Structural Engineering and Mechanics, Vol. 15, No. 4 (2003) 437-449
DOI: http://dx.doi.org/10.12989/sem.2003.15.4.437

Analytical model for the prediction of the eigen modes of a beam with open cracks and external strengthening

P. A. Ovigne†, M. Massenzio‡, E. Jacquelin‡ and P. Hamelin‡†

Laboratoire Mecanique Materiaux et Structures, Universite Claude Bernard, 82 boulevard Niels Bohr, Domaine scientifique de la Doua, 69622 Villeurbanne, France

(Received July 5, 2002, Accepted March 8, 2003)

Abstract. The aim of this study is to develop an analytical model of a beam with open cracks and external strengthening which is able to predict its modal scheme components (natural frequencies and mode shapes). The model is valid as far as the excitation level is low enough not to activate non linear effects. The application field of the model are either the prediction of the efficiency of the reinforcement or the non destructive assessment of the structural properties. The degrees of freedom associated to the fault lips must be taken into account in order to introduce the effect of the external strengthening. In a first step, an analytical formulation of a beam with thin notches is proposed according to the references. The model is then extended to incorporate the strengthening consisting in a longitudinal stiffness applied in the vicinity of the cracks. In a second step, the analytical results are compared with these obtained from a finite element simulation.

Key words: eigen modes; beam; open crack; external strengthening.

1. Introduction

The dynamical behavior of a structure depends on its modal scheme components and on the excitation source. The modal scheme components (natural frequencies and mode shapes) of a structural member are very dependent on its geometry and a crack is a kind of geometrical fault having a great influence on these parameters. A crack is subjected to breathing during vibration and that is leading to a non linear behavior of the beam with coupling between bending and longitudinal modes (Abraham 1993, Abraham and Brandon 1995 (a&b), Papadopoulos and Dimarogonas 1987).

This work deals with the influence of an external repair on the natural frequencies, observed at low excitation level, of a cracked homogeneous beam substructure. An analytical modeling of a cracked homogeneous beam for both the unrepaired and repaired configuration is formulated for this purpose. The leading assumptions made are the following:

---

† Doctor
‡ Assistant Professor
‡† Professor

- The structural member to be tested is subjected to static loading that imposes a permanent open state condition to the cracks. Regarding the excitation level, the beam undergoes small amplitude vibrations in the vicinity of the static position and the cracks are maintained open during vibration cycles.
- The repair consists in an external strengthening of the beam. The material used has a linear mechanical behavior for the static and dynamic excitation ranges considered here. The repair is applied on the vicinity of the crack when static opening conditions are applied for the cracks.

These assumptions lead to a linear formulation because the cracks are not subjected to breathing. Furthermore, all the materials used have a linear mechanical behaviour in the field of the study. As a consequence, no material or structural non-linearity is considered.

In a first step, the modeling of the influence of the repair imposes to take into account the degrees of freedom associated to the crack lips. Cracking causes the release of the degrees of freedom associated to the section and can be viewed as a local loss of stiffness for a given inertia. The formulation proposed here is based on an assembly of beam elements with connecting conditions derived from fracture mechanics equations (Dimarogonas 1976). Such an approach can be found in various studies of cracked beam vibration (Gudmundson 1983, Qian *et al.* 1990). Then, an external longitudinal stiffness applied on the vicinity of the crack lips is introduced in the modeling.

In a second step, the analytical results are compared to these obtained by finite element simulations. The effect of the external strengthening is discussed.

2. Modeling of a beam with open cracks and external strengthening

2.1 Open crack state connecting conditions

Cracking causes a loss of stiffness of the damaged section resulting in a local displacement field discontinuity. In the open state, a crack can be viewed as a notch which is thin enough as the inertia remains the same (Fig. 1).

When a section of a beam is affected by a thin notch, its stiffness is locally affected and a local flexibility can be derived from linear fracture mechanics. The degrees of freedom of the notch lips can be introduced by modeling the beam as an assembly of beam elements connected to each other under connecting conditions that are expressed in the transfer matrix of the damaged section (Fig. 2). The connecting conditions are based on generalized forces continuity and displacement discontinuity at the damaged section.

