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Design procedure for modal controllers for defective and nearly defective systems

Yu Dong Chen† and Su Huan Chen‡

Department of Mechanics, Jilin University, Nanling Campus, Changchun, 130025, P.R. China

Guang Yang†

Department of Basic Sciences, Changchun University of Technology, Changchun, 130012, P.R. China

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Abstract. This paper presents a procedure for designing feedback controllers for defective systems with repeated eigenvalues, and also for a nearly defective system with close eigenvalues. For the nearly defective system, we first transform it into a defective one, and then apply the same method to deal with the nearly defective system. A method for computing the gain matrices is discussed here. The methodologies proposed are based on the modal coordinate equation to avoid the tedious mathematical manipulation. As an application of the present procedure, a numerical example is given.

Key words: modal control; defective and nearly defective system.

1. Introduction

The complexity in the control of large flexible structures is that there may exist repeated or close eigenvalues in these systems, thus it is desirable to develop an approach for designing the feedback controller for such systems.

The conditions that the closed-loop eigenvectors have to satisfy in order to obtain the output feedback gain matrices and to enable the desired eigenvalue placements have been discussed (Kimura 1997). The techniques for synthesis of output feed-back gains have been developed (Srinathkumar 1978, Maghami and Juang 1990, Andry *et al*. 1983). Dissipative output feedback gain matrices were used to assign eigenproblem (Maghami and Gupta 1997). The measures of controllability and observability of the repeated modes are discussed (Liu *et al*. 1994), but it does not deal with the corresponding design of the feedback control laws. The standard design methods for feedback control laws can be found in Meirovitch (1990).

The above discussions on the design of the feedback control laws mainly involve the control problems of the non-defective system, which has the complete eigenvectors to span the eigenspace. However, in actual engineering problems, such as general damping systems, flutter analysis of

† Assistant Professor
‡ Professor

aeroelasticity, and so on, the system called defective system does not have a set of complete eigenvectors to span the eigenspace (Xu and Chen 1994). Recent papers in this field include the dynamic analysis of mobility and graspability of general manipulation systems (Prattichizzo and Bicchi 1998), and the consistent task specification for manipulation systems with general kinematics (Prattichizzo and Bicchi 1997). Recently, Chen *et al.* (2001) gave modal optimal control procedure for nearly defective systems, and discussed the quantitative measurements of modal controllability and observability of defective and nearly defective systems.

The defective systems differ from nondefective ones in that the state matrix $\boldsymbol{A}$ cannot be diagonalized. For this reason, the standard methods for designing the feedback controllers cannot be used to deal with the modal control problems of the defective and nearly defective systems.

This study will present an approach for designing modal controllers for the defective system with repeated eigenvalues based on the modal control equations, and also for the nearly defective system with close eigenvalues. For the nearly defective system, we first transform it into a defective one, and then use the same method to deal with the nearly defective system. The theory is illustrated by a numerical example to prove the validity.

2. Feedback control design of defective and nearly defective systems

Consider the control system indicated by the following state equation

$$\left.\begin{aligned} \dot{\boldsymbol{X}}(t) &= \boldsymbol{AX} + \boldsymbol{B}Z(t) \\ \boldsymbol{y}(t) &= \boldsymbol{CX}(t) \end{aligned}\right\} \tag{1}$$

where $\boldsymbol{A}$ is the state matrix. $\boldsymbol{X}(t) \in \boldsymbol{R}^{n\times 1}$ is the state vector, $Z(t)$ is the input, $\boldsymbol{y}(t) \in \boldsymbol{R}^{q\times 1}$ is the output vector, $\boldsymbol{B} \in \boldsymbol{R}^{n\times 1}$ and $\boldsymbol{C} \in \boldsymbol{R}^{q\times n}$ are called the actuator distribution matrix and sensor distribution matrix, respectively, indicating the locations of control forces and sensors.

Denote AM as the algebraic multiplicity of the eigenvalues of the $\boldsymbol{A}$, and GM the number of the linear independent eigenvectors corresponding to λ. If $AM=GM$ for the distinct or repeated eigenvalues, the system is non-defective; if $AM>GM$, the system with repeated eigenvalues is defective (Deif 1992).

In Eq. (1), we assumed that $\lambda_1 = \lambda_2 = \dots = \lambda_m = \lambda$ are defective repeated eigenvalues with m multiplicity, and rest of eigenvalues, $\lambda_{m+1}, \lambda_{m+2}, \dots, \lambda_n$, are distinct. The right and left modal matrices are expressed as the partitional form $\boldsymbol{U}=[\boldsymbol{U}_m, \boldsymbol{U}_{n-m}]$, $\boldsymbol{V}=[\boldsymbol{V}_m, \boldsymbol{V}_{n-m}]$. ξ_m and ξ_d are the modal coordinates corresponding to the repeated and distinct eigenvalues.

