{q}_j(t)$$
$$= 2\overline{\omega}_R\overline{\omega}_I\overline{q}_j(t), \qquad j = 1, 2, \ldots, n' \tag{34b}$$

The continuous derivation based on Eq. (34b) is placed in Appendix B. To write Eq. (34a) in matrix form gives

$$[\tilde{A}]\{\overline{q}_j\} = (\overline{\omega}_I^2 - \overline{\omega}_R^2)[\tilde{B}]\{\overline{q}_j\} \tag{35}$$

where

$$[\tilde{A}]_{n'\times n'} = [\diagdown\omega^2\diagdown]_{n'\times n'} + [\tilde{A}']_{n'\times n'} \tag{36a}$$

$$[\tilde{B}]_{n'\times n'} = [\diagdown I\diagdown]_{n'\times n'} = \ulcorner 11\cdots\cdots 11 \lrcorner_{n'\times n'} \tag{36b}$$

$$[\tilde{A}']_{n'\times n'} = -\sum_{\upsilon=1}^{r}(k_{eff,\,\upsilon} + \overline{\omega}_R \cdot C_{eff}) \cdot [\overline{W}_j(x_{e,\,\upsilon}, y_{e,\,\upsilon})]_{n'\times n'} \tag{36c}$$

$$[\overline{W}_j(x_{e,\,\upsilon}, y_{e,\,\upsilon})]_{n'\times n'} = \{\overline{W}_j(x_{e,\,\upsilon}, y_{e,\,\upsilon})\}_{n'\times 1} \cdot \{\overline{W}_j(x_{e,\,\upsilon}, y_{e,\,\upsilon})\}^T_{n'\times 1} \tag{36d}$$

$$\{\overline{W}_j(x_{e,\,\upsilon}, y_{e,\,\upsilon})\}_{n'\times 1} = \{\overline{W}_1(x_{e,\,\upsilon}, y_{e,\,\upsilon})\overline{W}_2(x_{e,\,\upsilon}, y_{e,\,\upsilon})\cdots\cdots\overline{W}_{n'}(x_{e,\,\upsilon}, y_{e,\,\upsilon})\}_{n'\times 1} \tag{36e}$$

$$\{\overline{q}_j\}_{n'\times 1} = \{\overline{q}_1\overline{q}_2\cdots\cdots\overline{q}_{n'}\}_{n'\times 1} \tag{36f}$$

$$[\diagdown\omega^2\diagdown]_{n'\times n'} = \ulcorner\omega_1^2\ \omega_2^2\ \cdots\ \omega_{n'}^2\lrcorner \tag{36g}$$

The value of $k_{eff,\,\upsilon}$ and $C_{eff,\,\upsilon}$ appearing in Eq. (36) is given by [c.f. Eqs. (17a)-(17b)]

$$k_{eff,\,\upsilon} = \left[-\frac{E_{1\upsilon}G_{1\upsilon} + F_{1\upsilon}H_{1\upsilon}}{G_{1\upsilon}^2 + H_{1\upsilon}^2} + \frac{F_{1\upsilon}G_{1\upsilon} - E_{1\upsilon}H_{1\upsilon}}{G_{1\upsilon}^2 + H_{1\upsilon}^2}\left(\frac{\overline{\omega}_R}{\overline{\omega}_I}\right)\right] \tag{37a}$$

$$C_{eff,\,\upsilon} = \left[-\frac{F_{1\upsilon}G_{1\upsilon} - E_{1\upsilon}H_{1\upsilon}}{G_{1\upsilon}^2 + H_{1\upsilon}^2}\left(\frac{1}{\overline{\omega}_I}\right)\right] \tag{37b}$$

where

$$E_{1\upsilon} = m_{e,\,\upsilon}[(\overline{\omega}_R^2 - \overline{\omega}_I^2) \cdot (C_{e,\,\upsilon}\overline{\omega}_R + k_{e,\,\upsilon}) - 2C_{e,\,\upsilon}\overline{\omega}_R\overline{\omega}_I^2] \tag{38a}$$

$$F_{1\upsilon} = m_{e,\,\upsilon}[2\overline{\omega}_R\overline{\omega}_I(C_{e,\,\upsilon}\overline{\omega}_R + k_{e,\,\upsilon}) + C_{e,\,\upsilon}\overline{\omega}_I(\overline{\omega}_R^2 - \overline{\omega}_I^2)] \tag{38b}$$

$$G_{1\upsilon} = [m_{e,\,\upsilon}(\overline{\omega}_R^2 - \overline{\omega}_I^2) + C_{e,\,\upsilon}\overline{\omega}_R + k_{e,\,\upsilon}] \tag{38c}$$

$$H_{1\upsilon} = [2\overline{\omega}_R\overline{\omega}_I m_{e,\,\upsilon} + C_{e,\,\upsilon}\overline{\omega}_I] \tag{38d}$$

After rewriting Eq. (35) to the form of Eq. (28), one may use the same technique as the one employed to solve Eq. (28) to treat the problem.

