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# A frictionless contact problem for two elastic layers supported by a Winkler foundation

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**Abstract.** The plane contact problem for two infinite elastic layers whose elastic constants and heights are different is considered. The layers lying on a Winkler foundation are acted upon by symmetrical distributed loads whose lengths are $2a$ applied to the upper layer and uniform vertical body forces due to the effect of gravity in the layers. It is assumed that the contact between two elastic layers is frictionless and that only compressive normal tractions can be transmitted through the interface. The contact along the interface will be continuous if the value of the load factor, $\lambda$, is less than a critical value. However, interface separation takes place if it exceeds this critical value. First, the problem of continuous contact is solved and the value of the critical load factor, $\lambda_{cr}$, is determined. Then, the discontinuous contact problem is formulated in terms of a singular integral equation. Numerical solutions for contact stress distribution, the size of the separation areas, critical load factor and separation distance, and vertical displacement in the separation zone are given for various dimensionless quantities and distributed loads.

**Key words:** continuous contact; discontinuous contact; separation integral equation; elastic layer; Winkler foundation; elasticity.

## 1. Introduction

The contact problems for a frictionless elastic layer lying on an elastic half-plane, elastic foundation or rigid foundation have attracted considerable attention in the past due to its applicability to a variety of important structures of practical interest (see, e.g., Civelek and Erdogan 1974, Erdogan and Ratwani 1974, Civelek and Erdogan 1975, Civelek and Erdogan 1976, Civelek *et al*. 1978, Gecit and Erdogan 1978, Gecit 1978, Gecit 1980, Gecit 1981, Gecit and Yapici 1986, Gecit 1990, Dempsey *et al*. 1990, Cakiroglu and Cakiroglu 1991, Dempsey *et al*. 1991, Kahya *et al*. 2001). However, there are few studies about the contact between two or more elastic layers made of different elastic materials and heights. Some of these are a frictionless contact problem between two elastic-viscoplastik bodies considered by Rochdi and Sofonea (1997), contact problems for two elastic layers resting on an elastic half-plane examined by Cakiroglu *et al*. (2001) and continuous and discontinuous contact problem for a layered composite resting on simple supports studied by Birinci and Erdol (2001). In such problems if the magnitude of the external load exceeds a certain critical value a separation takes place between the layers or the layer and the foundation.

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In this paper, the plane elastostatic contact problem of two infinite elastic layers having different elastic constants and heights lying on a Winkler foundation is considered according to theory of Elasticity. The frictionless layer-layer interface is assumed to transmit compressive normal tractions only. The upper elastic layer is subjected to a symmetrical distributed loads whose lengths are $2a$ on its top surface and the layers have a uniform body force due to the gravity. If the value of the load factor, $\lambda$, is less than a critical value, $\lambda_{cr}$, the normal stress along the entire interface is compressive and the contact is continuous. A separation takes place between the elastic layers when the applied load exceeds this critical value and the contact is discontinuous. Firstly, the continuous contact problem is solved and the value of the critical load factor, the critical separation distance, and the contact stress distribution are determined. Then, the discontinuous contact problem is formulated in terms of a singular integral equation. Solving the integral equation numerically by using appropriate Gauss-Chebyshev integration formula, the stress distribution along the interface, the initial and end distances of the separation, and the vertical displacement in the separation area are investigated for various dimensionless quantities. Finally, numerical results are analyzed and conclusions are drawn.

## 2. Formulation of the problem

Consider two infinite elastic layers of which thickness are $h_1$ and $h_2$ in smooth contact with each other. The geometry and coordinate system are shown in Fig. 1. Let $\rho_1 g$ and $\rho_2 g$ be the body forces acting vertically in the layers. Writing,

$$u_i(x,y) = u_{ip}(x) + u_{ih}(x,y), \tag{1a}$$

$$v_i(x,y) = v_{ip}(y) + v_{ih}(x,y), \qquad (i = 1, 2), \tag{1b}$$

the particular part of the displacement components corresponding to $\rho_1 g$ and $\rho_2 g$ for the layers may be obtained as (Civelek and Erdogan 1974)

$$u_{1p}(x) = \frac{3-\kappa_1}{8\mu_1}\frac{\rho_1 g h_1}{2}x, \tag{2a}$$

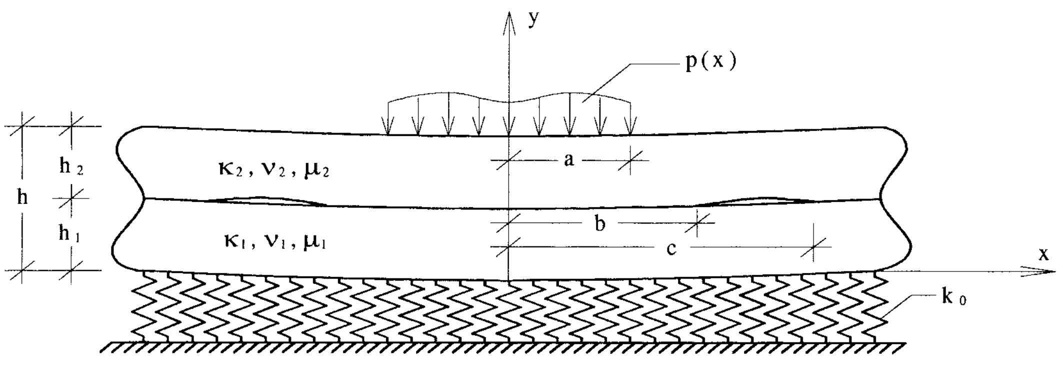


Fig. 1 Geometry of the frictionless contact problem with interface separation

$$v_{1p}(y) = \frac{\kappa_1 - 1}{\kappa_1 + 1}\frac{\rho_1 g y}{2\mu_1}(y - h_1) - \frac{1 + \kappa_1}{8\mu_1}y(\rho_2 g h_2 + \rho_1 g h_1/2) + E, \tag{2b}$$

