Structural Engineering and Mechanics, Vol. 14, No. 6 (2002) 679-698
DOI: http://dx.doi.org/10.12989/sem.2002.14.6.679

Comparative study on dynamic analyses of non-classically damped linear systems

Annalisa Greco†

Dipartimento di Ingegneria Civile ed Ambientale, University of Catania, Viale Andrea Doria 6, 95125 Catania, Italy

Adolfo Santini‡

Dipartimento di Meccanica e Materiali, University of Reggio Calabria, Feo di Vito, 89060 Reggio Calabria, Italy

(Received November 1, 2001, Accepted July 15, 2002)

Abstract. In this paper some techniques for the dynamic analysis of non-classically damped linear systems are reviewed and compared. All these methods are based on a transformation of the governing equations using a basis of complex or real vectors. Complex and real vector bases are presented and compared. The complex vector basis is represented by the eigenvectors of the complex eigenproblem obtained considering the non-classical damping matrix of the system. The real vector basis is a set of Ritz vectors derived either as the undamped normal modes of vibration of the system, or by the load dependent vector algorithm (Lanczos vectors). In this latter case the vector basis includes the static correction concept. The rate of convergence of these bases, with reference to a parametric structural system subjected to a fixed spatial distribution of forces, is evaluated. To this aim two error norms are considered, the first based on the spatial distribution of the load and the second on the shear force at the base due to impulsive loading. It is shown that both error norms point out that the rate of convergence is strongly influenced by the spatial distribution of the applied forces.

Key words: dynamic response; non-classical damping; Ritz method; Lanczos vectors; complex modal analysis.

1. Introduction

The evaluation of the dynamic response of practical engineering structures subjected to any type of loading, including earthquake excitation, requires the solution of a large number of coupled differential dynamic equilibrium equations (Clough and Penzien 1993, Chopra 1995).

The direct integration approach is numerically efficient only for short duration loads which excite a large number of natural frequencies. For long duration loads, like in an earthquake, the standard mode superposition analysis represents, for linear systems, the most appropriate numerical technique in order to reduce the computational effort. According to this method, the dynamic response is

† Assistant Professor
‡ Associate Professor

expressed as a linear combination of real undamped natural modes of vibration and the equations of motion are decoupled by transformation in terms of these modal co-ordinates. Furthermore a sufficient degree of approximation can be obtained considering only the first few modal contributions, but the accuracy of the dynamic response can be improved by using the procedures of static correction or mode acceleration (Clough and Penzien 1993, Chopra 1995, Cornwell *et al.* 1983, Hansteen and Bell 1979) which account for the contribution of truncated modes.

However it has not been proved that the real undamped exact eigenvectors provide the best basis for reducing the size of the problem. The Ritz method, in fact, has been widely used as an alternative approach for the dynamic analysis of engineering structures (Wilson *et al.* 1982). This procedure leads to an approximate solution of the equations of motion in which the displacements are expressed as a linear combination of admissible shape vectors, called Ritz vectors, which must be linearly independent and must satisfy the geometric boundary conditions. Moreover, when used in conjunction with the Rayleigh-Ritz method, the Ritz vectors provide a good approximation of the first natural modes of vibration of the system, which can be conveniently used for decoupling the equations of motion without solving the eigenvalue problem of the original structure.

Obviously the accuracy of the results is strongly influenced by the numerical technique used to generate the set of Ritz vectors. The load dependent vector algorithm (Clough and Penzien 1993, Chopra 1995, Nour-Humid and Clough 1984, Wilson *et al.* 1982) leads to particular Ritz vectors often called Lanczos vectors. These turn out to be very effective because the starting vector of the sequence, obtained as the static displacement due to the applied forces, includes the static correction effect, while the subsequent vectors account for the inertial effects on the dynamic response.

It is worth noticing that both exact real eigenvectors and Rayleigh-Ritz approximate vectors can rigorously decouple the equations of motions only when damping is of the form specified by Caughey and O'Kelly (1965). This physically means that the energy loss mechanisms are almost homogeneous throughout the structure. Systems satisfying this condition are said to be classically damped and the mode superposition method for such systems is referred to as standard modal analysis. In this case the damping properties can be specified in terms of modal damping ratios, thus avoiding the construction of the damping matrix. However a classical damping matrix can be easily derived in the form of Rayleigh damping (Clough and Penzien 1993, Chopra 1995), Caughey damping (Caughey 1960, Chopra 1995), or by superposition of modal damping matrices (Clough and Penzien 1993, Chopra 1995, Wilson and Penzien 1972).

Nevertheless there are structural systems for which the standard mode superposition analysis cannot be applied in the above-mentioned form. These are structures with non-uniform damping properties, such as soil-structure interacting systems or base-isolated ones, which are referred to as non-classically damped. For these structures the equations of motion cannot be decoupled by using the real undamped modes of vibration or the Rayleigh-Ritz vectors, because the generalised damping matrix presents off-diagonal terms.