4. Determining the eigenvalues of a constrained plate using FEM

In order to confirm the reliability of the presented theory, all the results obtained from the ANCM are checked by the ones obtained from the conventional finite element method (FEM). Fig. 4 shows the plate element carrying four spring-damper-mass systems at the nodes (A, B, C and D). The element mass matrix $[M]^{(e)}$, damping matrix $[C]^{(e)}$ and stiffness matrix $[K]^{(e)}$ are given by

$$[M]^e = \begin{array}{c} \begin{array}{ccccccccc} u_1 \;\cdots & u_4 \;\cdots & u_7 \;\cdots & u_{10} \;\cdots & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} \end{array} \\ \left[\begin{array}{ccccccccc} M_{1,1} & & & & M_{1,12} & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 \\ & M_{4,4} & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 \\ & & M_{7,7} & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 \\ & & & M_{10,10} & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & 0 \\ M_{12,1} & & & & M_{12,12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{e,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_{e,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{e,3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{e,4} \end{array}\right] \end{array} \begin{array}{c} \\ u_1 \\ \vdots \\ u_4 \\ \vdots \\ u_7 \\ \vdots \\ u_{10} \\ \vdots \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16} \end{array} \tag{39a}$$

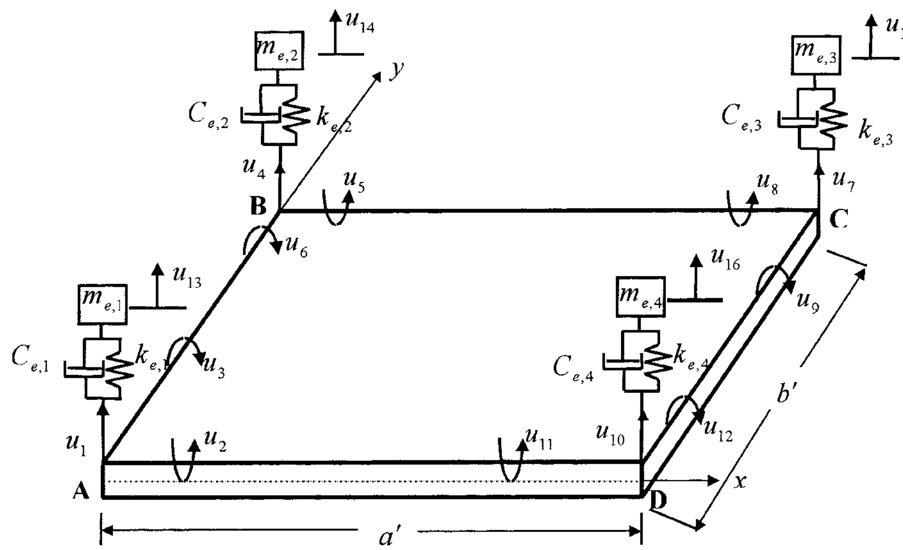

Fig. 4 A plate element attached by four spring-damper-mass systems at the four nodes A, B, C and D

$$
[K]^e = \begin{array}{c}
\begin{array}{ccccccccc} u_1 \ \cdots & u_4 \ \cdots & u_7 \ \cdots & u_{10} \ \cdots & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} \end{array} \\
\left[\begin{array}{ccccccccc}
K_{1,1}+k_{e,1} & & & & -K_{1,12} & -k_{e,1} & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 \\
 & K_{4,4}+k_{e,2} & & & & 0 & -k_{e,2} & 0 & 0 \\
 & & & & & 0 & 0 & 0 & 0 \\
 & & K_{7,7}+k_{e,3} & & & 0 & 0 & -k_{e,3} & 0 \\
 & & & & & 0 & 0 & 0 & 0 \\
 & & & K_{10,10}+k_{e,4} & & 0 & 0 & 0 & -k_{e,4} \\
 & & & & & 0 & 0 & 0 & 0 \\
K_{12,1} & & & & K_{12,12} & 0 & 0 & 0 & 0 \\
-k_{e,1} & 0 & 0 & 0 & 0 & k_{e,1} & 0 & 0 & 0 \\
0 & -k_{e,2} & 0 & 0 & 0 & 0 & k_{e,2} & 0 & 0 \\
0 & 0 & -k_{e,3} & 0 & 0 & 0 & 0 & k_{e,3} & 0 \\
0 & 0 & 0 & -k_{e,4} & 0 & 0 & 0 & 0 & k_{e,4}
\end{array}\right]
\end{array}
\begin{array}{c} \\ u_1 \\ \vdots \\ u_4 \\ \vdots \\ u_7 \\ \vdots \\ u_{10} \\ \vdots \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16} \end{array}
\tag{39b}
$$

$$
[C]^e = \begin{array}{c}
\begin{array}{cccccccccc} u_1 \ \cdots & u_4 \ \cdots & u_7 \ \cdots & u_{10} \ \cdots & & u_{12} & u_{13} & u_{14} & u_{15} & u_{16} \end{array} \\
\left[\begin{array}{cccccccccc}
c_{e,1} & & & & & & -c_{e,1} & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & c_{e,2} & & & & & 0 & -c_{e,2} & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & & c_{e,3} & & & & 0 & 0 & -c_{e,3} & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & & & c_{e,4} & & & 0 & 0 & 0 & -c_{e,4} \\
 & & & & & & 0 & 0 & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 \\
-c_{e,1} & 0 & 0 & 0 & & 0 & c_{e,1} & 0 & 0 & 0 \\
0 & -c_{e,2} & 0 & 0 & & 0 & 0 & c_{e,2} & 0 & 0 \\
0 & 0 & -c_{e,3} & 0 & 0 & 0 & 0 & 0 & c_{e,3} & 0 \\
0 & 0 & 0 & -c_{e,4} & 0 & 0 & 0 & 0 & 0 & c_{e,4}
\end{array}\right]
\end{array}
\begin{array}{c} \\ u_1 \\ \vdots \\ u_4 \\ \vdots \\ u_7 \\ \vdots \\ u_{10} \\ \vdots \\ u_{12} \\ u_{13} \\ u_{14} \\ u_{15} \\ u_{16} \end{array}
\tag{39c}
$$