$$u_{2p}(x) = \frac{3 - \kappa_2}{8\mu_2}\frac{\rho_2 g h_2}{2}x, \tag{2c}$$

$$v_{2p}(y) = -\frac{\rho_2 g y}{2\mu_2}\left[\frac{1 + \kappa_2}{8}h_2 - \frac{\kappa_2 - 1}{\kappa_2 + 1}(h_1 + h - y)\right] + F, \tag{2d}$$

where $u$ and $v$ are the $x$ and $y$ - components of the displacement vector, $\mu$ is shear modulus, $\nu$ is Poisson's ratio, and $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress and $\kappa = 3 - 4\nu$ for plane strain. The subscripts 1 and 2 refer to the lower and upper elastic layers, respectively. The constants $E$ and $F$ appearing in Eqs. (2b) and (2d) are the (yet unknown) rigid body displacement terms.

Observing that $x = 0$ is a plane symmetry, the homogenous part of the displacement components for the upper and lower elastic layers may be written as (Civelek and Erdogan 1974, Cakiroglu *et al*. 2001, Birinci and Erdol 2001),

$$u_{ih}(x, y) = \frac{2}{\pi}\int_0^\infty [(A_i + B_i y)e^{-\alpha y} + (C_i + D_i y)e^{\alpha y}]\sin(\alpha x)d\alpha, \tag{3a}$$

$$v_{ih}(x, y) = \frac{2}{\pi}\int_0^\infty \left\{\left[A_i + \left(\frac{\kappa_i}{\alpha} + y\right)B_i\right]e^{-\alpha y} + \left[-C_i + \left(\frac{\kappa_i}{\alpha} - y\right)D_i\right]e^{\alpha y}\right\}\cos(\alpha x)d\alpha, \tag{3b}$$

where $A_i$, $B_i$, $C_i$ and $D_i (i = 1,\ 2)$ are the unknown functions which will be determined from continuity and boundary conditions prescribed on $y = 0$, $y = h_1$ and $y = h$.

The stress components needed for the application of the boundary conditions may be obtained from Eqs. (1-3) using Hooke's law as follows (Civelek and Erdogan 1975):

$$\begin{aligned}\frac{1}{2\mu_1}\sigma_{y1}(x, y) = \frac{2}{\pi}\int_0^\infty &\left\{-\left[\alpha(A_1 + B_1 y) + \frac{1 + \kappa_1}{2}B_1\right]e^{-\alpha y}\right.\\ &\left.+\left[-\alpha(C_1 + D_1 y) + \frac{1 + \kappa_1}{2}D_1\right]e^{\alpha y}\right\}\cos(\alpha x)d\alpha\\ &+\frac{1}{2\mu_1}[-\rho_2 g h_2 + \rho_1 g(y - h_1)], \quad (0 \le y \le h_1),\end{aligned} \tag{4a}$$

$$\begin{aligned}\frac{1}{2\mu_2}\sigma_{y2}(x, y) = \frac{2}{\pi}\int_0^\infty &\left\{-\left[\alpha(A_2 + B_2 y) + \frac{1 + \kappa_2}{2}B_2\right]e^{-\alpha y}\right.\\ &\left.+\left[-\alpha(C_2 + D_2 y) + \frac{1 + \kappa_2}{2}D_2\right]e^{\alpha y}\right\}\cos(\alpha x)d\alpha\\ &+\frac{1}{2\mu_2}[\rho_2 g(y - h)], \qquad (h_1 \le y \le h),\end{aligned} \tag{4b}$$

$$\frac{1}{2\mu_i}\tau_{xyi}(x,y) = \frac{2}{\pi}\int_0^\infty\left\{-\left[\alpha(A_i+B_iy)+\frac{\kappa_i-1}{2}B_i\right]e^{-\alpha y}\right.$$

$$\left.+\left[\alpha(C_i+D_iy)-\frac{\kappa_i-1}{2}D_i\right]e^{\alpha y}\right\}\sin(\alpha x)d\alpha, \quad (i=1,2), \tag{4c}$$

## 3. The case of continuous contact

Referring to Fig. 1, let the upper elastic layer be subjected to a symmetrical distributed load, $p(x)$, about $y$ axis, along its boundary $y = h$. If the load factor, $\lambda$, is sufficiently small, the contact along the interface $y = h_1$ will be continuous and $A_i$, $B_i$, $C_i$ and $D_i (i = 1,2)$ should be determined from the following boundary and continuity conditions :

$$\tau_{xy2}(x,h) = 0, \qquad (0 \le x < \infty), \tag{5a}$$

$$\tau_{xy2}(x,h_1) = 0, \qquad (0 \le x < \infty), \tag{5b}$$

$$\tau_{xy1}(x,h_1) = 0, \qquad (0 \le x < \infty), \tag{5c}$$

$$\sigma_{y2}(x,h_1) = \sigma_{y1}(x,h_1), \qquad (0 \le x < \infty), \tag{5d}$$

$$\tau_{xy1}(x,0) = 0, \qquad (0 \le x < \infty), \tag{5e}$$

$$\sigma_{y2}(x,h) = \begin{cases} -p(x), & 0 \le x \le a \\ 0, & a < x < \infty \end{cases}, \tag{5f}$$

$$\sigma_{y1}(x,0) = k_0 v_1(x,0), \qquad (0 \le x < \infty), \tag{5g}$$

$$\frac{\partial}{\partial x}[v_2(x,h_1)-v_1(x,h_1)] = 0, \qquad (0 \le x < \infty), \tag{5h}$$