The advantages of decoupling the equations of motion may be retained by using the complex mode superposition method. This procedure was firstly developed by Foss (1958) and requires the solution of an unsymmetric, unbounded eigenvalue-eigenvector problem, the dimensions of which are twice the number of degrees of freedom of the system. Furthermore the eigensolutions, i.e., the natural frequencies and the damped modes of vibration, are complex and their orthogonality properties look very different from the corresponding ones of classically damped systems. Owing to the numerical difficulties in the evaluation of the complex frequencies and modes of vibration and also to the lack of physical understanding of the elements of the solution, this method has been rarely used in structural engineering practice in the form due to Foss. Subsequently Veletsos and Ventura (1986) have reformulated this procedure for discrete systems, simplifying its implementation and clarifying the physical meaning of all the elements of the dynamic response.

Although the complex modal superposition method is well established, at least in the case of discrete systems, it remains inherently more involved than the classical one. For these reasons several studies, aiming to perform approximately the dynamic analysis of such systems, have been carried out.

Some of these studies are concerned with a modal superposition technique based on the use of undamped modes of vibration. The approximate dynamic response can then be obtained simply by neglecting the off-diagonal terms of the generalised damping matrix and adjusting the modal damping ratios in order to account for the modal damping coupling, as suggested by Roesset *et al.* (1973), Bielak (1976) and Cronin (1976). However, the errors induced by this diagonalisation procedure are difficult to estimate, especially in the case of closely spaced natural frequencies. To this end several criteria have been developed, such as those presented by Hasselman (1976), Warburton and Soni (1977), Shahruz and Ma (1988), Xu and Igusa (1991) and Hwang and Ma (1993). Moreover Prater and Singh (1986) and Nair and Singh (1986) have worked out some numerical indices to determine quantitatively the extent of non-classical damping in discrete systems and the errors induced by the decoupling approximation.

A different technique has been proposed by Clough and Mojtahedi (1976) for discrete systems and subsequently extended by Warburton (1978) to continuous ones. Firstly the equations of motion are transformed in terms of undamped modal co-ordinates. Then, assuming that only the first few modal contributions are significant, the dynamic response is obtained by direct integration of a truncated set of these coupled equations. However Duncan and Eatock Taylor (1979) have shown that, for non-classically damped systems, coupling may occur between the first modes and a much higher one. For this reason accuracy in the results can be ensured only including in the analysis a much larger number of modal contributions than it is usually done for classically damped structures.

Of course, the method of Clough and Mojtahedi (1976) can also be applied using a Ritz vector basis, even if they are orthogonal only with respect to the mass matrix of the system. In this case the transformed equations of motion are coupled by the off-diagonal terms of generalised damping and stiffness matrices. More recently an alternative approach has been presented by Ibrahimbegovic *et al.* (1990), which is based on the use of a complex Ritz vector basis, also derived by the load depended vector algorithm.

The purpose of this paper is to compare the results of some techniques for the dynamic analysis of a non-classically damped linear system. These are the complex modal analysis and other methods based on the direct integration of a truncated set of transformed-coupled equations of motion using a Ritz vector basis.

In the following these methods will be firstly reviewed. Subsequently, with reference to a parametric non-classically damped structural linear system, the rate of convergence of the considered techniques will be shown as a function of the number of co-ordinates required to achieve the same accuracy. In particular the effects of the amount of the overall damping and of the extent of non-classical damping will be clearly outlined. Two error norms will be considered in order to obtain an estimate of the accuracy of the dynamic response analysis. The first is based on the spatial distribution of the applied forces, while the second on the shear force at the base due to impulsive loading. It will be shown that, whatever norm is considered, the rate of convergence is strongly influenced by the spatial distribution of the applied forces.

2. The Ritz method

For a discrete system, having N degrees of freedom, the equations of motion in terms of nodal displacements are expressed as

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{C}\dot{\boldsymbol{u}}(t) + \boldsymbol{K}\boldsymbol{u}(t) = \boldsymbol{f}(t) \tag{1}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are the $N \times N$ mass, damping and stiffness matrices, $\boldsymbol{f}(t)$ is the $N \times 1$ loading vector and $\boldsymbol{u}(t)$ is the $N \times 1$ nodal displacement vector which describes the dynamic response of the structure. It is worth noticing that, for complicated structural systems, N can be very large. In the case of linear systems, the Ritz method (Clough and Penzien 1993, Chopra 1995, Wilson *et al.* 1982, Nour-Humid and Clough 1984) can be employed in order to reduce the number of unknowns, leading to an approximate solution. The basic assumption is that the displacement vector $\boldsymbol{u}(t)$ can be expressed as a linear combination of a small set of assumed shapes $\boldsymbol{\varphi}_i$ of amplitude $y_i(t)$ as follows

$$\boldsymbol{u}(t) = \sum_{i=1}^{n} \boldsymbol{\varphi}_i y_i(t) = \boldsymbol{\Phi}\boldsymbol{y}(t) \tag{2}$$