In Eqs. (39a)-(39c), $k_{e,s}$, $c_{e,s}$ and $m_{e,s}$ (s = 1, 2, 3, 4) are the spring constants, damping coefficients and lumped masses of the four spring-damper-mass systems, respectively. K_{ij} and M_{ij} (i, j = 1-12) are the coefficients of the stiffness matrix and mass matrix for an "unconstrained" plate element (Przemieniecki 1968, Warburton 1976).

Assembling all the element property matrices ($[M]^{(e)}, [C]^{(e)}$ and $[K]^{(e)}$) and imposing the prescribed boundary conditions at the four sides of the plate, one obtains the following equation of motion for a plate carrying any number of spring-damper-mass systems

$$
[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = 0 \tag{40}
$$

where $[M]$, $[C]$ and $[K]$ are the overall mass, damping and stiffness matrices, while $\{\ddot{U}\}$, $\{\dot{U}\}$ and $\{U\}$ are the overall node acceleration, velocity and displacement vectors, respectively.

To solve the problem, Eq. (40) is rewritten as (Tse, Morse and Hinkle 1978)

$$\begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix}_{2n\times 2n} \begin{Bmatrix} \ddot{U} \\ \dot{U} \end{Bmatrix}_{2n\times 1} + \begin{bmatrix} -[M] & [0] \\ [0] & [K] \end{bmatrix}_{2n\times 2n} \begin{Bmatrix} \dot{U} \\ U \end{Bmatrix}_{2n\times 1} = 0 \tag{41a}$$

or

$$\{\dot{\phi}\} - [\hat{K}]\{\phi\} = 0 \tag{41b}$$

where n represents the total degrees of freedom for the constrained plate and

$$\{\phi\} = \begin{Bmatrix} \dot{U} \\ U \end{Bmatrix} \tag{42a}$$

$$-[\hat{K}] = \begin{bmatrix} -[\hat{K}_{11}] & -[\hat{K}_{12}] \\ [I] & [0] \end{bmatrix}, \tag{42b}$$

$$[\hat{K}_{11}] = [M]^{-1}[C], \qquad [\hat{K}_{12}] = [M]^{-1}[K] \tag{42c}$$

In Eq. (42b), $[I]$ is a unit matrix of order n.
For harmonic free vibration, one has

$$\{\phi\} = \{\Phi\}e^{\gamma t} \tag{43}$$

From Eqs. (41b) and (43) one obtains the eigen equation

$$(\gamma[\hat{I}] - [\hat{K}])\{\Phi\} = 0 \tag{44}$$

where $[\hat{I}]$ is a unit matrix of order $2n$. To solve Eq. (44), the EISPACK computer package of MATLAB (Inman and Daniel 1994) is used. The eigenvalues of Eq. (44) are complex numbers, its real parts denote the decaying parameters of vibrations and its imaginary parts denote the natural frequencies of the constrained plate.

5. Numerical results and discussions

In this section, four support (boundary) conditions of the constrained plate are studied. For convenience, a four-letter acronym is used to designate the type of support, starting at the left edge and proceeding in a counterclockwise direction. Hence, if the clamped, free and simply supported edges are denoted by C, F and S, respectively, then the boundary conditions of Figs. 5(a)-5(d) are represented by SSSS, SSSC, SCSC and SFSF, respectively.

The dimensions and physical properties for the rectangular plate are : $a = 2.0$ m, $b = 3.0$ m, $h = 0.005$ m, $\nu = 0.3$, $\rho = 7850.$ kg/m^3, $\rho_A = \rho h = 39.25$ kg/m^2, $E = 2.051\times10^{11}$ N/m^2, $D_E = Eh^3/$

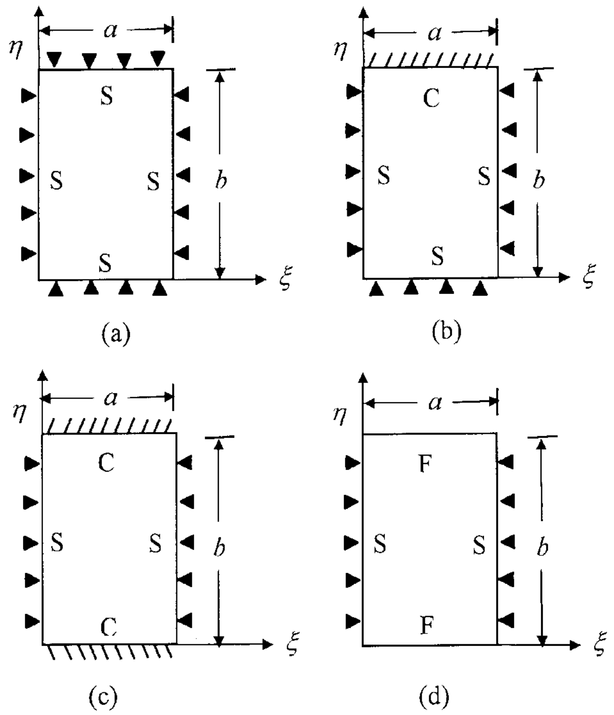

Fig. 5 The four support conditions of the rectangular plate studied: (a) SSSS, (b) SSSC, (c) SCSC, (d) SFSF