where $k_0$ is the stiffness of the foundation and Eq. (5h) is equivalent to $v_2(x,h_1) = v_1(x,h_1)$. By making use of conditions (5), $A_i$, $B_i$, $C_i$ and $D_i (i = 1,2)$ functions may be calculated and by substituting the values of the functions into Eqs. (4a) or (4b), the contact pressure along the interface $y = h_1$ becomes

$$\sigma_y(x,h_1) = -\rho_2 g h_2 - \frac{4}{\pi}(1+\kappa_2)\int_0^\infty \frac{P_i}{\Delta(\alpha)}e^{-\alpha h}e^{-\alpha h_1}Y1(\alpha)[4\alpha Y2(\alpha)$$

$$+\,k(1+\kappa_1)Y3(\alpha)]\cos(\alpha x)d\alpha, \tag{6}$$

in which $P_i$ and $\Delta(\alpha)$ are defined as,

$$P_i = \int_0^\infty p(x)\cos(\alpha x)dx, \qquad (i = 1, 2, 3), \tag{7a}$$

$$\Delta(\alpha) = Y4(\alpha)[-4\alpha(1+\kappa_2)Y2(\alpha) - k(1+\kappa_1+\kappa_2+\kappa_1\kappa_2)Y3(\alpha)]$$
$$+\frac{\mu_2}{\mu_1}Y5(\alpha)[4\alpha(1+\kappa_1)Y3(\alpha)+k(1+2\kappa_1+\kappa_1^2)Y6(\alpha)], \tag{7b}$$

where,

$$Y1(\alpha) = e^{-2\alpha h_1} - e^{-2\alpha h} + (\alpha h - \alpha h_1)(e^{-2\alpha h_1} + e^{-2\alpha h})$$
$$Y2(\alpha) = 1 - e^{2\alpha h_1}(2 + 4\alpha^2 h_1^2 - e^{2\alpha h_1})$$
$$Y3(\alpha) = -1 + e^{2\alpha h_1}(4\alpha h_1 + e^{2\alpha h_1})$$
$$Y4(\alpha) = e^{-4\alpha h} - e^{-4\alpha h_1} - 2e^{-2\alpha h_1}e^{-2\alpha h}(2\alpha h - 2\alpha h_1)$$
$$Y5(\alpha) = e^{-4\alpha h} + e^{-4\alpha h_1} - e^{-2\alpha h_1}e^{-2\alpha h}(2 + 4\alpha^2 h^2 + 4\alpha^2 h_1^2 - 8\alpha^2 h h_1)$$
$$Y6(\alpha) = 1 - e^{2\alpha h_1}(2 - e^{2\alpha h_1}) \tag{7c}$$

and

$$k = \frac{k_0}{\mu_1}. \tag{7d}$$

For each loading case, $P_i (i = 1, 2, 3)$ is calculated as,
a) Loading case 1 : $p(x) = p_0 =$ constant

$$P_1 = \frac{p_0}{\alpha}\sin(\alpha a), \tag{8a}$$

b) Loading case 2 : $p(x) = p_0(1 - x^2/a^2)$

$$P_2 = \frac{2p_0}{\alpha^3 a^2}[\sin(\alpha a) - \alpha a\cos(\alpha a)], \tag{8b}$$

c) Loading case 3 : $p(x) = p_0(1 - x/a)$

$$P_3 = \frac{p_0}{\alpha^2 a}[1 - \cos(\alpha a)]. \tag{8c}$$

By substituting the expressions in Eq. (8) into Eq. (6) and replacing $\omega = \alpha h$, $r = h_1/h$, the normalized contact pressure along the interface is obtained as,

$$\frac{\sigma_y(x,h_1)}{\rho_2 g h_2} = -1 - \lambda\frac{4}{\pi}(1+\kappa_2)\int_0^\infty \frac{f_i(\omega, a/h)}{\Delta(\omega,r)} e^{-\omega}e^{-\omega r}Y1(\omega,r)\Big[4\frac{\omega}{h}Y2(\omega,r)$$
$$+ k(1+\kappa_1)Y3(\omega,r)\Big]\cos\left(\omega\frac{x}{h}\right)d\omega, \tag{9}$$

where $\lambda$ being the load factor is defined as,

$$\lambda = \frac{p_0}{\rho_2 g h_2}, \tag{10}$$

and $Y1(\omega, r)$, $Y2(\omega, r)$, $Y3(\omega, r)$, and $\Delta(\omega, r)$ may be obtained by substituting $\omega = \alpha h$ and $r = h_1/h$ into Eqs. (7b) and (7c). Additionally, each $f_i(\omega, a/h)(i = 1, 2, 3)$ function corresponding to a loading case is defined as,

$$f_1\left(\omega, \frac{a}{h}\right) = \frac{1}{\omega}\sin\left(\omega\frac{a}{h}\right), \tag{11a}$$

$$f_2\left(\omega, \frac{a}{h}\right) = \frac{2}{\omega^3}\left(\frac{h}{a}\right)^2\left[\sin\left(\omega\frac{a}{h}\right) - \omega\frac{a}{h}\cos\left(\omega\frac{a}{h}\right)\right], \tag{11b}$$

$$f_3\left(\omega, \frac{a}{h}\right) = \frac{1}{\omega^2}\left(\frac{h}{a}\right)\left[1 - \cos\left(\omega\frac{a}{h}\right)\right]. \tag{11c}$$