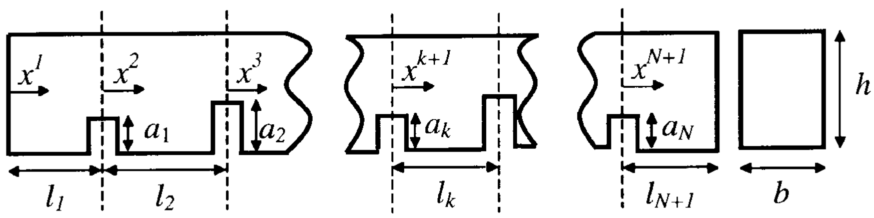

Fig. 1 General configuration

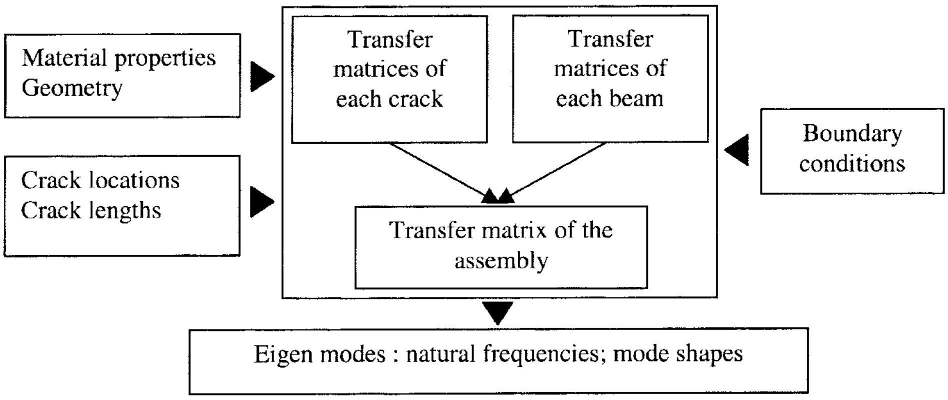

Fig. 2 General configuration

In a multi-damage case, consisting in a beam with N notches, the transfer matrix of k^{th} notched section linking the generalized forces and displacements on both sides can be expressed as:

$$\begin{bmatrix} v \\ \theta \\ P \\ M \end{bmatrix}_{right}^{k-1} = \begin{bmatrix} 1 & 0 & C_{11} & C_{12} \\ 0 & 1 & C_{21} & C_{22} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}^{k} \begin{bmatrix} v \\ \theta \\ P \\ M \end{bmatrix}_{left}^{k} \tag{1}$$

The loss of stiffness due to the fault is taken into account in the terms C_{ij} (i, j = 1, 2) that constitute the flexibility matrix due to the presence of the fault. The flexibility matrix is derived from the strain energy associated to the notch:

$$C_{ij} = \frac{\partial^2 W_{notch}}{\partial P_i \partial P_j} \qquad i, j = 1, 2 \qquad P_1 = P \qquad P_2 = M \tag{2}$$

Where P is the shear force and M is the bending moment.

The strain energy W_{notch} is derived from the Griffith principle by taking into account the opening modes of the fault. For a planar formulation, the opening modes considered are the two in-plane opening modes, as shown in Fig. 3.

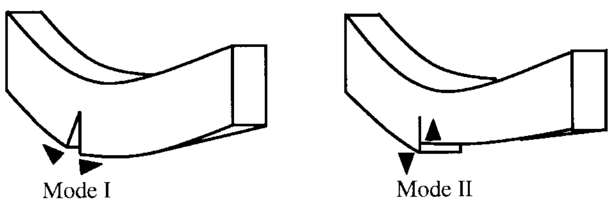

Fig. 3 Opening modes of the notch

The strain energy W_{notch} is computed by integrating the energy loss factor along the length of the notch (Tada *et al.* 1973):

$$W_{notch} = b\int_0^a \frac{1-\nu^2}{E}[(K_{IM}+K_{IP})^2 + K_{IIP}^2]da \tag{3}$$