where $n < N$. The Ritz vectors $\boldsymbol{\varphi}_i$, which are the columns of the $N \times n$ matrix $\boldsymbol{\Phi}$, must be linearly independent vectors satisfying the geometric boundary conditions, while $\boldsymbol{y}(t)$ is the vector of the n generalised co-ordinates $y_i(t)$ representing the approximate solution. It can be noticed that Eq. (2) carries out the projection of the original space of nodal co-ordinates $\boldsymbol{u}(t)$ onto the subspace of generalised co-ordinates $\boldsymbol{y}(t)$. Therefore $\boldsymbol{\varphi}_i$ $(i = 1, ..., n)$ can be interpreted as the set of projection vectors, that is the subspace vector basis.

The introduction of co-ordinate transformation (2) into Eq. (1) leads to a new set of differential equations defined on the subspace

$$\hat{\boldsymbol{M}}\ddot{\boldsymbol{y}}(t) + \hat{\boldsymbol{C}}\dot{\boldsymbol{y}}(t) + \hat{\boldsymbol{K}}\boldsymbol{y}(t) = \hat{\boldsymbol{f}}(t), \tag{3}$$

where

$$\hat{\boldsymbol{M}} = \boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{\Phi}, \quad \hat{\boldsymbol{C}} = \boldsymbol{\Phi}^T\boldsymbol{C}\boldsymbol{\Phi}, \quad \hat{\boldsymbol{K}} = \boldsymbol{\Phi}^T\boldsymbol{K}\boldsymbol{\Phi}, \quad \hat{\boldsymbol{f}}(t) = \boldsymbol{\Phi}^T\boldsymbol{f}(t) \tag{4}$$

are respectively the $n \times n$ mass, damping and stiffness generalised matrices, usually fully populated, and the $n \times 1$ generalised load vector.

Being $n < N$, the main advantage of this co-ordinate transformation is the drastic reduction in the number of equations to be solved. Obviously the accuracy of the results depends on the choice of the set of Ritz vectors.

2.1 The standard mode superposition method

If the real undamped modes of vibration of the system, obtained by solving the original eigenvalue-eigenvector problem

$$(\boldsymbol{K} - \omega^2\boldsymbol{M})\boldsymbol{\varphi} = \boldsymbol{0}, \tag{5}$$

are employed in the co-ordinate transformation (2), the standard mode superposition method is recovered. In this case, it is well known that in the Eq. (3) the mass and stiffness generalised matrices, $\hat{\boldsymbol{M}}$ and $\hat{\boldsymbol{K}}$, are diagonal, while the damping generalised matrix, $\hat{\boldsymbol{C}}$, usually presents off-diagonal terms.

However, as shown by Caughey and O'Kelly (1965), if the damping matrix $\boldsymbol{C}$ satisfies the identity

$$\boldsymbol{C}\boldsymbol{M}^{-1}\boldsymbol{K} = \boldsymbol{K}\boldsymbol{M}^{-1}\boldsymbol{C}, \tag{6}$$

the generalised damping matrix $\hat{\boldsymbol{C}}$ is also diagonal and the system of Eq. (3) is uncoupled in modal co-ordinates as follows

$$\ddot{y}_i + 2\xi_i\omega_i\dot{y}_i + \omega_i^2 y_i = f_i(t), \qquad i = 1, 2, \ldots, n. \tag{7}$$

Structural systems which satisfy Eq. (6) are said to be classically damped and this physically means that damping mechanisms are uniformly distributed along the structure.

2.2 The load dependent vector algorithm

A very effective set of Ritz vectors, often called Lanczos vectors, can be derived by the load dependent vector algorithm (Clough and Penzien 1993, Chopra 1995, Wilson *et al.* 1982, Nour-Humid and Clough 1984). Compared to the standard mode superposition method, this procedure presents two main advantages: (i) the eigenvalue-eigenvector problem (5) needs not to be solved, (ii) the rate of convergence to the exact solution is increased because, as it will be shown later, the static correction method is automatically included.

The load dependent Ritz vectors are generated starting from the equations of motion (1) without taking into account the dissipative term, that is

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{K}\boldsymbol{u}(t) = \boldsymbol{f}(t). \tag{8}$$

Furthermore it is assumed that the loading term is of the form

$$\boldsymbol{f}(t) = \boldsymbol{p}f(t) \tag{9}$$

where the vector $\boldsymbol{p}$ represents the constant spatial distribution of forces and $f(t)$ is a dimensionless scalar function of time t which defines the time dependence of all forces. It is important to note that load vectors in the form of Eq. (9) are appropriate for many important applications, e.g., earthquake excitations.