From Eq. (9), it is seen that $\sigma_y(x, h_1)/\rho_2 g h_2 = -1$ for $\lambda = 0$ and up to a certain value of $\lambda$, it remains negative and the contact on $y = h_1$ at which the interface separation starts at $x_{cr}$ can be obtained from Eq. (9) by using the condition $\sigma_y(x, h_1)/\rho_2 g h_2 = 0$ as,

$$\frac{1}{\lambda_{cr}} = -\frac{4}{\pi}(1 + \kappa_2)\int_0^\infty \frac{f_i(\omega, a/h)}{\Delta(\omega, r)} e^{-\omega} e^{-\omega r} Y1(\omega, r)\left[4\frac{\omega}{h} Y2(\omega, r)\right.$$

$$\left. + \, k(1 + \kappa_1) Y3(\omega, r)\right]\cos\left(\omega\frac{x}{h}\right) d\omega. \tag{12}$$

After $f_i(\omega, a/h)$ $(i = 1, 2, 3)$ values corresponding to each loading case are substituting into Eq. (12) and the integral is calculated numerically, $x_{cr}$ values, for which initial separation takes place, and corresponding $\lambda_{cr}$ values may be obtained for various proportions of material constants, $a/h$, the stiffness of the foundation and the thickness of the layers.

## 4. The case of discontinuous contact

Since the interface cannot carry tensile tractions, for $\lambda > \lambda_{cr}$ there will be separation in the neighborhood of $x = x_{cr}$ on the contact plane $y = h_1$ as shown in Fig. 1. Eqs. (1)-(4) and the conditions (5a-g) are still valid. However, (5h) should be replaced by the following mixed boundary conditions;

$$\sigma_y(x, h_1) = 0, \qquad (b < x < c), \tag{13a}$$

$$\frac{\partial}{\partial x}[v_2(x, h_1) - v_1(x, h_1)] = \varphi(x), \qquad (b < x < c), \tag{13b}$$

where $(b < x < c)$ is described as the separation area where $b$ and $c$ are unknown, and are functions of $\lambda$. Note that the unknown function $\varphi(x)$ in Eq. (13b) may be replaced by

$$\varphi(x) = 0, \qquad (0 \le x < b,\ c < x < \infty) \tag{14a}$$

and

$$\int_b^c \varphi(x)dx = 0. \tag{14b}$$

Utilizing the conditions (5a-g) and (13b), the new functions $A_i$, $B_i$, $C_i$ and $D_i$ ($i = 1, 2$) which appear in Eqs. (3) and (4) may be obtained in terms of the unknown function $\varphi(x)$. Then, Eq. (13a) gives the following singular integral equation which $\varphi(x)$ is the unknown function;

$$\frac{4\mu_1}{(1+\kappa_1)\rho_2 g h_2}\frac{1}{(1+m)}\frac{1}{\pi}\int_b^c\left[\frac{1}{t+x}+\frac{1}{t-x}+k_1(x,t)\right]\varphi(t)dt - \frac{4}{\pi}\lambda k_2(x) = 1, \tag{15}$$

where,

$$k_1(x,t) = \int_0^\infty\left\{\frac{1}{\Delta(\alpha)}\left(1+\frac{1}{m}\right)(1+\kappa_2)[4\alpha Y2(\alpha)+k(1+\kappa_1)Y3(\alpha)]Y5(\alpha) - 1\right\}$$

$$*\{\sin[\alpha(t+x)] + \sin[\alpha(t-x)]\}d\alpha, \tag{16a}$$

$$k_2(x) = \int_0^\infty \frac{P_i}{\Delta(\alpha)}e^{-\alpha h}e^{-\alpha h_1}(1+\kappa_2)Y1(\alpha)[4\alpha Y2(\alpha)+k(1+\kappa_1)Y3(\alpha)]\cos(\alpha x)d\alpha, \tag{16b}$$

$$m = \frac{\mu_1}{\mu_2}\frac{1+\kappa_2}{1+\kappa_1}. \tag{16c}$$

One may notice that because of the smooth contact at the end points $b$ and $c$, the function $\varphi(x)$ is zero at the ends and the index of the integral Eq. (15) is equal to $-1$ (Muskhelishvili 1953). In this case the two relations which are needed to determine the unknown constants $b$ and $c$ are the single-valuedness condition (14b) and the consistency condition of the integral Eq. (15) defined as follows:

$$\int_b^c\left\{\frac{4\mu_1}{(1+\kappa_1)\rho_2 g h_2}\frac{1}{(1+m)}\frac{1}{\pi}\int_b^c\left[\frac{1}{t+x}+k_1(x,t)\right]\varphi(t)dt\right.$$

$$\left. -1 - \frac{4}{\pi}\lambda k_2(x)\right\}\frac{dx}{[(x-b)(c-x)]^{1/2}} = 0. \tag{17}$$

Defining the following dimensionless quantities;

$$\eta = \frac{2t}{c-b} - \frac{c+b}{c-b}, \tag{18a}$$

$$\xi = \frac{2x}{c-b} - \frac{c+b}{c-b}, \tag{18b}$$

$$g(\eta) = \varphi(t)\frac{4\mu_1}{(1+\kappa_1)\rho_2 g h_2}, \tag{18c}$$

and substituting Eq. (18) into the integral Eq. (15), the single-valuedness condition (14b), and the consistency condition (17), the following equations are obtained:

$$\frac{1}{(1+m)}\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{\eta-\xi}+\frac{1}{\eta+\xi+2\dfrac{c+b}{c-b}}+\frac{c-b}{2h}k_1^*(\xi,\eta)\right]g(\eta)d\eta = \frac{4}{\pi}\lambda k_2^*(\xi)+1, \tag{19a}$$

$$\int_{-1}^{1} g(\eta)d\eta = 0, \tag{19b}$$

$$\int_{-1}^{1}\frac{d\xi}{(1-\xi^2)^{1/2}}\left\{1+\frac{4}{\pi}\lambda k_2^*(\xi)-\frac{1}{(1+m)}\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{\eta+\xi+2\dfrac{c+b}{c-b}}+\frac{c-b}{2h}k_1^*(\xi,\eta)\right]g(\eta)d\eta\right\}=0, \tag{19c}$$