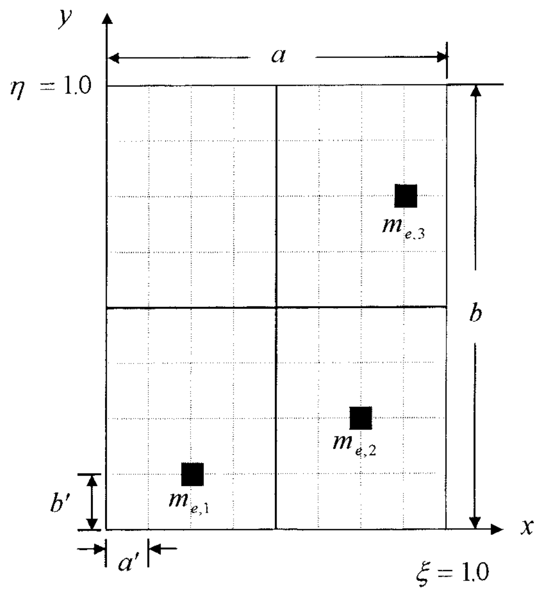

Fig. 6 A uniform rectangular plate carrying three spring-damper-mass systems

$[12(1-\nu^2)] = 2.3478 \times 10^3$ N-m, $m_p = \rho hab = 235.5$ kg and $k_p = D_E/a^2 = 5.8695 \times 10^2$ N/m, $c_p = m_p/a^2 \cdot \sqrt{D_E/\rho_A} = 455.346$ N-s/m. Note that m_p is the total mass of plate and k_p is the stiffness of plate. Since m_p, k_p and c_p are the important parameters for mass, stiffness and damping of the plate, respectively, they are used as the bases of the dimensionless parameters, $m^*_{e,\upsilon}(= m_{e,\upsilon}/m_p)$, $k^*_{e,\upsilon}(= k_{e,\upsilon}/k_p)$ and $c^*_{e,\upsilon}(= C_{e,\upsilon}/c_p)$, where $\upsilon = 1, 2, \ldots$, in the following discussions.

For the present problem, the accuracy of the lowest five eigenvalues obtained from the ANCM with 30 modes (i.e., $n' = 30$) is approximately equal to that obtained from the FEM with 64 plate elements (i.e., $n_e = 64$). Therefore, the following comparisons are based on $n' = 30$ for the ANCM and $n_e = 64$ for the FEM [the size of each element is $a' \times b' = (a/8) \times (b/8)$, see Fig. 6]. This criterion is the same as that adopted by Wu and Luo (1997).

5.1 Reliability of the theory and the computer programs

In this subsection, the lowest five eigenvalues of a SSSS uniform rectangular plate carrying one elastically mounted concentrated mass (i.e., a spring-damper-mass system with $C_{e,\upsilon} = 0$) studied by Avalos *et al.* (1993) are calculated with ANCM and FEM, and then compared with the results given by Avalos *et al.* (1993), as shown in Table 1. It can be seen that the reliability of both the theory and the computer programs of this paper is satisfactory. It is noted that the eigenvalues given by Avalos *et al.* (1993) are the frequency coefficients Ω_j. However, the eigenvalues shown in Table 1 are the actual natural frequencies $\bar{\omega}_j$ and the relationship between them is given by $\bar{\omega}_j = (\Omega_j/a^2)\sqrt{D_E/\rho_A}(j = 1, 2, \ldots)$.

Table 1 The lowest five eigenvalues of the SSSS plate with a spring-damper-mass system as shown in Fig. 1

Location of spring-damper-mass system (ξ_{e1}, η_{e1})	$k^*_{e,1}=\frac{k_{e,1}a^2}{D_E}$	$m^*_{e,1}=\frac{m_{e,1}}{m_p}$	$c^*_{e,1}=\frac{C_{e,1}}{c_p}$	Methods	Eigenvalues $\bar{\omega}_j = \bar{\omega}_{jR} \pm \bar{\omega}_{jI} \cdot \bar{i}$				
					$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
(0.75, 0.75)	0.5	0.25	0.0	FEM	$\pm 3.86459\bar{i}$	$\pm 93.5355\bar{i}$	$\pm 157.6363\bar{i}$	$\pm 260.2014\bar{i}$	$\pm 325.2037\bar{i}$
				ANCM	$+3.86527\bar{i}$	$+95.4344\bar{i}$	$+152.6880\bar{i}$	$+248.0860\bar{i}$	$+324.4220\bar{i}$
				Avalos *et al.* 1993	3.86505	95.4347	152.6884	248.0864	324.4224
	1.0	0.5	0.0	FEM	$\pm 3.86263\bar{i}$	$\pm 93.6035\bar{i}$	$\pm 157.7170\bar{i}$	$\pm 260.2226\bar{i}$	$\pm 325.9596\bar{i}$
				ANCM	$+3.86299\bar{i}$	$+95.4529\bar{i}$	$+152.7130\bar{i}$	$+248.0930\bar{i}$	$+324.4330\bar{i}$
				Avalos *et al.* 1993	3.86306	95.4541	152.7129	248.0940	324.4339