Using the modal transformation, we obtain modal control equations corresponding to the defective repeated eigenvalues and distinct eigenvalues

$$\dot{\xi}_m = \boldsymbol{J}\xi_m + \boldsymbol{V}_m^H \boldsymbol{B} Z_m(t) \tag{2}$$

$$\dot{\xi}_d = \boldsymbol{\Lambda}_d \xi_d + \boldsymbol{V}_{n-m}^H \boldsymbol{B} Z_d(t) \tag{3}$$

$$\boldsymbol{y}_m = \boldsymbol{C}\boldsymbol{U}_m \xi_m \tag{4}$$

$$\boldsymbol{y}_d = \boldsymbol{C}\boldsymbol{U}_{n-m} \xi_d \tag{5}$$

Eqs. (2) to (5) can be written as

$$\dot{\xi}_m = \boldsymbol{J}\xi_m + \boldsymbol{P}_m Z_m(t), \quad \dot{\xi}_d = \boldsymbol{\Lambda}_d \xi_d + \boldsymbol{P}_d Z_d(t) \tag{6}$$

$$\boldsymbol{y}_m = \boldsymbol{C}_m \xi_m, \quad \boldsymbol{y}_d = \boldsymbol{C}_d \xi_d \tag{7}$$

where

$$\boldsymbol{P}_m = \boldsymbol{V}_m^H \boldsymbol{B}, \quad \boldsymbol{P}_d = \boldsymbol{V}_{n-m}^H \boldsymbol{B} \tag{8}$$

$$\boldsymbol{C}_m = \boldsymbol{C}\boldsymbol{U}_m, \quad \boldsymbol{C}_d = \boldsymbol{C}\boldsymbol{U}_{n-m} \tag{9}$$

If the control loops which generate the input vector by linear feedback of the state vector of the system are introduced, then the response characteristic of the closed-loop system will be different from that of the open loop system. Thus, it is possible to re-assign a closed-loop system eigenvalues, which correspond to the controllable modes of the repeated defective system so that the closed-loop response characteristic is superior to the defective characteristics of the original uncontrolled system.

Since Eqs. (6) and (7) are much simpler than the state Eq. (1), the gain matrix of the close-loop system can be derived directly without the tedious mathematical manipulation.

Here we assume that the modes corresponding to the m defective repeated eigenvalues of the defective system and the distinct eigenvalues are controllable.

If the direct output feedback control is used, the modal control forces are given as follows

$$\boldsymbol{P}_m Z_m(t) = \boldsymbol{V}_m^H \boldsymbol{B} \boldsymbol{G}_m^T \xi_m, \quad \boldsymbol{P}_d Z_d(t) = \boldsymbol{V}_{n-m}^H \boldsymbol{B} \boldsymbol{G}_d^T \xi_d \tag{10}$$

where

$$\boldsymbol{G}_m^T = [GM_1, GM_2, \cdots, GM_m], \qquad \boldsymbol{G}_d^T = [GD_1, GD_2, \cdots, GD_{n-m}] \tag{11}$$

Substituting Eq. (10) into Eq. (6), yields

$$\dot{\xi}_m = (\boldsymbol{J} + \boldsymbol{P}_m \boldsymbol{G}_m^T)\xi_m, \qquad \dot{\xi}_d = (\boldsymbol{\Lambda}_d + \boldsymbol{P}_d \boldsymbol{G}_d^T)\xi_d \tag{12}$$

or

$$\begin{bmatrix} \dot{\xi}_m \\ \cdots \\ \dot{\xi}_d \end{bmatrix} = \begin{bmatrix} \boldsymbol{J} + \boldsymbol{P}_m \boldsymbol{G}_m^T & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & \boldsymbol{\Lambda}_d + \boldsymbol{P}_d \boldsymbol{G}_d^T \end{bmatrix} \begin{bmatrix} \xi_m \\ \cdots \\ \xi_d \end{bmatrix} \tag{13}$$

Eq. (13) indicates that the effect of the input variable given by Eq. (10) is to change the Jordan matrix $\boldsymbol{J}$ and $\boldsymbol{\Lambda}_d$ into new matrices $\boldsymbol{H}_m$ and $\boldsymbol{H}_d$ given by

$$\boldsymbol{H}_m = \boldsymbol{J} + \boldsymbol{P}_m \boldsymbol{G}_m^T, \qquad \boldsymbol{H}_d = \boldsymbol{\Lambda}_d + \boldsymbol{P}_d \boldsymbol{G}_d^T \tag{14}$$

Eq. (14) indicates that the defective repeated eigenvalues, $\lambda_1 = \lambda_2 = ... = \lambda_m$ are not the eigenvalues of matrix $\boldsymbol{H}_m$, the $\boldsymbol{\Lambda}_d$ are not the eigenvalues of $\boldsymbol{H}_d$.