where

$$k_1^*(\xi,\eta) = k_1^*(x,t), \tag{20a}$$

$$k_2^*(\xi) = k_2^*(x), \tag{20b}$$

and, in which

$$k_1^*(x,t) = \int_0^\infty\left\{\frac{1}{\Delta(\omega,r)}\left(1+\frac{1}{m}\right)(1+\kappa_2)\left[4\frac{\omega}{h}Y2(\omega,r)+k(1+\kappa_1)Y3(\omega,r)\right]Y5(\omega,r)-1\right\}$$
$$*\left\{\sin\left[\frac{\omega}{h}(t+x)\right]+\sin\left[\frac{\omega}{h}(t-x)\right]\right\}d\omega, \tag{21a}$$

$$k_2^*(x) = \int_0^\infty\frac{f_i(\omega,a/h)}{\Delta(\omega,r)}e^{-\omega}e^{-\omega r}(1+\kappa_2)Y1(\omega,r)\left[4\frac{\omega}{h}Y2(\omega,r)\right.$$
$$\left.+\,k(1+\kappa_1)Y3(\omega,r)\right]\cos\left(\omega\frac{x}{h}\right)d\omega. \tag{21b}$$

$Y5(\omega, r)$ in Eq. (21a) may be obtained by substituting $\omega = \alpha h$ and $r = h_1/h$ into Eq. (7c).

To insure smooth contact at the end points of the separation area, let

$$g(\eta) = G(\eta)(1-\eta^2)^{1/2}, \qquad (-1<\eta<1), \tag{22}$$

where $G(\eta)$ is a bounded function. Using the appropriate Gauss-Chebyshev integration formula (Erdogan and Gupta 1972), Eqs. (19a) and (19b) become

$$\sum_{k=1}^{n}\frac{1-\eta_k^2}{n+1}\left\{\frac{1}{(1+m)}\left[\frac{1}{\eta_k-\xi_j}+\frac{1}{\eta_k+\xi_j+2\frac{c+b}{c-b}}+\frac{c-b}{2h}k_1^*(\xi_j,\eta_k)\right]G(\eta_k)\right\}$$

$$= \frac{4}{\pi}\lambda k_2^*(\xi_j)+1, \qquad (j=1,\ldots,n+1), \tag{23a}$$

$$\sum_{k=1}^{n}\frac{1-\eta_k^2}{n+1}G(\eta_k) = 0, \tag{23b}$$

where

$$\eta_k = \cos\left(\frac{k\pi}{n+1}\right), \qquad (k=1,\ldots,n), \tag{24a}$$

$$\xi_j = \cos\left(\frac{\pi}{2}\,\frac{2j-1}{n+1}\right), \qquad (j=1,\ldots,n+1). \tag{24b}$$

Eq. (23) give $(n+2)$ equations to determine the $(n+2)$ unknowns $G(\eta_k)$, $(k=1, ..., n)$, $b$ and $c$. Note that the consistency condition of the integral equation such as (19c) is automatically satisfied since the Gauss-Chebyshev integration formula is used (Erdogan and Gupta 1972). The equations are linear $G(\eta_k)$ in but highly nonlinear in $b$ and $c$. Therefore, an interpolation and iteration scheme had to be used to obtain these two unknowns.

It should be noted that Eq. (15) gives the stress $\sigma_y(x, h_1)$ outside as well as inside the separation region $(b, c)$. Thus, once the functions $G(\eta_k)$, and the constants $b$ and $c$ are determined, the contact stress may be easily evaluated for the discontinuous contact case.

Another quantity of some practical interest may be the displacement component in the separation zone $(b, c)$. The separation between two elastic layers may be expressed as Eq. (13b) or

$$\bar{v}(x,h_1) = v_2(x,h_1)-v_1(x,h_1) = \int_b^x \varphi(t)dt, \qquad (b<x<c). \tag{25}$$

From Eq. (18), Eq. (25) may be written as,

$$\frac{4\mu_1}{\rho_2 g h_2}\frac{\bar{v}(x,h_1)}{(1+\kappa_1)} = \frac{c-b}{2h}\int_{-1}^{\xi} g(\eta)d\eta, \qquad (-1<\xi<1). \tag{26}$$

Using appropriate Gauss-Chebyshev integration formula, the following expression may be written for the separation:

$$\frac{4\mu_1}{\rho_2 g h_2}\frac{\bar{v}(x_i,h_1)}{(1+\kappa_1)\pi} = \frac{c-b}{2h}\sum_{k=1}^{i-1}\frac{1-\eta_k^2}{n+1}G(\eta_k), \qquad (i=2,\ldots,n+1) \tag{27}$$

where $\eta_k$ is given by Eq. (24a).

## 5. Numerical results

Some of calculated results obtained from the solution of the continuous and discontinuous contact problems described in the previous sections for various dimensionless quantities such as $a/h$, $\mu_2/\mu_1$, $h_1/h$, $\lambda$ and $k = k_0/\mu_1$ are shown in Figs. 2-8 and Tables 1-2.

Fig. 2 shows the variations of the critical load factor $\lambda_{cr}$ and the separation initiation distance $x_{cr}$ with the stiffness of the Winkler foundation, $k = k_0/\mu_1$. As it can be seen in the Fig., both the critical load factor and the separation initiation distance goes to infinity as $k$ approaches 0. That $k$ approaches 0 is that the stiffness of the foundation goes to zero, but this is impossible physically. As $k$ increases, that is the stiffness of the foundation increases, both the critical load factor and the separation initiation distance decrease, and they approaches a constant value after the determined value of $k$. These results can be also seen in Table 1 which shows the variations of the critical load factor and the separation initiation distance with $k$ for various values of $\mu_2/\mu_1$. Table 1 also shows that the critical load factor decreases, but the separation initiation distance increases with increasing $\mu_2/\mu_1$.