Where K_{IP} and K_{IM} are the strain intensity factors (SIF) related to the participation of the shear forces and moment to the first opening mode. K_{IIP} is the SIF related to the second opening mode due to the shear forces. The expressions of the different SIF used here are (Tada *et al.* 1973):

$$K_{IM} = M\frac{6}{bh^2}\sqrt{\pi a\frac{\tan\varsigma}{\varsigma}}\frac{0{,}923 + 0{,}199(1-\sin\varsigma)^4}{\cos\varsigma} \tag{4}$$

$$K_{IP} = P\frac{1}{bh}\sqrt{\pi a\frac{\tan\zeta}{\varsigma}}\frac{0{,}752 + 2{,}02\left(\frac{2\varsigma}{\pi}\right) + 0{,}37(1-\sin\varsigma)^2}{\cos\varsigma} \tag{5}$$

$$K_{IIP} = P\frac{1}{bh}\sqrt{\pi a}\frac{2\varsigma}{\varsigma}\left(\varsigma - \frac{4\varsigma}{\pi}\right)\frac{1{,}122 - 0{,}56\left(\frac{2\varsigma}{\pi}\right) - 0{,}085\left(\frac{2\varsigma}{\pi}\right)^2 + 0{,}18\left(\frac{2\varsigma}{\pi}\right)^3}{\sqrt{1-\left(\frac{2\varsigma}{\pi}\right)}} \tag{6}$$

Where:

$$\varsigma = \frac{\pi a}{2h} \tag{7}$$

Eqs. (2) to (6) are used to compute the transfer matrix T related to each notched section. This matrix tends to the transfer matrix of an undamaged section when the length of the notch tends to zero.

2.2 Open crack with external strengthening: connecting conditions

The bridging element considered here, consists in a longitudinal spring which stiffness is k_{repair} · (150 MNm^{-1}). This value is imposed by applications concerning beam strengthening that are undertaken. It is applied on the cracked face of the beam and then limits the relative displacements of the crack lips (Fig. 4). Its mass can be neglected.

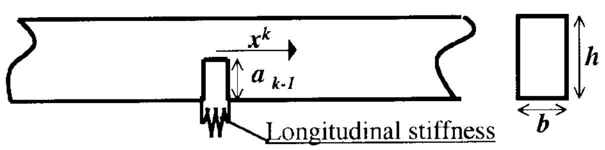

Fig. 4 External strengthening of a cracked section

The beam with fully open crack undergoes linear vibrations. So the bridging effect can be superposed to this of the crack resulting in a strengthening of the equivalent elastic hinge. The stiffness of the elastic hinge takes into account the longitudinal stiffness of the bridging element k_{repair}. The rotational stiffness of the cracked bridged section is given by:

$$k^{\theta}_{repair} = h^2 k_{repair} \tag{8}$$

The stiffness of the hinge corresponding to the notched section with bridging element is then:

$$k^{\theta}_{hinge} = \frac{1}{C_{22}} + k^{\theta}_{repair} \tag{9}$$

And the resulting flexibility associated to the rotation discontinuity is given by:

$$\overline{C_{22}} = \frac{1}{k^{\theta}_{hinge}} \tag{10}$$

The transfer matrix of the notched section with external strengthening is then:

$$\begin{bmatrix} v \\ \theta \\ P \\ M \end{bmatrix}^{k-1}_{right} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \overline{C_{22}} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}^{k} \begin{bmatrix} v \\ \theta \\ P \\ M \end{bmatrix}^{k}_{left} \tag{11}$$

2.3 Generalized displacements and forces vector

The Timoshenko beam cinematic is described by two independent variables which are the transverse displacement v (x, t) and the rotation of the section θ (x, t). All the sections remain plane but are not perpendicular to the mid fiber (Fig. 5).