Denote the assigned new distinct eigenvalues as ρ_j ($j = 1, 2, \cdots, m$) and corresponding eigenvectors as $\boldsymbol{W}_j$, they satisfy the following eigenvalue problem

$$[\boldsymbol{J} + \boldsymbol{P}_m \boldsymbol{G}_m^T]\boldsymbol{W}_j = \rho_j \boldsymbol{W}_j \qquad (j = 1, 2, \cdots, m) \tag{15}$$

or

$$[\boldsymbol{J} + \boldsymbol{P}_m \boldsymbol{G}_m^T - \rho_j \boldsymbol{I}]\boldsymbol{W}_j = 0 \qquad (j = 1, 2, \cdots, m) \tag{16}$$

Since $\boldsymbol{W}_j \neq 0$, the eigen-determinant of the matrix is zero

$$det[\boldsymbol{J} + \boldsymbol{P}_m \boldsymbol{G}_m^T - \rho_j \boldsymbol{I}] = 0 \tag{17}$$

Considering the Eqs. (8) and (11), we have

$$\boldsymbol{P}_m \boldsymbol{G}_m^T = \begin{bmatrix} p_1 GM_1 & p_1 GM_2 & \cdots & p_1 GM_m \\ p_2 GM_1 & p_2 GM_2 & \cdots & p_2 GM_m \\ & \cdots & & \\ p_m GM_1 & p_m GM_2 & \cdots & p_m GM_m \end{bmatrix} \tag{18}$$

where $\boldsymbol{P}_m^T = [p_1, p_2, \cdots, p_m]$.

After introduction of Eq. (18) into Eq. (17), Eq. (17) becomes

$$det\begin{bmatrix} \lambda - \rho_j + GM_1 p_1 & 1 + GM_2 p_1 & \cdots & GM_m p_1 \\ GM_1 p_2 & \lambda - \rho_j + GM_2 p_2 & \cdots & GM_m p_2 \\ \cdots & & & \\ GM_1 p_m & GM_2 p_m & \cdots & \lambda - \rho_j + GM_m p_m \end{bmatrix} = 0 \qquad (j = 1, 2, \cdots, m) \tag{19}$$

Expanding Eq. (19), yields

$$(\lambda - \rho_j)^m \left[1 + \sum_{l=0}^{m-1} \sum_{s=1}^{m} \frac{GM_s p_{l+s}}{(\lambda - \rho_j)^{l+1}} \right] = 0 \qquad (j = 1, 2, \cdots, m) \tag{20}$$

If $\rho_j \neq \lambda$, from Eq. (20), we have

$$-\sum_{l=0}^{m-1} \sum_{s=1}^{m} \frac{GM_s p_{l+s}}{(\lambda - \rho_j)^{l+1}} = 1 \qquad (j = 1, 2, \cdots, m) \tag{21}$$

In order to obtain a convenient form, we introduce the following notations

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_1 \\ \vdots \\ \boldsymbol{F}_m \end{bmatrix} \tag{22}$$

where

$$\boldsymbol{F}_j = \left[\frac{1}{\rho_j - \lambda}, \frac{1}{(\rho_j - \lambda)^2}, \cdots, \frac{1}{(\rho_j - \lambda)^m}\right] \tag{23}$$

and

$$\boldsymbol{P} = \begin{bmatrix} p_1 & p_2 & \cdots & p_m \\ p_2 & p_3 & \cdots & 0 \\ \cdots\cdots & & & \\ p_m & 0 & \cdots & 0 \end{bmatrix} \tag{24}$$

where $p_1, p_2, \cdots, p_m$ are the elements of the $\boldsymbol{P}_m$ in Eq. (8).