The first vector of the sequence, $\boldsymbol{q}_1$, is the deflected shape due to the application of the constant load distribution $\boldsymbol{p}$ and is obtained as the solution of the static equilibrium equation

$$\boldsymbol{K}\boldsymbol{q}_1 = \boldsymbol{p}. \tag{10}$$

Subsequently this vector is scaled by the normalising factor

$$\beta_1 = \sqrt{\boldsymbol{q}_1^T\boldsymbol{M}\boldsymbol{q}_1} \tag{11}$$

in order to obtain the vector

$$\boldsymbol{\varphi}_1 = \frac{1}{\beta_1}\boldsymbol{q}_1 \tag{12}$$

which provides a unit generalised mass, that is

$$\boldsymbol{\varphi}_1^T\boldsymbol{M}\boldsymbol{\varphi}_1 = 1. \tag{13}$$

It is worth noticing that, because the vector $\boldsymbol{\varphi}_1$ corresponds to the static displaced shape, the static correction procedure is automatically taken into account.

The other vectors of the sequence account for the inertial effects and are generated using the dynamic matrix $\boldsymbol{K}^{-1}\boldsymbol{M}$. Assuming that the first j Lanczos vectors have been determined, the vector $\boldsymbol{q}_{j+1}$ is obtained as the deflected shape due to the inertial load $\boldsymbol{M}\boldsymbol{\varphi}_j$ through the equation

$$\boldsymbol{q}_{j+1} = \boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{\varphi}_j. \tag{14}$$

This vector contains components from each of the preceding vectors and needs to be purified. To this end the Grahm-Schmidt procedure is used, which makes the new vector $\boldsymbol{M}$-orthogonal to all the previous vectors. Denoting the purified vector as $\tilde{\boldsymbol{q}}_{j+1}$, it follows that

$$\tilde{\boldsymbol{q}}_{j+1} = \boldsymbol{q}_{j+1} - \alpha_j\boldsymbol{\varphi}_j - \beta_j\boldsymbol{\varphi}_{j-1} - \gamma_j\boldsymbol{\varphi}_{j-2} - \delta_j\boldsymbol{\varphi}_{j-3} - \dots \tag{15}$$

where, considering the $\boldsymbol{M}$-orthogonality of the Lanczos vectors, the first two coefficients, α_j, β_j, take the form

$$\alpha_j = \boldsymbol{\varphi}_j^T\boldsymbol{M}\boldsymbol{q}_{j+1}, \quad \beta_j = \boldsymbol{\varphi}_{j-1}^T\boldsymbol{M}\boldsymbol{q}_{j+1}, \tag{16}$$

while all the others vanish (Clough and Penzien 1993, Nour-Humid and Clough 1984). Therefore, being $\gamma_j = \delta_j = \dots = 0$, the Grahm-Schmidt procedure has to be applied only to the previous two vectors. Moreover β_j turns out to be equal to the preceding normalising factor, i.e.,

$$\beta_j = \sqrt{\tilde{\boldsymbol{q}}_j^T\boldsymbol{M}\tilde{\boldsymbol{q}}_j} \tag{17}$$

Afterwards the vector $\boldsymbol{\varphi}_{j+1}$, orthonormal with respect to the mass matrix $\boldsymbol{M}$, is obtained normalising $\tilde{\boldsymbol{q}}_{j+1}$ as follows

$$\boldsymbol{\varphi}_{j+1} = \frac{1}{\beta_{j+1}}\tilde{\boldsymbol{q}}_{j+1} \tag{18}$$

where

$$\beta_{j+1} = \sqrt{\tilde{\boldsymbol{q}}_{j+1}^T\boldsymbol{M}\tilde{\boldsymbol{q}}_{j+1}}. \tag{19}$$

By this procedure any desired number of vectors can be obtained. Nevertheless, because the orthogonality is ensured with only the two preceding vectors at each step of the sequence, loss of orthogonality with respect to earlier vectors can occur due to round-off errors. When such errors reach a critical size, they must be corrected imposing the orthogonality with respect to all preceding vectors. Further details can be found in the literature.

2.3 Transformed equations of motion

The equations of motion transformed in terms of generalised co-ordinates by using the load dependent Ritz vectors can be written as

$$\boldsymbol{I}\ddot{\boldsymbol{y}}(t) + \hat{\boldsymbol{C}}\dot{\boldsymbol{y}}(t) + \hat{\boldsymbol{K}}\boldsymbol{y}(t) = \hat{\boldsymbol{f}}(t), \tag{20}$$

where $\hat{\boldsymbol{M}} = \boldsymbol{I}$ due to the $\boldsymbol{M}$-orthogonality property of these vectors. Furthermore this property allows obtaining a set of transformed equations of motion which look slightly different from Eq. (20) and which may be, sometimes, more efficient from a computational point of view. In fact, premultiplying all the terms of Eq. (1) by $\boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{K}^{-1}$ and considering the co-ordinate transformation (2), the following equation is obtained

$$\boldsymbol{T}_n\ddot{\boldsymbol{y}} + \boldsymbol{C}_n\dot{\boldsymbol{y}} + \boldsymbol{I}_n\boldsymbol{y} = \boldsymbol{g}_n f(t) \tag{21}$$