For fixed values of $a/h$, $h_1/h$ and $k$, the variation of the critical load factor $\lambda_{cr}$ with $\mu_2/\mu_1$ is shown in Fig. 3 for various loading cases. As it can be seen in the graphic, the critical load factor is the biggest in the case of $\mu_2/\mu_1 = 0.20$ for each loading cases and it is always the smallest in the loading case 1 independently $\mu_2/\mu_1$. Fig. 4 shows the variation of the contact stress distribution $\sigma_y(x, h_1)$ with various loading cases along the interface for the case of the continuous contact ($\lambda = \lambda_{cr}$) described in Section 3. It appears that both the critical load factor and the separation initiation distance are different for each loading cases in spite of the same $a/h$. While the smallest value of $\lambda_{cr}$

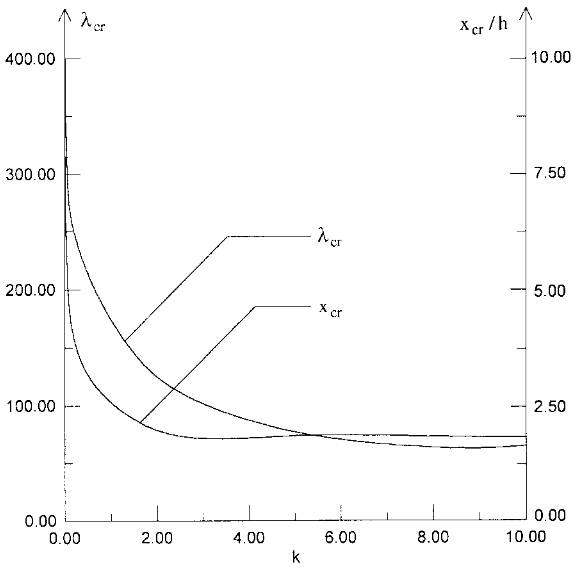


Fig. 2 The critical load factor $\lambda_{cr}$ and the separation initiation distance $x_{cr}$ as functions of $k(a/h = 1.00,\ \mu_2/\mu_1 = 1.740,\ h_1/h = 0.70,\ \nu_1 = \nu_2 = 0.34,\ p(x) = p_0)$

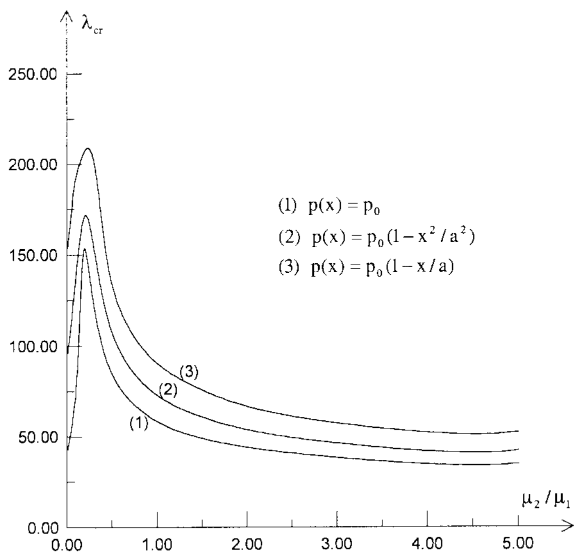


Fig. 3 Variation of the critical load factor with $\mu_2/\mu_1$ for various loading cases, $p(x)$ $(a/h = 2.00,\ k = 0.50,\ h_1/h = 0.50,\ \nu_1 = \nu_2 = 0.30)$

occurs in the loading case 1, $p(x) = p_0$ = constant, the smallest value of $x_{cr}$ occurs in the loading case 3, $p(x) = p_0(1 - x/a)$.

In Figs. 5-6, the normalized contact stress distributions $\sigma_y(x, h_1)$ at the interface of two elastic layers are given for the case of the continuous and discontinuous contact described in Sections 3

Table 1 Variations of the critical load factor $\lambda_{cr}$ and the separation initiation distance $x_{cr}$ with $k = k_0/\mu_1$ for various values of $\mu_2/\mu_1$ ($a/h = 1.00$, $p(x) = p_0$, $h_1/h = 0.50$, $\nu_1 = \nu_2 = 0.34$)

| $k$ | $\mu_2/\mu_1 = 0.575$ | | $\mu_2/\mu_1 = 1.000$ | | $\mu_2/\mu_1 = 1.740$ | |
|---|---|---|---|---|---|---|
| $\Downarrow$ | $x_{cr}/h$ | $\lambda_{cr}$ | $x_{cr}/h$ | $\lambda_{cr}$ | $x_{cr}/h$ | $\lambda_{cr}$ |
| 0.01 | 6.7840 | 136.0002 | 7.1842 | 106.1545 | 7.7506 | 90.1967 |
| 0.05 | 4.6646 | 102.9774 | 4.9258 | 79.2366 | 5.2968 | 66.3644 |
| 0.10 | 4.0054 | 93.7405 | 4.2210 | 71.5691 | 4.5282 | 59.4143 |
| 0.50 | 2.9086 | 81.8042 | 3.0530 | 61.3516 | 3.2510 | 49.7341 |
| 1.00 | 2.5474 | 76.8183 | 2.7010 | 58.8175 | 2.8770 | 47.6897 |
| 2.00 | 2.2408 | 67.3158 | 2.4182 | 55.4055 | 2.5872 | 46.0245 |
| 5.00 | 2.0174 | 54.7155 | 2.1700 | 49.7007 | 2.3254 | 43.6516 |
| 10.00 | 1.9418 | 49.3543 | 2.0684 | 46.3911 | 2.2088 | 42.0911 |
| 50.00 | 1.8770 | 44.8567 | 1.9770 | 43.1235 | 2.0972 | 40.3418 |
| 100.00 | 1.8682 | 44.2993 | 1.9646 | 42.6894 | 2.0818 | 40.0948 |
| $\rightarrow\infty$ | 1.8594 | 43.7491 | 1.9520 | 42.2557 | 2.0658 | 39.8456 |