$$\boldsymbol{G}_m^T = [GM_1, GM_2, \cdots, GM_m] \tag{25}$$

$$\boldsymbol{E}^T = [1, 1, \cdots, 1] \tag{26}$$

Using these notations, the m Eq. (21) can be written in a matrix equation

$$\boldsymbol{FPG}_m = \boldsymbol{E} \tag{27}$$

It is possible to solve (27) for the gain vector $\boldsymbol{G}_m$, i.e.,

$$\boldsymbol{G}_m = \boldsymbol{P}^{-1}\boldsymbol{F}^{-1}\boldsymbol{E} \tag{28}$$

This is the solution for the gain vector of the defective systems with repeated eigenvalues. The control law of the defective system is given by

$$Z(t) = \boldsymbol{G}^T\xi(t) \tag{29}$$

where $\boldsymbol{G}^T = [G_m^T \vdots G_d^T]$

Using the modal transformation

$$\boldsymbol{x}(t) = \boldsymbol{U}\xi(t) \tag{30}$$

one has

$$\xi(t) = \boldsymbol{V}^H\boldsymbol{x}(t) \tag{31}$$

Thus, Eq. (29) becomes

$$Z(t) = [G_m^T \ \vdots\ G_d^T]\begin{bmatrix} \xi_m \\ \cdots \\ \xi_d \end{bmatrix} = [G_m^T \ \vdots\ G_d^T]\boldsymbol{V}^H\boldsymbol{x}(t) \tag{32}$$

If the eigenvalues, λ_1, λ_2, ..., are distinct, the gain matrix $\boldsymbol{G}_d$ can be obtained by Meirovitch (1990)

$$GD_j = \prod_{k=1}^{n-m}(\rho_k - \lambda_j)/p_j\prod_{\substack{k\neq j\\ j=1}}^{n-m}(\lambda_k - \lambda_j) \qquad (j = 1, 2, \cdots, n-m) \tag{33}$$

where $\rho_k (k = 1, 2, \cdots, n-m)$ are the assigned new eigenvalues, $\lambda_j\,(j = 1, 2, \cdots, n-m)$ are eigenvalues associated with the controllable modes.

It should be pointed out that if some small changes of parameters of the defective systems are introduced, the system with the defective repeated eigenvalues can be perturbed into nearly defective one with close eigenvalues. For such a case, from a mathematical view point, although the close eigenvalues are distinct, the dynamic characteristic of the system is still defective.

In a similar way to the deduction presented in Chen *et al.* (2001) for nearly defective system with close eigenvalues, the following equation can be obtained

$$\dot{\xi}_m(t) = (\boldsymbol{J}_0 + \delta\boldsymbol{J}_0)\xi_m(t) + \boldsymbol{P}_m\boldsymbol{Z}_m(t) \approx \boldsymbol{J}_0\xi_m(t) + \boldsymbol{P}_m\boldsymbol{Z}_m(t) \tag{34}$$

where

$$\lambda_0 = \frac{1}{n}\sum_{i=1}^{n}\lambda_i \tag{35}$$

$$\boldsymbol{J}_0 = \begin{bmatrix} \lambda_0 & 1 & & \\ & \lambda_0 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_0 \end{bmatrix} \tag{36}$$

$$\delta\boldsymbol{J}_0 = \begin{bmatrix} \lambda_1 - \lambda_0 & -1 & & \\ & \lambda_2 - \lambda_0 & \ddots & \\ & & \ddots & -1 \\ & & & \lambda_n - \lambda_0 \end{bmatrix} \tag{37}$$

Eq. (34) shows that the feedback control design problem of the nearly defective system with close eigenvalues can be transformed into one of the defective system with repeated eigenvalues, which are equal to the average value of the close eigenvalues.

3. Eigenvalue perturbation analysis for the closed-loop systems

From the above discussion, it can be shown that feedback control design of the nearly defective system with close eigenvalues can be transformed into one of the defective system with a repeated eigenvalue, which is equal to the average value of the close eigenvalues. If the feedback control law given by Eq. (32) is applied to the nearly defective system with close eigenvalues, the assigned eigenvalues will have some perturbations. In this section we present the eigenvalue perturbation analysis of the closed-loop system. These are induced by the error matrix $\delta\boldsymbol{J}_0$ in Eq. (37).