For such a beam the generalize displacements and forces are:

$$\theta(x, t) = \frac{\partial v(x, t)}{\partial x} + \beta(x, t) \tag{12}$$

$$M = EI\frac{\partial \theta}{\partial x} \tag{13}$$

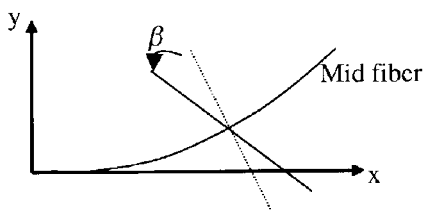

Fig. 5 Section rotation within a Timoshenko beam

$$P = kAG\left(\frac{\partial v}{\partial x} - \theta\right) \tag{14}$$

where k is the shear factor that can be expressed for a rectangular section as:

$$k = 10\frac{(1+\nu)}{(12+11\nu)} \tag{15}$$

Separating the time and space variables gives:

$$v(x,t) = v(x)e^{i\omega t} \tag{16}$$

$$\theta(x,t) = \theta(x)e^{i\omega t} \tag{17}$$

The governing equations can be written as:

$$\begin{cases} (m\omega^2 + kAGD^2)v - kAGD\theta = 0 \\ kAGDv + (EID^2 - kAG + mr^2\omega^2)\theta = 0 \end{cases} \tag{18}$$

where r is the gyration radius given by

$$r = \sqrt{\frac{I}{A}} \tag{19}$$

and D is the differential operator:

$$D = \frac{\partial}{\partial x} \tag{20}$$

The equation can be written in a matricial form as:

$$\begin{bmatrix} D^2 + a_1 & -D \\ a_3 D & D^2 + a_2 \end{bmatrix}\begin{bmatrix} v \\ \theta \end{bmatrix} \tag{21}$$

where:

$$a_1 = \frac{m\omega^2}{kAG} \tag{22}$$

$$a_2 = \frac{mr^2\omega^2 - kAG}{EI} \tag{23}$$

$$a_3 = \frac{kAG}{EI} \tag{24}$$

Assuming that:

$$v(x) = \sum_{j=1}^{4} A_j^k e^{p_j x} \tag{25}$$

$$\theta(x) = \sum_{j=1}^{4} B_j^k e^{p_j x} \tag{26}$$

Where A_j^k and B_j^k are functions of the wave numbers p_j. Introducing Eqs. (25) and (26) into Eq. (21) and searching the roots of the matrix determinant leads to the characteristic polynomial associated to the beam. The wave numbers p_j are the roots of the characteristic polynomial:

$$(p_j^2 + a_1)(p_j^2 + a_2) + a_3 p_j^2 = 0 \tag{27}$$

This is a four-degree polynomial then, there are four wave numbers p_j with $j = 1$ to 4.
B_j^k is related to A_j^k by considering Eq. (21) that gives:

$$A_j^k(p_j^2 + a_1) - p_j B_j^k = 0 \tag{28}$$

Then B_j^k can be expressed by a function f_B and A_j^k by:

$$B_j^k = f_B A_j^k = \frac{p_j^2 + a_1}{p_j} A_j^k \tag{29}$$

As consequences, the generalized forces are related to A_j^k:

$$P(x) = \sum_{j=1}^{4} C_j^k e^{p_j x} \tag{30}$$

$$C_j^k = f_C A_j^k = kAG\left(\frac{a_1}{p_j}\right) A_j^k \tag{31}$$

and

$$M(x) = \sum_{j=1}^{4} D_j^k e^{p_j x} \tag{32}$$

$$D_j^k = f_D A_j^k = EI(p_j^2 + a_1) A_j^k \tag{33}$$

As a consequence, the generalized displacements and forces vector can be written as:

$$\begin{bmatrix} v \\ \theta \\ T \\ M \end{bmatrix}^k = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & C_3 & C_4 \\ D_1 & D_2 & D_3 & D_4 \end{bmatrix}^k \begin{bmatrix} e^{p_1 x^k} \\ e^{p_2 x^k} \\ e^{p_3 x^k} \\ e^{p_4 x^k} \end{bmatrix} \tag{34}$$

$$0 \le x^k \le L_k \qquad k = 1 \text{ to } (N+1)$$

At each side of the beam, the generalized forces are given by:

$$M_{left}^{k} = -\sum_{j=1}^{4} D_j^k \tag{35}$$

$$P_{left}^{k} = \sum_{j=1}^{4} C_j^k \tag{36}$$

$$M_{right}^{k} = \sum_{j=1}^{4} D_j^k e^{p_j L_k} \tag{37}$$

$$P_{right}^{k} = -\sum_{j=1}^{4} C_j^k e^{p_j L_k} \tag{38}$$

2.4 Assembly of the beam elements

The assembly of the beam elements number k and $k + 1$ follows the connecting conditions of the notch number k. That can be written as:

$$\sum_{j=1}^{4} A_j^k e^{p_j L_k} - \sum_{j=1}^{4} A_j^{k+1}(1 + C_{11}^k f_C - C_{12}^k f_D) = 0 \tag{39}$$

$$\sum_{j=1}^{4} A_j^k f_B e^{p_j L_k} - \sum_{j=1}^{4} A_j^{k+1}(f_B + C_{21}^k f_C - C_{22}^k f_D) = 0 \tag{40}$$

$$-\sum_{j=1}^{4} A_j^k f_C e^{p_j L_k} + \sum_{j=1}^{4} A_j^{k+1} f_C = 0 \tag{41}$$

$$\sum_{j=1}^{4} A_j^k f_D e^{p_j L_k} - \sum_{j=1}^{4} A_j^{k+1} f_D = 0 \tag{42}$$

2.5 Boundary conditions

For free-free boundary conditions the generalized forces are zeroes at the beam tips:

$$\sum_{j=1}^{4} A_j^1 f_C = 0 \tag{43}$$

$$-\sum_{j=1}^{4} A_j^1 f_D = 0 \tag{44}$$

$$\sum_{j=1}^{4} A_j^{N+1} f_C e^{p_j L_{N+1}} = 0 \tag{45}$$

$$\sum_{j=1}^{4} A_j^{N+1} f_D e^{p_j L_{N+1}} = 0 \tag{46}$$

2.6 Global equation system

Eqs. (39) to (42), written for N notches and added to the four boundary conditions, Eqs. (43) to (46), lead to a $4(N+1) \times 4(N+1)$ system in terms of A_j^k coefficients Eq. (47).

$$\begin{bmatrix} Mat \\ size \quad 4(N+1) \times 4(N+1) \end{bmatrix} \begin{bmatrix} A_1^1 \\ A_2^1 \\ \bullet \\ \bullet \\ \bullet \\ A_3^{N+1} \\ A_4^{N+1} \end{bmatrix} = [0] \tag{47}$$

2.7 Natural frequencies and mode shapes of the beam

The natural pulsation ω of the beam set the determinant of the matrix [Mat] equal to zero

$$Det[Mat](\omega) = 0 \tag{48}$$

For a given pulsation, the A_j^k coefficients vector in Eq. (47) is computed as a function of one arbitrary A_j^k coefficient and the mode shape is computed with Eq. (25).

3. Finite element validation

The analytical results are compared with these obtained from finite element simulations (ANSYS code) for three configurations of a beam: intact, notched and notched with external strengthening. A planar analysis is done and the natural frequencies are calculated in a plane stress configuration. A quadrangular type of solid elements is used. This element is defined by four nodes having two degrees of freedom at each node and includes extra displacement shapes. Only results concerning thin -plane bending modes are presented.

3.1 Specimens

The beam is 765 mm long and has a section of 39 × 20 mm (Fig. 6). The material has a Young modulus of 216 GPa and a Poisson ration of 0,3. A central notch is modeled (length: 30 mm, width: 2 mm). The longitudinal stiffness (150 10^6N/m) is applied on the vicinity of the crack (Fig. 6).

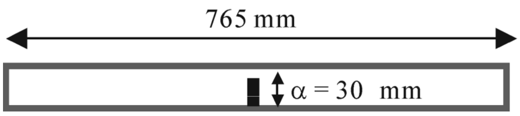

Fig. 6 Beam with a central notch

3.2 Results

The results are given in Table 1 for the intact beam, in Table 2 for the notched beam and in Table 3 for the notched beam with external strengthening.