Note: $\bar{i} = \sqrt{-1}$, $a = 2.0$ m, $b = 1.0$ m, $h = 0.005$ m, $\upsilon = 0.3$, $\rho = 7850$ Kg/m^3, $m_p = \rho hab = 78.5$ Kg, $E = 2.051\times10^{11}$ N/m^2, $c_p = m_p/a^2 \cdot \sqrt{D_E/\rho_A}$, $D_E = Eh^3/[12(1-\upsilon)] = 2.3478\times10^3$ N–m

5.2 Eigenvalues for a rectangular plate with one spring-damper-mass system

Fig. 1 shows a uniform rectangular plate carrying one spring-damper-mass system located at $(\xi_{e1}, \eta_{e1}) = (0.75, 0.75)$, where $(\xi_{ei}, \eta_{ei}) = (x_{ei}/a, y_{ei}/b)$. The lowest five eingenvalues $\bar{\omega}_j$ $(j = 1\sim5)$ and the dimensions and physical properties are shown in Table 2. From Table 2 one sees that the results obtained from the ANCM are very close to those from the conventional FEM.

Table 2 The lowest five eigenvalues of a rectangular plate with a spring-damper-mass system ($k^*_{e,1}$ = 1.0, $m^*_{e,1}$ = 0.5, $c^*_{e,1}$ = 0.5) as shown in Fig. 1

Location of spring-damper-mass system (ξ_{e1}, η_{e1})	Boundary conditions	Methods	Eigenvalues $\overline{\omega}_j = \overline{\omega}_{jR} + \overline{\omega}_{jI} \cdot \overline{i}$					CPU time (sec)
			$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$	
(0.75,0.75)	SSSS	FEM	–0.48405 ±25.73792 $\overline{i}$	–0.96648 ±53.50331 $\overline{i}$	–0.96732 ±85.01285 $\overline{i}$	–0.48594 ±97.50865 $\overline{i}$	–1.94557 ±112.9208 $\overline{i}$	146.64
		ANCM	–0.48664 +27.6138 $\overline{i}$	–0.97119 +53.06409 $\overline{i}$	–0.97186 +84.84805 $\overline{i}$	–0.48809 +95.42911 $\overline{i}$	–1.93020 +110.3193 $\overline{i}$	30.52
	SSSC	FEM	–0.26793 ±28.60939 $\overline{i}$	–0.88967 ±60.92763 $\overline{i}$	–0.67580 ±86.55126 $\overline{i}$	–0.95553 ±109.7843 $\overline{i}$	–1.90408 ±118.1543 $\overline{i}$	153.45
		ANCM	–0.26776 +30.12523 $\overline{i}$	–0.89451 +60.08168 $\overline{i}$	–0.68693 +86.16808 $\overline{i}$	–0.97715 +107.1034 $\overline{i}$	–1.86806 +114.9671 $\overline{i}$	31.03
	SCSC	FEM	–0.38592 ±32.47550 $\overline{i}$	–1.01113 ±69.58031 $\overline{i}$	–0.84660 ±88.48082 $\overline{i}$	–0.19543 ±123.3243 $\overline{i}$	–2.74523 ±124.1396 $\overline{i}$	134.92
		ANCM	–0.38717 +33.59601 $\overline{i}$	–1.01642 +68.34100 $\overline{i}$	–0.84948 +87.84042 $\overline{i}$	–0.18099 +119.9873 $\overline{i}$	–2.89282 +120.0001 $\overline{i}$	27.79
	SFSF	FEM	–0.47289 ±15.23623 $\overline{i}$	–0.40447 ±23.82540 $\overline{i}$	–0.01584 ±45.22968 $\overline{i}$	–0.88698 ±75.06602 $\overline{i}$	–0.65858 ±80.50251 $\overline{i}$	246.64
		ANCM	–0.46868 +18.76158 $\overline{i}$	–0.39061 +25.10635 $\overline{i}$	–0.02067 +44.38113 $\overline{i}$	–0.95822 +75.61264 $\overline{i}$	–0.57918 +78.03154 $\overline{i}$	29.33

Note: $\overline{i} = \sqrt{-1}$, a = 2.0 m, b = 3.0 m, h = 0.005 m, υ = 0.3, ρ = 7850 Kg/m^3, $m_p = \rho hab$ = 235.5 Kg, E = 2.051×10^{11} N/m^2, $k_p = D_E/a^2$ = 5.8695×10^2 N/m, $D_E = Eh^3/[12(1-\upsilon)]$ = 2.3478×10^3 N–m, $c_p = m_p/a^2 \cdot \sqrt{D_E/\rho_A}$ = 455.34611 N–s/m

5.3 Eigenvalues for a rectangular plate carrying multiple spring-damper-mass systems

Fig. 6 shows a uniform plate carrying three spring-damper-mass systems with locations, magnitudes, and physical properties of the three spring-damper-mass systems shown in Table 3. The lowest five eigenvalues $\overline{\omega}_j = \overline{\omega}_{jR} \pm \overline{\omega}_{jI} \cdot \overline{i}$ (j = 1-5), obtained from the conventional FEM and those from the ANCM, are listed in Table 4. From Table 4 one sees that the lowest five eigenvalues obtained from the two methods are also in good agreement.