If the feedback control law (32) is applied to nearly defective system with close eigenvalues, from Eq. (34) we obtain

$$\dot{\boldsymbol{x}} = \left(\boldsymbol{U}\begin{bmatrix} \boldsymbol{J}_0 + \boldsymbol{P}_m\boldsymbol{G}_m^T & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & \boldsymbol{\Lambda}_d + \boldsymbol{P}_d\boldsymbol{G}_d^T \end{bmatrix}\boldsymbol{V}^H + \boldsymbol{U}\delta\boldsymbol{J}_0\boldsymbol{V}^H\right)\boldsymbol{x} = (\boldsymbol{AA} + \delta\boldsymbol{A})\boldsymbol{x} \tag{38}$$

The eigenproblem corresponding to Eq. (38) is

$$(\boldsymbol{AA} + \delta\boldsymbol{A})\tilde{\boldsymbol{u}} = \tilde{\rho}\tilde{\boldsymbol{u}} \tag{39}$$

where $\boldsymbol{AA} = \boldsymbol{U}\begin{bmatrix} \boldsymbol{J}_0 + \boldsymbol{P}_m\boldsymbol{G}_m^T & \vdots & 0 \\ \cdots\cdots & \vdots & \cdots\cdots \\ 0 & \vdots & \boldsymbol{\Lambda}_d + \boldsymbol{P}_d\boldsymbol{G}_d^T \end{bmatrix}\boldsymbol{V}^H, \qquad \delta\boldsymbol{A} = \boldsymbol{U}\delta\boldsymbol{J}_0\boldsymbol{V}^H$

The eigenvalue $\tilde{\rho}$ and eigenvector $\tilde{\boldsymbol{u}}$ of $(\boldsymbol{AA} + \delta\boldsymbol{A})$ can be expressed in the following form (Chen 1999)

$$\tilde{\rho}_i = \rho_i + \varepsilon\rho_{1i} + \cdots \tag{40}$$

$$\tilde{\boldsymbol{u}}_i = \boldsymbol{u}_i + \varepsilon\boldsymbol{u}_{1i} + \cdots \tag{41}$$

where $\rho_i = -\alpha_i + j\beta_i (j = \sqrt{-1})$, $(i = 1, 2, \cdots)$ are the eigenvalues of the matrix $\boldsymbol{AA}$, ρ_{1i} and $\boldsymbol{u}_{1i}$ are the corresponding 1st order perturbations.

It can be shown that the 1st order perturbation, ρ_{1i}, is

$$\rho_{1i} = \boldsymbol{V}_i^H\delta\boldsymbol{A}\boldsymbol{u}_i \qquad (i = 1, 2, \cdots, m) \tag{42}$$

If the following condition

$$\alpha_i + \delta_{\min} \le \eta_i \le \alpha_i + \delta_{\max} \qquad (i = 1, 2, \cdots, m) \tag{43}$$

is satisfied, the closed-loop system will have good dynamic stability, where η_i is the modal damping ratio, and

$$\delta_{\min} = \min \boldsymbol{R}_e(\tilde{\rho}_i), \qquad \delta_{\max} = \max \boldsymbol{R}_e(\tilde{\rho}_i) \tag{44}$$

It is obvious that as long as α_i $(i = 1, 2, \cdots, m)$ are large enough for designing the feedback control law of the defective system with repeated eigenvalues, the closed-loop system may have the dynamic stability we need. This indicates that the present procedure for designing the feedback control law of the nearly defective system with close eigenvalues is available.

The procedure of feedback control design for the nearly defective systems with close eigenvalues is summarized as follows:

1) Form state matrix $\boldsymbol{A}$ of nearly defective system and compute m close eigenvalues, λ_1, ..., λ_m;

2) Compute

$$\lambda_0 = \frac{1}{m}\sum_{j=1}^{m}\lambda_j$$

3) Compute generalized modal matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ using the invariant subspace recursive procedure presented in Chen (1999);
4) Form a approximate defective system using Eq. (34);
5) Compute $\boldsymbol{G}_{m0}$ and $\boldsymbol{G}_d$ from Eqs. (28) and (33) for the approximate system with defective repeated eigenvalue λ_0;
6) Eigenvalue perturbation analysis of the closed-loop systems using Eq. (40).

4. Numerical example

In order to illustrate the application of the present procedure, a numerical example of the defective system is given as follows.

We consider the flutter problem of an airfoil in simplified formulation. The airfoil is replaced by a rigid rectangular panel with two degrees of freedom, the vertical displacement h and the rotation α. It is assumed that aerodynamic lift force is proportional to the angle of attack α and to the square of the velocity v of flight. The differential equations of motion are Shi *et al.* (1989)

$$m\ddot{h} + s\ddot{\alpha} + K_h h = -\rho v^2 ab\alpha$$

$$s\ddot{h} + J_\alpha\ddot{\alpha} + K_\alpha\alpha = \rho v^2 abe\alpha$$

where m is the mass of the panel, s the static moment of the cross section area of the panel, J_α the moment of inertia, K_h the bending stiffness, K_α the torsional stiffness, respectively.