where

$$\boldsymbol{T}_n = \boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{\Phi} = \begin{bmatrix} \alpha_1 & \beta_2 & 0 & \dots & 0 & 0 \\ \beta_2 & \alpha_2 & \beta_3 & \dots & 0 & 0 \\ 0 & \beta_3 & \alpha_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{n-1} & \beta_n \\ 0 & 0 & 0 & \dots & \beta_n & \alpha_n \end{bmatrix} \tag{22}$$

is the $n \times n$ three-diagonal matrix made up by the coefficients of Eq. (15), $\boldsymbol{C}_n = \boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{C}\boldsymbol{\Phi}$ is the $n \times n$ generalised damping matrix, $\boldsymbol{I}_n = \boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{\Phi}$ is the $n \times n$ unit matrix and $\boldsymbol{g}_n = \boldsymbol{\Phi}^T\boldsymbol{M}\boldsymbol{K}^{-1}\boldsymbol{p} = [\beta_1 \;\; 0 \;\; \dots \;\; 0]^T$ is the $n \times 1$ load vector, the first component of which is the only one different from zero.

The coupled transformed equations of motion, in the form (20) or (21), can be conveniently solved by a numerical time-stepping algorithm.

3. The complex mode superposition method

For non-classically damped systems the decoupling of the equations of motion (1) can be rigorously performed by means of the complex mode superposition method, which will be briefly described in this section.

3.1 Eigenvalues and eigenvectors

The natural frequencies and modes of vibration of the damped system can be evaluated solving the following linear eigenvalue-eigenvector problem of size $2N$

$$(\boldsymbol{B} + s\boldsymbol{A})\boldsymbol{z} = \boldsymbol{0} \tag{23}$$

where

$$\boldsymbol{A} = \left[\begin{array}{c|c} \boldsymbol{0} & \boldsymbol{M} \\ \hline \boldsymbol{M} & \boldsymbol{C} \end{array}\right], \quad \boldsymbol{B} = \left[\begin{array}{c|c} -\boldsymbol{M} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{K} \end{array}\right], \quad \boldsymbol{z} = \left[\begin{array}{c} s\boldsymbol{\psi} \\ \hline \boldsymbol{\psi} \end{array}\right]. \tag{24}$$

In Eqs. (23) and (24) s is the generic eigenvalue, $\boldsymbol{z}$ is the generic eigenvector, the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ describe the dynamic properties of the system, and $\boldsymbol{\psi}$ and $s\boldsymbol{\psi}$ represent the generic mode of vibration in terms of displacements and velocities of the degrees of freedom. Usually the eigenvalues and the eigenvectors of problem (23) occur in complex conjugate pairs, but for highly damped systems an even number of them can be real (Inman and Andry jr. 1980).

Moreover for each couple of eigenvectors corresponding to distinct eigenvalues, even if conjugate, the following orthogonality conditions hold (Veletsos and Ventura 1986)

$$(s_j + s_k)\boldsymbol{\psi}_j^T \boldsymbol{M}\boldsymbol{\psi}_k + \boldsymbol{\psi}_j^T \boldsymbol{C}\boldsymbol{\psi}_k = 0 \tag{25}$$

$$\boldsymbol{\psi}_j^T \boldsymbol{K}\boldsymbol{\psi}_k - s_j s_k \boldsymbol{\psi}_j^T \boldsymbol{M}\boldsymbol{\psi}_k = 0 \tag{26}$$

3.2 Modal impulsive response functions

For an impulsive loading term of the form

$$\boldsymbol{f}(t) = \boldsymbol{p}\delta(t) \tag{27}$$

where $\delta(t)$ is the Dirac delta function, the equations of motion can be written as

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{C}\dot{\boldsymbol{u}}(t) + \boldsymbol{K}\boldsymbol{u}(t) = \boldsymbol{p}\delta(t) \tag{28}$$

According to the standard mode superposition method, the displacement vector can be expressed as a linear combination of the natural modes of vibration as follows

$$\boldsymbol{u}(t) = \sum_{j=1}^{2N} \boldsymbol{\psi}_j q_j(t) \tag{29}$$

where $\boldsymbol{\psi}_j$ and $q_j(t)$ are, in general, complex. Owing to the impulsive nature of the excitation, the generic generalised co-ordinate can be assumed of the form

$$q_j(t) = B_j \exp\{s_j t\}. \tag{30}$$

Substituting Eq. (29) into Eq. (28) it follows that

$$\sum_{j=1}^{2N} [\ddot{q}_j(t)\boldsymbol{M}\boldsymbol{\psi}_j + \dot{q}_j(t)\boldsymbol{C}\boldsymbol{\psi}_j + q_j(t)\boldsymbol{K}\boldsymbol{\psi}_j] = \boldsymbol{p}\delta(t). \tag{31}$$