Table 3 The locations and magnitudes of the three spring-damper-mass systems shown in Fig. 6

Locations of spring-damper-mass system $(\xi_{ei}, \eta_{ei}) = (x_{ei}/a, y_{ei}/b)$			Magnitudes of point masses			Magnitudes of spring constants			Magnitudes of damping coefficients		
(ξ_{e1}, η_{e1})	(ξ_{e2}, η_{e2})	(ξ_{e3}, η_{e3})	$m^*_{e,1}$	$m^*_{e,2}$	$m^*_{e,3}$	$k^*_{e,1}$	$k^*_{e,2}$	$k^*_{e,3}$	$c^*_{e,1}$	$c^*_{e,2}$	$c^*_{e,3}$
(0.25, 0.125)	(0.75,0.25)	(0.875,0.75)	0.2	0.2	0.2	3.0	3.0	3.0	0.5	0.5	0.5

Note: a = 2.0 m, b = 3.0 m, h = 0.005 m, υ = 0.3, ρ = 7850 Kg/m^3, $m_p = \rho hab$ = 235.5 Kg, E = 2.051×10^{11} N/m^2, $k_p = D_E/a^2$ = 5.8695×10^2 N/m, $D_E = Eh^3/[12(1-\upsilon)]$ = 2.3478×10^3 N–m, $c_p = m_p/a^2 \cdot \sqrt{D_E/\rho_A}$ = 455.34611 N–s/m

Table 4 The lowest five eigenvalues of a rectangular plate with three spring-damper-mass systems shown in Table 3

Boundary conditions	Methods	Eigenvalues $\overline{\omega}_j = \overline{\omega}_{jR} + \overline{\omega}_{jI} \cdot \overline{i}$					CPU time (sec)
		$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$	
SSSS	FEM	-0.82241 $\pm 25.80911\,\overline{i}$	-1.77021 $\pm 53.56939\,\overline{i}$	-1.74180 $\pm 85.11316\,\overline{i}$	-1.46803 $\pm 97.53119\,\overline{i}$	-3.90973 $\pm 113.1908\,\overline{i}$	176.75
	ANCM	-0.81442 $+27.63009\,\overline{i}$	-1.76858 $+53.07409\,\overline{i}$	-1.74523 $+84.85454\,\overline{i}$	-1.45339 $+95.44775\,\overline{i}$	-3.89574 $+110.2912\,\overline{i}$	32.75
SSSC	FEM	-0.97373 $\pm 28.72900\,\overline{i}$	-1.85080 $\pm 60.96206\,\overline{i}$	-1.86181 $\pm 86.69183\,\overline{i}$	-1.49127 $\pm 109.68709\,\overline{i}$	-4.13114 $\pm 118.4227\,\overline{i}$	169.11
	ANCM	-0.96943 $+30.19194\,\overline{i}$	-1.84549 $+60.13853\,\overline{i}$	-1.85843 $+86.20791\,\overline{i}$	-1.47606 $+107.1321\,\overline{i}$	-4.110165 $+115.0027\,\overline{i}$	32.29
SCSC	FEM	-0.57071 $\pm 32.51269\,\overline{i}$	-1.55265 $\pm 69.61637\,\overline{i}$	-1.40180 $\pm 88.57062\,\overline{i}$	-1.66957 $\pm 122.9936\,\overline{i}$	-3.66433 $\pm 124.67933\,\overline{i}$	144.28
	ANCM	-0.56719 $+33.63182\,\overline{i}$	-1.55154 $+68.38661\,\overline{i}$	-1.40795 $+87.87315\,\overline{i}$	-1.63303 $+120.0114\,\overline{i}$	-3.64451 $+120.0279\,\overline{i}$	31.08
SFSF	FEM	-1.33312 $\pm 15.50366\,\overline{i}$	-1.47714 $\pm 23.86676\,\overline{i}$	-0.36612 $\pm 45.27458\,\overline{i}$	-2.31420 $\pm 75.39186\,\overline{i}$	-0.53978 $\pm 80.6846\,\overline{i}$	264.58
	ANCM	-1.24375 $+18.87423\,\overline{i}$	-1.45495 $+25.21089\,\overline{i}$	-0.35793 $+44.40181\,\overline{i}$	-2.46981 $+75.66413\,\overline{i}$	-0.76917 $+78.05496\,\overline{i}$	32.95