If the parameters are given as follows: $m/(\rho ab^2) = 5$, $s/(mb) = 0.25$, $J_\alpha/(mb^2) = 0.5$, $e/b = 0.4$, $K_h/m = 0.25$, $K_\alpha/J_\alpha = 1$, and $u = v(J_\alpha/K_\alpha)^{1/2}/b$, then the above differential equations become

$$\boldsymbol{M}\ddot{q} + \boldsymbol{K}q = 0$$

where

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 0.5 \end{bmatrix} \qquad \boldsymbol{K} = \begin{bmatrix} 0.25 & 0.2u^2 \\ 0 & 0.5 - 0.08u^2 \end{bmatrix}$$

If the parameter $u = 1.32567735$, the state matrix has the following form

$$\boldsymbol{A} = \begin{bmatrix} 0 & -\boldsymbol{M}^{-1}\boldsymbol{K} \\ \boldsymbol{I} & 0 \end{bmatrix} = \begin{bmatrix} 0.0 & 0.0 & -0.28571428571429 & -0.19632103395740 \\ 0.0 & 0.0 & 0.14285714285714 & -0.62065221321282 \\ 1.0 & 0.0 & 0.00000000000000 & 0.00000000000000 \\ 0.0 & 1.0 & 0.00000000000000 & 0.00000000000000 \end{bmatrix}$$

The control matrix **B** in Eq.(1) for single-input control force is

$$\boldsymbol{B} = \begin{bmatrix} 0 \\ \cdots \\ \boldsymbol{M}^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.00000000000000 \\ 0.00000000000000 \\ -0.57142857142857 \\ 2.28571428571429 \end{bmatrix}$$

The flutter of the airfoil is characterized by the conditions: if $R_e(\lambda) = 0, I_m(\lambda) \neq 0$, which describe the critical state of the flutter; if $R_e(\lambda) > 0, I_m(\lambda) \neq 0$, which describe the flutter occurs, and the eigenvalue is also the corresponding flutter frequency.

From the above discussion, we see how important it is to know the behaviours of eigenvalues of systems.

The eigenvalues of $\boldsymbol{A}$ are

$$\lambda_1 = 0.67318886946616i \qquad \lambda_2 = 0.67318886946616i$$

$$\lambda_3 = -0.67318886946616i \qquad \lambda_4 = -0.67318886946616i$$

where $i = \sqrt{-1}$. This system is defective. Because $R_e(\lambda_i) = 0, I_m(\lambda_i) \neq 0$, the system is in the critical state of the flutter. The main problem of the control is to stabilize the system, i.e., make it more safe.

The Jordan matrix of this system is

$$\boldsymbol{J} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

The right and left modal matrices $\boldsymbol{U}$ and $\boldsymbol{V}$ are

$$\boldsymbol{U} = \begin{bmatrix} -0.46540318867956 & -0.27308418690627 & -0.46540318867956 & -0.27308418690627 \\ -0.39700584910218 & 0.55929647822929 & -0.39700584910218 & 0.55929647822929 \\ 0.69134117215010i & 0.66056754098396i & -0.69134117215010i & -0.66056754098396i \\ 0.58973918736331i & -0.61336913663528i & -0.58973918736331i & 0.61336913663528i \end{bmatrix}$$

$$\boldsymbol{V} = \begin{bmatrix} -0.68374618664713 & -0.53836500812307 & -0.68374618664713 & -0.53836500812307 \\ -0.45788328680308 & 0.63111613367462 & -0.45788328680308 & 0.63111613367462 \\ 0.32665905473259i & 0.36242133822492i & -0.32665905473259i & -0.36242133822492i \\ 0.46489559029220i & -0.42486035051943i & -0.46489559029220i & 0.42486035051943i \end{bmatrix}$$

and

$$\boldsymbol{P}_{m1} = \boldsymbol{V}_{m1}^H \boldsymbol{B} = \begin{bmatrix} -0.87595617510641i \\ 1.17820728017294i \end{bmatrix}$$

$$\boldsymbol{P}_{m2} = \boldsymbol{V}_{m2}^H \boldsymbol{B} = \begin{bmatrix} 0.87595617510641i \\ -1.17820728017294i \end{bmatrix}$$

Taking the singular-value decomposition of $\boldsymbol{P}_{m1}$ and $\boldsymbol{P}_{m2}$ yields (Chen *et al.* 2001)

$$\Sigma_1 = diag(\sigma_1^1 = 1.46815244976792, \quad \sigma_2^1 = 0)$$

$$\Sigma_2 = diag(\sigma_1^2 = 1.46815244976792, \quad \sigma_2^2 = 0)$$

Since $\sigma_1^1 > 0, \sigma_2^1 = 0, \sigma_1^2 > 0, \sigma_2^2 = 0$, the 1st and 3rd modes are controllable, the 2nd and 4th modes are uncontrollable.