Considering the relationships between the generalised co-ordinates (30) and their derivatives, the Eq. (31) can be rewritten as

$$\sum_{j=1}^{2N} q_j(t)[s_j^2 \boldsymbol{M}\boldsymbol{\psi}_j + s_j \boldsymbol{C}\boldsymbol{\psi}_j + \boldsymbol{K}\boldsymbol{\psi}_j] = \boldsymbol{p}\delta(t). \tag{32}$$

Premultiplying by $\boldsymbol{\psi}_k^T$ both terms of Eq. (32) it follows that

$$\sum_{j=1}^{2N} q_j(t)[s_j^2 \boldsymbol{\psi}_k^T \boldsymbol{M}\boldsymbol{\psi}_j + s_j \boldsymbol{\psi}_k^T \boldsymbol{C}\boldsymbol{\psi}_j + \boldsymbol{\psi}_k^T \boldsymbol{K}\boldsymbol{\psi}_j] = \boldsymbol{\psi}_k^T \boldsymbol{p}\delta(t). \tag{33}$$

Taking into account the orthogonality condition (26), Eq. (33) can be written as

$$\sum_{j=1}^{2N} s_j q_j(t)[(s_j + s_k)\boldsymbol{\psi}_k^T \boldsymbol{M}\boldsymbol{\psi}_j + \boldsymbol{\psi}_k^T \boldsymbol{C}\boldsymbol{\psi}_j] = \boldsymbol{\psi}_k^T \boldsymbol{p}\delta(t). \tag{34}$$

Because the term in square brackets represents the orthogonality condition (25), only one term in the summation is different from zero, thus decoupling the equations of motion. These can be written in the form

$$2M_j\ddot{q}_j(t) + C_j\dot{q}_j(t) = L_j\delta(t) \qquad j = 1, 2, \ldots, 2N \tag{35}$$

where

$$M_j = \boldsymbol{\psi}_j^T\boldsymbol{M}\boldsymbol{\psi}_j; \qquad C_j = \boldsymbol{\psi}_j^T\boldsymbol{C}\boldsymbol{\psi}_j; \qquad L_j = \boldsymbol{\psi}_j^T\boldsymbol{p}. \tag{36}$$

Integrating the modal Eq. (35) in the interval $[0^-, \; 0^+]$ it follows that

$$2M_j\dot{q}_j(0^+) + C_jq_j(0^+) = L_j \tag{37}$$

from which the modal constant B_j can be derived

$$B_j = \frac{L_j}{2s_jM_j + C_j} \tag{38}$$

Therefore the modal impulsive response functions take the form

$$\boldsymbol{h}_j(t) = B_j\boldsymbol{\psi}_j\exp\{s_jt\}. \tag{39}$$

It is worth noticing that these functions are complex when the couple s_j and $\boldsymbol{\psi}_j$ is complex, while they are real in the other case. However, each complex modal impulsive response function has a correspondent conjugate one and, therefore, the total response is obviously real. In fact, adding each complex modal contribution to its conjugate, it follows that

$$\boldsymbol{h}_n^c(t) = 2\mathrm{Re}[B_n\boldsymbol{\psi}_n\exp\{s_nt\}] \qquad n = 1, 2, \ldots, N_c \tag{40}$$

where $N_c \le N$ is the number of complex conjugate pairs of eigenvalues and eigenvectors. Furthermore, introducing the following real vectors

$$\boldsymbol{b}_n = 2\mathrm{Re}[B_n\boldsymbol{\psi}_n], \quad \boldsymbol{c}_n = 2\mathrm{Im}[B_n\boldsymbol{\psi}_n], \quad \boldsymbol{a}_n = \xi_n\boldsymbol{b}_n - \boldsymbol{c}_n\sqrt{1-\xi_n^2}, \tag{41}$$

Eq. (40) can be written in terms of real algebra as follows

$$\boldsymbol{h}_n^c(t) = \boldsymbol{a}_n|\omega_n|h_n(t) + \boldsymbol{b}_n\dot{h}_n(t) \tag{42}$$

where

$$h_n(t) = \frac{1}{\omega_{Dn}}\exp\{-\xi_n|\omega_n|t\}\sin\omega_{Dn}t \tag{43}$$

represents the impulsive response function of a single degree of freedom viscous linear system having natural frequency $|\omega_n| = |s_n|$, damping ratio $\xi_n = -\mathrm{Im}[s_n]/|s_n|$ and damped frequency $\omega_{Dn} = |\omega_n|\sqrt{1-\xi_n^2}$ (Veletsos and Ventura 1986).