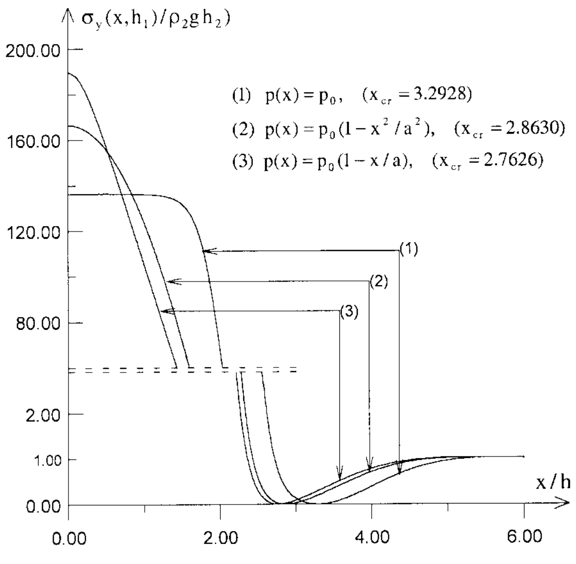


Fig. 4 Contact stress distribution between two elastic layers for the case of continuous contact ($a/h = 2.00$, $k = 1.25$, $h_1/h = 0.60$, $\mu_2/\mu_1 = 0.75$, $\nu_1 = \nu_2 = 0.30$)

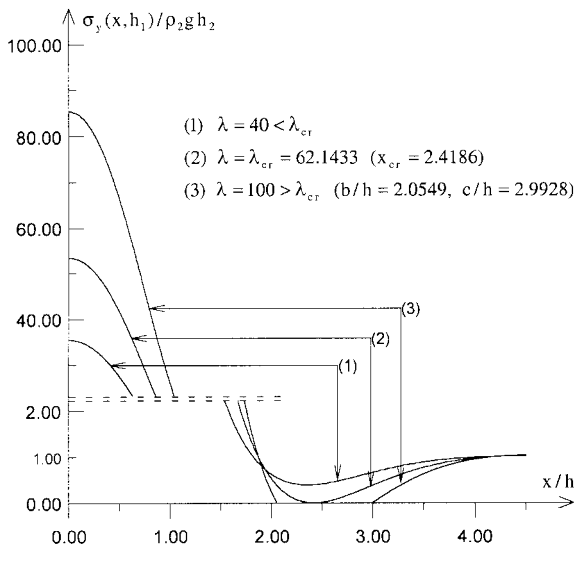


Fig. 5 Contact stress distributions for the case of continuous ($\lambda \le \lambda_{cr}$) and discontinuous ($\lambda > \lambda_{cr}$) contact along the interface ($a/h = 1.00$, $k = 1.00$, $h_1/h = 0.40$, $\mu_2/\mu_1 = 0.575$, $p(x) = p_0(1 - x^2/a^2)$, $\nu_1 = \nu_2 = 0.30$)

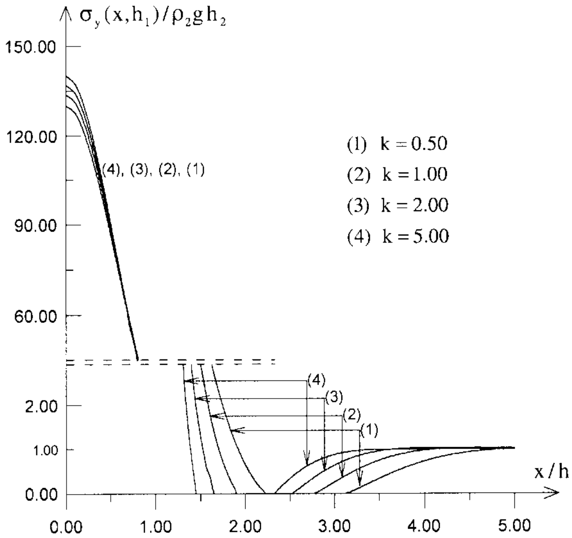


Fig. 6 Contact stress distribution between two elastic layers for the case of discontinuous contact ($a/h = 1.00$, $p(x) = p_0(1 - x/a)$, $h_1/h = 0.50$, $\mu_2/\mu_1 = 0.575$, $\lambda = 175 > \lambda_{cr}$, $\nu_1 = \nu_2 = 0.30$)