In order to improve the defective characteristic of the original uncontrolled system, the new eigenvalues ρ_1 and ρ_3 can be assigned as $\rho_1 = -0.25 + 1.0i$ and $\rho_3 = -0.25 - 1.0i$.

Because the 2nd mode is uncontrollable, the modal control force can be given by

$$\boldsymbol{P}_{m1}\boldsymbol{Z}_{m1}(t) = \boldsymbol{V}_{m1}^H \boldsymbol{B} \boldsymbol{G}_{m1}^H \xi_{m1}$$

where $\boldsymbol{V}_{m1}$ contains only the first 2 columns of $\boldsymbol{V}$, $\boldsymbol{G}_{m1} = [GM_1, 0]^T$, $\xi_{m1} = [\xi_1, \xi_2]^T$.

From Eq. (14), one has

$$\boldsymbol{P}_{m1}\boldsymbol{G}_{m1}^T = \begin{bmatrix} p_1 GM_1 & 0 \\ p_2 GM_1 & 0 \end{bmatrix}$$

The eigendeterminat (19) becomes

$$det\begin{bmatrix} \lambda_1 - \rho_1 + GM_1 p_1 & 1 \\ GM_1 p_2 & \lambda - \rho_1 \end{bmatrix} = 0$$

Expanding this equation, yields

$$(\lambda_1 - \rho_1 + GM_1 p_1)(\lambda_1 - \rho_1) - GM_1 p_2 = 0$$

or

$$(\rho_1 - \lambda_1)^2 \left[1 - \frac{GM_1 p_1}{\rho_1 - \lambda_1} - \frac{GM_1 p_2}{(\rho_1 - \lambda_1)^2}\right] = 0$$

If $\rho_1 \neq \lambda_1$, we have

$$\frac{GM_1 p_1}{\rho_1 - \lambda_1} + \frac{GM_1 p_2}{(\rho_1 - \lambda_1)^2} = 1$$

It follows that the gain vector, where $\boldsymbol{G}_{m1} = [GM_1, 0]^T$, where GM_1 is given by

$$GM_1 = \frac{(\rho_1 - \lambda_1)^2}{p_1(\rho_1 - \lambda_1) + p_2} = -0.11847597715311 + 0.00743570028943i$$

where $\rho_1 = -0.25 + 1.0i$.

The required control law for the λ_1 is

$$Z_{m1}(t) = \boldsymbol{G}_{m1}^T \boldsymbol{V}_{m1}^T \boldsymbol{X}(t) = \begin{bmatrix} 0.08100749758773 - 0.00508413171795i \\ 0.05424816982607 - 0.00340468288821i \\ 0.00242893882782 + 0.03870125070536i \\ 0.00345682427529 + 0.05507895933404i \end{bmatrix}^T \boldsymbol{X}(t)$$

If $\rho_3 = -0.25 - 1.0i$, the required control law for the λ_3 can be also obtained

$$Z_{m2}(t) = \boldsymbol{G}_{m2}^T \boldsymbol{V}_{m2}^H \boldsymbol{X}(t) = \begin{bmatrix} 0.08100749758773 + 0.00508413171795i \\ 0.05424816982607 + 0.00340468288821i \\ 0.00242893882782 - 0.03870125070536i \\ 0.00345682427529 - 0.05507895933404i \end{bmatrix}^T \boldsymbol{X}(t)$$

It can be verified that state matrix of the closed-loop system in Eq. (13) is

$$\boldsymbol{H} = \boldsymbol{J} + \begin{bmatrix} \boldsymbol{P}_{m1}\boldsymbol{G}_{M1}^T & \vdots & 0 \\ \cdots & \vdots & \cdots \\ & \vdots & \boldsymbol{P}_{m2}\boldsymbol{G}_{m2}^T \end{bmatrix} =$$

$$\begin{bmatrix} 0.006513 + 0.776968i & 1.0 & 0.000000 & 0.000000 \\ -0.008760 - 0.139589i & 0.0 & 0.000000 & 0.000000 \\ 0.000000 & 0.0 & 0.006513 - 0.776968i & 1.000000 \\ 0.000000 & 0.0 & -0.008760 + 0.139589i & -0.673188i \end{bmatrix}$$

and that the eigenvalues of this matrix is

$$\lambda_1 = -0.25 + 1.0i, \qquad \lambda_2 = 0.25651334758476 + 0.45015750272135i$$

$$\lambda_3 = -0.25 - 1.0i, \qquad \lambda_4 = 0.25651334758476 - 0.45015750272135i$$