Even for real eigenvalues and eigenvectors the modal impulsive response functions can be written in the form of Eq. (42). To this aim it is worth noticing that these real eigenvalues, each associated with a real eigenvector, occur in an even number of negative values and, therefore, they can be grouped in N_r couples. Let s_j, $\boldsymbol{\psi}_j$ and s_k, $\boldsymbol{\psi}_k$ be a couple of such eigenvalues and eigenvectors with

$|s_k| > |s_j|$. For computational purposes it is convenient to express this pair of eigenvalues in the form

$$s_j = -\xi_j\omega_j + \tilde{\omega}_j, \qquad s_k = -\xi_j\omega_j - \tilde{\omega}_j \tag{44}$$

in which

$$\omega_j = \sqrt{s_j s_k}, \quad \xi_j = -\frac{s_j + s_k}{2\omega_j} = -\frac{s_j + s_k}{2\sqrt{s_j s_k}} > 1, \quad \tilde{\omega}_j = \omega_j\sqrt{\xi_j^2 - 1} = \frac{s_j - s_k}{2} \tag{45}$$

Introducing the real vectors

$$\boldsymbol{b}_j^r = B_k\boldsymbol{\psi}_k + B_j\boldsymbol{\psi}_j, \quad \boldsymbol{c}_j^r = B_k\boldsymbol{\psi}_k - B_j\boldsymbol{\psi}_j, \quad \boldsymbol{a}_j^r = \xi_j\boldsymbol{b}_j^r - \boldsymbol{c}_j^r\sqrt{\xi_j^2 - 1}, \tag{46}$$

where B_j and B_k are determined from Eq. (38), it follows that

$$\boldsymbol{h}_j^r(t) = \boldsymbol{a}_j^r\omega_j h_j^r(t) + \boldsymbol{b}_j^r\dot{h}_j^r(t) \tag{47}$$

where

$$h_j^r(t) = \frac{1}{\tilde{\omega}_j}\exp\{-\xi_j\omega_j t\}\sinh\tilde{\omega}_j t \tag{48}$$

represents the impulsive response function of a single degree of freedom overcritically-damped viscous linear system having natural frequency ω_j and damping ratio ξ_j, with $\tilde{\omega}_j = \omega_j\sqrt{\xi_j^2 - 1}$.

3.3 Response to an arbitrary force distribution

The dynamic response to an arbitrary force distribution

$$\boldsymbol{f}(t) = \boldsymbol{p}f(t) \tag{49}$$

can be obtained by means of the convolution integrals

$$\boldsymbol{u}(t) = \sum_{n=1}^{N_c}\int_0^t f(\tau)\boldsymbol{h}_n^c(t-\tau)d\tau + \sum_{j=1}^{N_r}\int_0^t f(\tau)\boldsymbol{h}_j^r(t-\tau)d\tau \tag{50}$$

which refer to the contribution of complex and real modes respectively. For computational purposes it is convenient to express the dynamic response in terms of response integrals as follows

$$\boldsymbol{u}(t) = \sum_{n=1}^{N_c}[\boldsymbol{a}_n|\omega_n|D_n(t) + \boldsymbol{b}_n\dot{D}_n(t)] + \sum_{j=1}^{N_r}[\boldsymbol{a}_j^r\omega_j D_j^r(t) + \boldsymbol{b}_j^r\dot{D}_j^r(t)] \tag{51}$$

where

$$D_n(t) = \int_0^t f(\tau)h_n(t-\tau)d\tau; \qquad \dot{D}_n(t) = \int_0^t f(\tau)\dot{h}_n(t-\tau)d\tau \tag{52}$$

$$D_j^r(t) = \int_0^t f(\tau)h_j^r(t-\tau)d\tau; \qquad \dot{D}_j^r(t) = \int_0^t f(\tau)\dot{h}_j^r(t-\tau)d\tau. \tag{53}$$

It is worth noticing that $D_n(t)$ and $\dot{D}_n(t)$ represent the response in terms of displacement and velocity of a single degree of freedom viscous linear system having natural frequency $|\omega_n|$ and damping ratio ξ_n excited by the force $f(t)$, while $D_j^r(t)$ and $\dot{D}_j^r(t)$ represent the response in terms of displacement and velocity of a single degree of freedom overcritically-damped viscous linear system having natural frequency ω_j and damping ratio ξ_j excited by the force $f(t)$.

4. Numerical applications

Some numerical applications have been conducted in order to compare the use of load dependent Ritz vectors with the use of real and complex exact eigenvectors in dynamic mode superposition analysis of non-classically damped systems.

4.1 The structural model

A planar N-storey shear-type building frame, shown in Fig. 1, has been considered. The mass of the first floor and the damping and stiffness coefficients of the first interstorey are denoted with m_1, c_1 and k_1 respectively. For the other storeys, the mass, damping and stiffness coefficients have been taken constant and are indicated with m, c and k.

For this system the mass, damping and stiffness matrices take the following form

$$\boldsymbol{M} = \begin{bmatrix} m_1 & & & \\ & m & & \\ & & \ddots & \\ & & & m \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} c_1 + c & -c & & \\ -c & 2c & -c & \\ & \cdots & \cdots & \cdots \\ & & -c & c \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} k_1 + k & -k & & \\ -k & 2k & -k & \\ & \cdots & \cdots & \cdots \\ & & -k & k \end{bmatrix}, \tag{54}$$

while the loading term will be assumed in the form of Eq. (9).