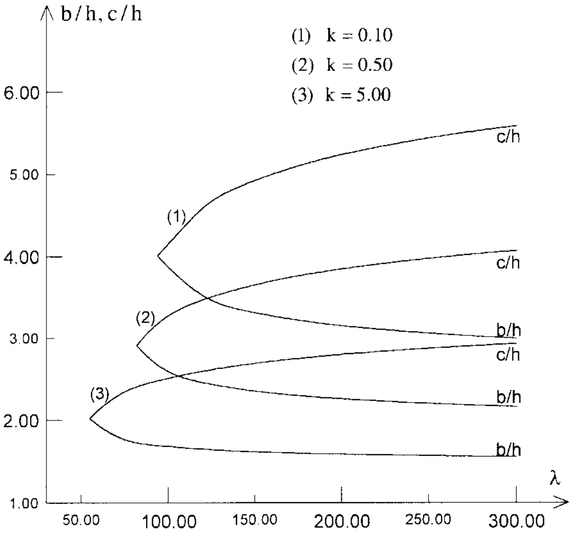


Fig. 7 Separation distances $b$ and $c$ along the interface as a function of the load factor $\lambda$ for various values of $k$ ($a/h = 1.00$, $\mu_2/\mu_1 = 0.575$, $h_1/h = 0.50$, $p(x) = p_0$, $\nu_1 = \nu_2 = 0.34$)

and 4. As it can be seen in graphics, there are three regions in the discontinuous contact along the interface. These are the continuous contact region, separation zone, and also the continuous contact region where the effect of the external load decreases and disappears infinitely. In Figs. 4-6, different scales have been used in order to include the entire pressure distribution and to give sufficient details in compact forms. Based on the numerical values obtained, it can be observed that, for each loading case and for the same $a/h$, the initial separation point occurs at the a longer distance from the origin with decreasing $k$. Similarly, the longer value of $a/h$, the longer the distance of initial separation point from the origin. These results are in agreement with the study in (Cakiroglu and Cakiroglu 1991).

Fig. 7 and Table 2 show results giving the distances $b$ and $c$ which define the separation zone along the interface (see Fig. 1). It appears in Fig. 7 that, as the stiffness of the foundation, $k = k_0/\mu_1$, increases, the separation zone, $(c - b)/h$, seems to decrease and $b/h$ approaches a constant asymptotic value with increasing load factor $\lambda$ for each value of $k$. Sharp points in Fig. 7 are corresponding to the initial separation loads and the initial separation points. In addition, it can be seen in Table 2 that for fixed $a/h$, the separation zone decreases and the initial separation point occurs at a shorter distance from the origin as $h_1/h$ increases.

Some sample results calculated from Eq. (27) giving the displacement $\bar{v}(x, h_1)$ in the separation zone $b < x < c$, $y = h_1$, are shown in Fig. 8 as a function of $x$ for various values of $\lambda$. As expected, the separation zone and the separation displacement $\bar{v}(x, h_1)$ increase with increasing load factor $\lambda$.

## 6. Conclusions

Based on the numerical results obtained in the previous section, it is observed that the stiffness of

Table 2 Variations of the separation distances $b$ and $c$ with $h_1/h$ along the interface ($a/h = 1.00$, $p(x) = p_0(1 - x/a)$, $k = 5.0$, $\lambda = 150 > \lambda_{cr}$, $\mu_2/\mu_1 = 0.575$, $\nu_1 = \nu_2 = 0.3$)

| $h_1/h$ | $b/h$ | $c/h$ | $(c-b)/h$ |
|---|---|---|---|
| 0.30 | 1.5525 | 2.6712 | 1.1187 |
| 0.40 | 1.4919 | 2.4794 | 0.9875 |
| 0.50 | 1.4366 | 2.2312 | 0.7946 |
| 0.60 | 1.3985 | 1.8460 | 0.4475 |

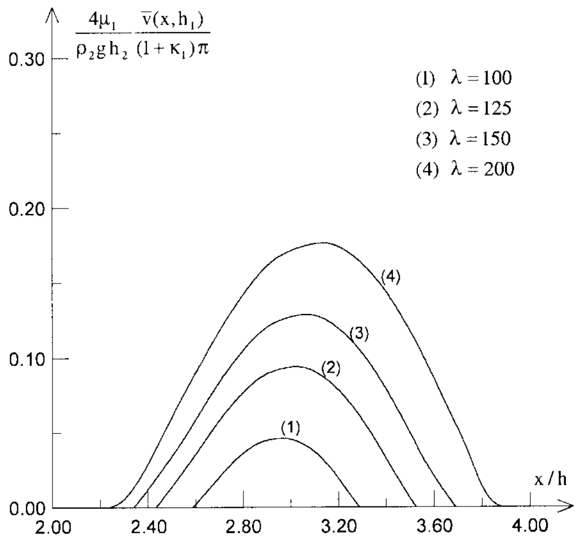


Fig. 8 Separation displacement $\bar{v}(x, h_1)$ between two elastic layers as a function of $x$ for various values of the load factor $\lambda$ ($a/h = 1.00$, $p(x) = p_0$, $k = 0.50$, $h_1/h = 0.50$, $\mu_2/\mu_1 = 0.575$, $\nu_1 = \nu_2 = 0.34$)

the Winkler foundation, the elastic properties and thickness of the layers and the loading cases play a very important role in the formation of the continuous and discontinuous contact area, the stress distribution on the contact surface, the critical load factor, the separation initiation distance, the separation zone and the separation displacement. From this study the following conclusions may be written:

- Both the critical load factor and the separation initiation distance decrease as the stiffness of the foundation increases and they approaches a constant value after the determined value of the stiffness.
- The critical load factor decreases but the separation initiation distance increases with increasing $\mu_2/\mu_1$.
- There are three regions in the discontinuous contact between two elastic layers. These are the continuous contact region, the separation zone and also the continuous contact region where the effect of the external load decreases and disappears infinitely.
- For each loading cases and the same $a/h$, the initial separation point occurs a longer distance from the origin with decreasing the stiffness of the foundation.
- The longer value of $a/h$, the longer the distance of the initial separation point from the origin.

- As the stiffness of the foundation increases, the separation zone decreases and the initial separation point approaches a constant asymptotic value with increasing load factor for fixed stiffness of the foundation.
- For fixed $a/h$, the separation zone decreases and the initial separation point occurs at a shorter distance from the origin as $h_1/h$ increases.
- As the load factor increases, both the separation zone and the separation displacement increases.

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