The results show that λ_1 and λ_3 are the required eigenvalues. The original defective system with repeated eigenvalues is changed into nondefective one with distinct eigenvalues. It should noted that because of the coupling between the 1st and 2nd modes, the 2nd eigenvalue is changed into $0.25651334758476 + 0.45015750272135i$ from $0.67318886946616i$. For the uncontrollable mode 4, the similar results can be also obtained. From the results we see that since $R_e(\lambda_2) > 0$, and $R_e(\lambda_4) > 0$, the 2nd and 4th modes of the closed-loop system obtained by the 1st stage design can not be stabilized. To stabilize the system the 2nd stage design is necessary. After the 1st stage design, the system is changed into nondefective one with distinct eigenvalues, it is easy to obtain the gain matrix G_d with Eq. (33).

5. Conclusions

The vibration control of the systems with repeated or close eigenvalues is an important problem in engineering. This paper focuses on the case of the defective or nearly defective systems with repeated or close eigenvalues, and presents the design methods of the modal controller based on the generalized modal coordinates, thus avoiding the tedious mathematic manipulation. From mathematical view point, although the close eigenvalues of the nearly defective system are distinct, the dynamic characteristic of the system is still defective. For such case, the methods for computing the gain vector of the distinct eigenvalues can not be used, and we have to use the methods presented by this paper, so as to obtain the effective results. The conclusions are supported by the given numerical example.

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References

Andry, A.N., Shapiro, E.Y. and Chung, J.C. (1983), "Eigenvalue assignment for linear systems", *IEEE Transactions on Aerospace and Electronic Systems*, **19**(5), 711-729.

Chen, S.H. (1999), *Matrix Perturbation Theory in Structural Dynamic Design* (in Chinese), Science Press, Beijing.

Chen, Y.D., Chen, S.H. and Liu, Z.S. (2001), "Modal optimal control procedure for near defective systems", *J. Sound Vib.*, **245**(1), 113-132.

Chen, Y.D., Chen, S.H. and Liu, Z.S. (2001), "Quantitative measurements of modal controllability and observability in vibration control of defective and nearly defective systems", *J. Sound Vib.*, **248**(3), 413-426.

Deif, A.S. (1982), *Advanced Matrix Theory for Scientist and Engineers*, Abacus House, England.

Kimura, H. (1977), "A further result on the problem of pole assignment by output feedback", *IEEE Transactions on Automatic Control*, **22**(3), 458-463.

Liu, Z.S., Wang, D.J., Hu, H.C. and Yu, M. (1994), "Measure of modal controllability and observability in vibration control of flexible structures", *Journal of Guidance, Control and Dynamics*, **17**(6), 1377-1380.

Maghami, P.G. and Juang, J.-N. (1990), "Efficient eigenvalue assignment for large space structures", *Journal of Guidance, Control, and Dynamics*, **13**(6), 1033-1039.

Maghami, P. and Gupta, S. (1997), "Design of constant gains dissipative controllers for eigensystem assignment in passive systems", *Journal of Guidance, Control, and Dynamics*, **20**(4), 648-657.

Meirovitch, L. (1990), *Dynamics and Control*, Wiley, New York.

Prattichizzo, D. and Bicchi, A. (1997), "Consistent task specification for manipulation systems with general kinematics", *Journal of Dynamic Systems Measurement and Control*, *Transaction of the ASME*, **119**(4), 760-767.

Prattichizzo, D. and Bicchi, A. (1998), "Dynamic analysis of mobility and graspability of general manipulation systems", *IEEE Transations on Robotics and Automation APR*, **14**(2), 241-258.

Shi, G.Q. and Zhu, D.C. (1989), "Generalized modal theory for linear vibration defective systems (in Chinese)", *ACTA Mechanica Sinica*, **21**(2), 212-217.

Srinathkumar, S. (1978), "Eigenvalue/eigenvector assignment using output feedback", *IEEE Transactions on Automatic Control*, **23**(1), 79-81.

Xu, T. and Chen, S.H. (1994), "Perturbation sensitivity of generalized modes of defective systems", *Comput. Struct.*, **52**(2), 1377-1380.