The formulation proposed is valid and is able to predict the frequency changes induced by the presence of a thin notch with external strengthening. As expected the even modes are not affected by a central notch.

Table 1 Finite element validation: intact beam

Mode number	Analytical results	Finite element results	Difference (%)
1	357.14	357.30	0.04
2	968.07	970.11	0.21
3	1853.9	1864.4	0.56
4	2976.4	3013.6	1.23
5	4299.1	4402.6	2.35

Table 2 Finite element validation: notched beam

Mode number	Analytical results	Finite element results	Difference (%)
1	185.38	178.30	3.97
2	968.09	969.25	0.12
3	1481.5	1414.40	4.74
4	2976.4	3016.90	1.34
5	3707.9	4204.10	11.8

Table 3 Finite element validation: notched beam with external strengthening

Mode number	Analytical results	Finite element results	Difference (%)
1	316.14	303.28	4.24
2	968.07	969.39	0.14
3	1719.0	1678.30	2.42
4	2976.4	3008.00	1.05
5	4038.5	4245.9	4.88

3.3 Discussion

During calculations it clearly appears that only the C_{22} or $\overline{C_{22}}$ term associated to the rotation discontinuity plays an important role in the flexibility matrix in Eq. (1). As a consequence the transfer matrix of the notched section can be simplified considering only this term, leading to Eq. (49).

$$\begin{bmatrix} v \\ \theta \\ P \\ M \end{bmatrix}_{right}^{k-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & C_{22} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}^{k} \begin{bmatrix} v \\ \theta \\ P \\ M \end{bmatrix}_{left}^{k} \tag{49}$$

Eq. (49) gives the expression of the transfer matrix of an elastic hinge. The stiffness of the hinge equals the inverse of C_{22}. Indeed, regarding the importance of bending, shearing effects can be neglected. Furthermore, when the length of the notch is small, C_{22} tends to zero and the transfer matrix of the cracked section tends to the one of an intact one with both continuity of the generalized forces and displacements.

The contribution η_{repair} of the external strengthening to the stiffness of the section can be defined by Eq. (50):

$$\eta_{repair} = \frac{k_{repair}^{\theta}}{k_{hinge}^{\theta}} \times 100 \tag{50}$$

Eq. (50) and Fig. 7 show that η_{repair} increases with the crack length. This can be seen on Fig. 7.

η_{repair} is quickly increasing with the crack length. In that case (Fig. 7), the external strengthening represents half the stiffness of the hinge. For a small crack length, this contribution is null and then, the transfer matrix of the section equals the one of an intact section. The rotation stiffness of the notched section is then infinite and the effect of the bridging element k_{repair}^{θ} can be neglected (Eq. 47). As expected, the natural frequencies of a beam with notches and bridging elements tend to these of an intact beam when the length of the notches tends to zero (Table 4).

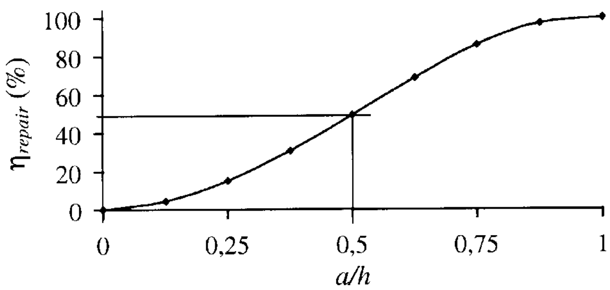

Fig. 7 External strengthening contribution to the stiffness of the section

Table 4 Natural frequencies (Hz) with small length notch (a = 1 10^{-4} m). Modeling

Mode number	Intact beam	Beam with small notches and bridging	Increase (%)
1	357	357	0.00
2	969	969	0.00
3	1853	1855	0.11
4	2976	2978	0.07
5	4292	4301	0.21

4. Conclusions

An analytical model of a cracked Timoshenko beam based on an assembly of beam elements with connecting conditions derived from linear fracture mechanics principle is formulated according to the bibliography. The effect of an external strengthening is modeled. It consists in a negligible mass linear stiffness applied in the vicinity of the crack. This model is able to predict with a good accuracy the natural frequencies of a beam with open cracks observed at low excitation level. The formulation proposed is validated by finite element simulation. As expected, strengthening enhances the natural frequencies of the cracked beam and reduces the displacement discontinuity in the vicinity of the crack. The contribution of the external stiffening increases with the crack length.