Table 5 The locations and magnitudes of the five spring-damper-mass systems

Locations of spring-damper-mass system $(\xi_{ei}, \eta_{ei}) = (x_{ei}/a, y_{ei}/b)$					Magnitudes of point Masses					Magnitudes of spring constants					Magnitudes of damping coefficients				
(ξ_{e1}, η_{e1})	(ξ_{e2}, η_{e2})	(ξ_{e3}, η_{e3})	(ξ_{e4}, η_{e4})	(ξ_{e5}, η_{e5})	m_1^*	m_2^*	m_3^*	m_4^*	m_5^*	k_1^*	k_2^*	k_3^*	k_4^*	k_5^*	c_1^*	c_2^*	c_3^*	c_4^*	c_5^*
(0.25, 0.125)	(0.25, 0.625)	(0.5, 0.5)	(0.75, 0.25)	(0.875, 0.75)	0.2	0.2	0.2	0.2	0.2	3.0	3.0	3.0	3.0	3.0	0.5	0.5	0.5	0.5	0.5

Note: $a = 2.0$ m, $b = 3.0$ m, $h = 0.005$ m, $\upsilon = 0.3$, $\rho = 7850$ Kg/m^3, $m_p = \rho hab = 235.5$ Kg, $E = 2.051\times10^{11}$ N/m^2, $k_p = D_E/a^2 = 5.8695\times10^2$ N/m, $D_E = Eh^3/[12(1-\upsilon)] = 2.3478\times10^3$ N–m, $c_p = m_p/a^2 \cdot \sqrt{D_E/\rho_A} = 455.34611$ N–s/m

If all situations are kept unchanged except that two additional spring-damper-mass systems are placed on the uniform plate with locations, magnitudes, and physical properties shown in Table 5, then the lowest five eigenvalues of the constrained plate are shown in Table 6. From Tables 4 and 6 one sees that the damped natural frequencies of the uniform plate carrying "five" spring-damped-mass systems are larger than those carrying "three" systems, while the damping effect of the former is larger than that of the latter. These are the reasonable results; because the physical properties of each spring-damped-mass system are the identical each other, the total stiffness and the total damping of the uniform plate with "five" spring-damped-mass systems will be larger than those of the uniform plate with "three" systems, and the natural frequencies of a uniform plate are directly proportional to the square root of the stiffness, while the damping effect of a vibrating system is directly proportional to the magnitude of the damping.

From the final columns of Tables 2, 4 and 6 one sees that the CPU time required by the ANCM is only about one-fifths of that required by the FEM.

Table 6 The lowest five eigenvalues of uniform plate with five spring-damper-mass systems shown in Table 5

Boundary conditions	Methods	Eigenvalues $\overline{\omega}_j = \overline{\omega}_{jR} + \overline{\omega}_{jI} \cdot \bar{i}$					CPU time (sec)
		$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$	
SSSS	FEM	–3.80220 ±26.04139 $\bar{i}$	–2.25030 ±53.65154 $\bar{i}$	–3.42360 ±85.15887 $\bar{i}$	–3.56308 ±97.56985 $\bar{i}$	–4.85767 ±113.2567 $\bar{i}$	186.64
	ANCM	–3.76838 +27.72069 $\bar{i}$	–2.25177 +53.08273 $\bar{i}$	–3.42396 +84.86156 $\bar{i}$	–3.54296 +95.45700 $\bar{i}$	–4.85760 +110.2786 $\bar{i}$	37.69
SSSC	FEM	–3.89560 ±28.91471 $\bar{i}$	–2.92752 ±61.11834 $\bar{i}$	–3.40290 ±86.61762 $\bar{i}$	–3.26219 ±109.79577 $\bar{i}$	–5.63104 ±118.5161 $\bar{i}$	172.25
	ANCM	–3.87280 +30.28070 $\bar{i}$	–2.92844 +60.14888 $\bar{i}$	–3.40099 +86.21340 $\bar{i}$	–3.23649 +107.14104 $\bar{i}$	–5.61473 +114.9776 $\bar{i}$	38.06
SCSC	FEM	–3.99153 ±32.74383 $\bar{i}$	–2.24154 ±69.74141 $\bar{i}$	–3.22557 ±88.52757 $\bar{i}$	–3.71697 ±123.06192 $\bar{i}$	–4.93844 ±124.7931 $\bar{i}$	159.30
	ANCM	–3.98362 +33.71109 $\bar{i}$	–2.24432 +68.39705 $\bar{i}$	–3.22063 +87.88583 $\bar{i}$	-3.68585 +120.01548 $\bar{i}$	–4.90131 +120.02185 $\bar{i}$	36.41
SFSF	FEM	–3.04654 ±15.76099 $\bar{i}$	–1.54438 ±23.94770 $\bar{i}$	–2.55889 ±45.34587 $\bar{i}$	–3.32948 ±75.58326 $\bar{i}$	–1.44252 ±80.29457 $\bar{i}$	294.43
	ANCM	–2.78451 +18.97477 $\bar{i}$	–1.50279 +25.21646 $\bar{i}$	–2.60375 +44.47374 $\bar{i}$	–3.70522 +75.66683 $\bar{i}$	–1.05019 +78.07101 $\bar{i}$	37.07

6. Conclusions

1. The analytical-and-numerical-combined method (ANCM) is available for the determination of eigenvalues of a uniform plate carrying any number of spring-damper-mass systems.
2. The effective spring constant k_{eff} and the effective damping coefficient C_{eff} of the massless "equivalent spring-damper system" are two parameters composed of the effects due to the linear spring constant k_e, the damping coefficient C_e and the concentrated mass m_e of the "original spring-damper-mass system".
3. The imaginary parts of the eigenvalues for a uniform plate carrying any number of spring-damper-mass systems represent the damped natural frequencies of the constrained plate, $\overline{\omega}_{dj}$. The influence on $\overline{\omega}_{dj}$ of the magnitudes of the damping coefficients of the spring-damper-mass systems, $C_{e,\upsilon}$, is negligible.