Various applications of this model will be undertaken

- Analysis of the influence of a carbon-epoxy composite external strengthening on the natural frequencies of a cracked homogeneous beam.
- Natural frequencies prediction of a reinforced concrete beam with cracking induced by static loading for both the unrepaired and repaired configuration. In that case, the reinforcement is both due to the steel rebars within concrete and by epoxy-carbon composite plate.

References

Abraham, O.N.L. (1993), “Dynamic modelling of cracked Timoshenko beams”, Thesis, University of Wales, Cardiff.

Abraham, O.N.L. and Brandon, J.A. (1995), “The modelling of the opening and closing of a crack”, *Transactions of the American Society of Mechanical Engineers: Journal of Vibration and Acoustics*, **117**, 370-377.

Brandon, J.A. and Abraham, O.N.L. (1995), “Counter-intuitive quasi-periodic motion in autonomous vibration of cracked Timoshenko beams”, *J. Sound Vib.*, **185**(3), 415-430.

Dimarogonas, A.D. (1976), *Vibration Engineering*. St Paul, West Publishers.

Gudmundson, P. (1983), “The dynamic behavior of slender structures with cross sectional cracks”, *J. Mech. Phy. Sol.*, **31**(4), 329-345.

Papadopoulos, C.A. and Dimarogonas, A.D. (1987), “Coupled longitudinal and bending vibrations of a rotating shaft with an open crack”, *J. Sound Vib.*, **117**(1), 81-93.

Qian, G.L., Gu, S.N. and Jiang, J.S. (1990), “The dynamic behavior and crack detection of a beam with a crack”, *J. Sound Vib.*, **138**(2), 233-243.

Tada, H., Paris, P. and Irwin, G. (1973), *The Stress Analysis of Cracks Handbook.* Hellertown, Pennsylvania, Del Research Corporation.

Notation

A, A_{compo}	: Beam cross sectional area, composite cross sectional area
A_j^k	: Coefficients associated to $v(x)$
a, a_k	: Notch length, notch length of the k^{th} notch
B_j^k	: Coefficients associated to $\theta(x)$
b	: Beam width
C_{ij}	: Flexibility matrix term
C_j^k	: Coefficients associated to P
D	: Differential operator with respect to x
D_j^k	: Coefficients associated to M
Det	: Determinant operator
E, E_{compo}	: Young modulus, composite longitudinal Young modulus
f_B, f_C, f_D	: Function associated to B_j^k, C_j^k, D_j^k
f_i	: Natural frequency associated to the mode i
G	: Shear modulus
h	: Beam height
I	: Beam inertia
K_{IM} , K_{IP}	: Strain intensity factor associated to M and to P for the first opening mode
K_{IIP}	: Strain intensity factor associated to P for the second opening mode
k	: Shear factor
k_{hinge}^{θ}, k_{repair}^{θ}	: Rotation stiffness of the elastic hinge, external strengthening rotation stiffness
k_{repair}	: External strengthening longitudinal stiffness
L_{active}	: Composite active length
M	: Bending moment
Mat	: Global equation system associated to the assembly and boundary conditions
m	: Mass
N	: Number of notches
P	: Shear force
P_i	: Generalized force $P_1 = M$ and $P_2 = P$
p_j	: Wave number
r	: Gyration radius
t	: Time
$v\ (x, t)$, $v\ (x)$	: Transverse displacement, transverse displacement amplitude
W_{notch}	: Strain energy associated to a notch
x^k	: Axial coordinate associated to the k^{th} beam element
β	: Shear rotation
η_{repair}	: Contribution of the external strengthening to the rotation stiffness of the section
$\theta\ (x, t)$, $\theta\ (x)$	: Global rotation, global rotation amplitude
ν	: Poisson ratio
ω	: Natural angular frequency
ζ	: Nondimensional parameter associated to a