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Appendix A

Imaginary part for the equation of motion of a uniform plate carrying a spring-damper-mass system

From Eq. (24b) one has

$$(\overline{\omega}_I \cdot C_{eff}) \cdot \sum_{k=1}^{n'} \overline{W}_k(x_e, y_e)\overline{W}_k(x_e, y_e)\overline{q}_j(t)$$
$$= 2\overline{\omega}_R\overline{\omega}_I\overline{q}_j(t), \qquad j = 1, 2, \ldots, n' \tag{A1}$$

or in matrix form

$$[A]\{\overline{q}_j\} = (2\overline{\omega}_R\overline{\omega}_I)[B]\{\overline{q}_j\} \tag{A2}$$

where

$$[B]_{n' \times n'} = [\diagdown I \diagdown]_{n' \times n'} = \ulcorner 11 \cdots\cdots 11 \lrcorner_{n' \times n'} \tag{A3}$$

$$[A]_{n' \times n'} = (\overline{\omega}_I \cdot C_{eff}) \cdot [\overline{W}_j(x_e, y_e)]_{n' \times n'} \tag{A4}$$

$$[\overline{W}_j(x_e, y_e)]_{n' \times n'} = \{\overline{W}_j(x_e, y_e)\}_{n' \times 1} \cdot \{\overline{W}_j(x_e, y_e)\}^T_{n' \times 1} \tag{A5}$$

$$\{\overline{W}_j(x_e, y_e)\}_{n' \times 1} = \{\overline{W}_1(x_e, y_e)\overline{W}_2(x_e, y_e)\cdots\cdots\overline{W}_{n'}(x_e, y_e)\}_{n' \times 1} \tag{A6}$$

$$\{\overline{q}_j\}_{n' \times 1} = \{\overline{q}_1\overline{q}_2\cdots\cdots\overline{q}_{n'}\}_{n' \times 1} \tag{A7}$$

The value of C_{eff} appearing in Eq. (A4) is defined by Eqs. (17a)-(17b).

Appendix B

Imaginary part for the equation of motion of a uniform plate carrying any number of spring-damper-mass systems

From Eq. (34b) one has

$$\sum_{\upsilon=1}^{r} (\overline{\omega}_I \cdot C_{eff,\upsilon}) \sum_{k=1}^{n'} \overline{W}_k(x_{e,\upsilon}, y_{e,\upsilon})\overline{W}_k(x_{e,\upsilon}, y_{e,\upsilon})\overline{q}_j(t)$$
$$= 2\overline{\omega}_R\overline{\omega}_I\overline{q}_j(t), \qquad j = 1, 2, \ldots, n' \tag{A8}$$

or in matrix form

$$[\tilde{A}]\{\overline{q}_j\} = (2\overline{\omega}_R\overline{\omega}_I)[\tilde{B}]\{\overline{q}_j\} \tag{A9}$$

where

$$[\tilde{B}]_{n' \times n'} = [\diagdown I \diagdown]_{n' \times n'} = \ulcorner 11 \cdots\cdots 11 \lrcorner_{n' \times n'} \tag{A10}$$

$$[\tilde{A}]_{n' \times n'} = \sum_{\upsilon=1}^{r} (\overline{\omega}_I \cdot C_{eff,\upsilon}) \cdot [\overline{W}_j(x_{e,\upsilon}, y_{e,\upsilon})]_{n' \times n'} \tag{A11}$$

$$[\overline{W}_j(x_{e,\upsilon}, y_{e,\upsilon})]_{n' \times n'} = \{\overline{W}_j(x_{e,\upsilon}, y_{e,\upsilon})\}_{n' \times 1} \cdot \{\overline{W}_j(x_{e,\upsilon}, y_{e,\upsilon})\}^T_{n' \times 1} \tag{A12}$$

$$\{\overline{W}_j(x_{e,\upsilon}, y_{e,\upsilon})\}_{n' \times 1} = \{\overline{W}_1(x_{e,\upsilon}, y_{e,\upsilon})\overline{W}_2(x_{e,\upsilon}, y_{e,\upsilon})\cdots\cdots\overline{W}_{n'}(x_{e,\upsilon}, y_{e,\upsilon})\}_{n' \times 1} \tag{A13}$$

$$\{\overline{q}_j\}_{n' \times 1} = \{\overline{q}_1\overline{q}_2\cdots\cdots\overline{q}_{n'}\}_{n' \times 1} \tag{A14}$$

The value of $C_{eff,\upsilon}$ appearing in Eq. (A11) is defined by

$$C_{eff,\upsilon} = \left[-\frac{F_{1\upsilon}G_{1\upsilon} - E_{1\upsilon}H_{1\upsilon}}{G_{1\upsilon}^2 + H_{1\upsilon}^2}\left(\frac{1}{\overline{\omega}_I}\right)\right] \tag{A15}$$

For the values of $E_{1\upsilon}$, $F_{1\upsilon}$, $G_{1\upsilon}$ and $H_{1\upsilon}$ one may refer to Eqs. (38a)-(38d).