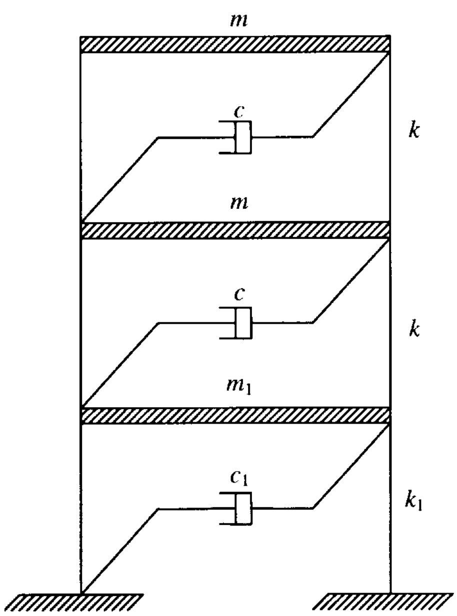

Fig. 1 The structural system

4.2 Dimensionless parameters

The dynamic behaviour of the system may be described in terms of few dimensionless parameters. These have been identified as follows

N	number of storeys
$\alpha = m_1/m$	floor mass ratio
$\beta = c_1/c$	interstorey damping ratio
$\gamma = k_1/k$	interstorey stiffness ratio
$\eta = c/\sqrt{km}$	damping parameter

It can be noted that the ratio $\delta = \beta/\gamma$ represents the extent of non-classical damping within the system. In fact, for $\delta = 1$ the damping matrix is proportional to the stiffness matrix and the system is classically damped. On the contrary, the more δ is lower or greater than one, the more the damping of the system is non-classical.

4.3 Error estimation

An appropriate number of Ritz vectors should be included in the dynamic analysis in order to ensure a sufficient degree of accuracy. Two error norms will be considered, which are based on the spatial distribution of forces and on the base shear respectively.

4.3.1 Spatial distribution of forces

Ritz vectors should represent accurately the vector $\boldsymbol{p}$ that defines the spatial distribution of forces. In the case of a real vector basis, because the Ritz vectors are $\boldsymbol{M}$-orthonormal, the vector $\boldsymbol{p}$ can be expanded as

$$\boldsymbol{p} = \sum_{i=1}^{N} \Gamma_i \boldsymbol{M} \boldsymbol{\psi}_i, \tag{55}$$

where

$$\Gamma_i = \frac{\boldsymbol{\psi}_i^T \boldsymbol{p}}{\boldsymbol{\psi}_i^T \boldsymbol{M} \boldsymbol{\psi}_i} = \boldsymbol{\psi}_i^T \boldsymbol{p} \tag{56}$$

is the modal participation factor, which is a measure of the degree of participation of the ***i***-th vector in the dynamic response. Therefore the error vector in the representation of the loading vector by a smaller number of Ritz vectors is given by

$$\boldsymbol{e}_n = \boldsymbol{p} - \sum_{i=1}^{n} \Gamma_i \boldsymbol{M} \boldsymbol{\psi}_i, \tag{57}$$

and an error norm can be defined as (Chopra 1995, Wilson *et al.* 1982)

$$e_n = \frac{\boldsymbol{p}^T \boldsymbol{e}_n}{\boldsymbol{p}^T \boldsymbol{p}}. \tag{58}$$

It can be noted that the error norm e_n is equal to zero when all N Ritz vectors are considered, while it is equal to one when no Ritz vectors are considered.

In the case of a complex vector basis the following expansion holds

$$\boldsymbol{g} = \sum_{i=1}^{2N} \lambda_i \boldsymbol{A} \boldsymbol{z}_i = 2\mathrm{Re}\left[\sum_{n=1}^{N_c} \lambda_n \boldsymbol{A} \boldsymbol{z}_n\right] + \sum_{j=1}^{N_r} \lambda_j \boldsymbol{A} \boldsymbol{z}_j, \tag{59}$$

where $\boldsymbol{g}^T = [\boldsymbol{0}^T \mid \boldsymbol{p}^T]$, $\boldsymbol{A}$ and $\boldsymbol{z}_i$ are given by expression (24), $N_c \le N$ is the number of complex conjugate pairs of eigenvalues and eigenvectors, $N_r = N - N_c$ is the number of pairs of real eigenvalues and eigenvectors and λ_i is the complex or real participation factor given by

$$\lambda_i = \frac{\boldsymbol{z}_i^T \boldsymbol{g}}{\boldsymbol{z}_i^T \boldsymbol{A} \boldsymbol{z}_i} = \frac{\boldsymbol{\psi}_i^T \boldsymbol{p}}{2 s_i \boldsymbol{\psi}_i^T \boldsymbol{M} \boldsymbol{\psi}_i + \boldsymbol{\psi}_i^T \boldsymbol{C} \boldsymbol{\psi}_i} = \frac{L_i}{2 s_i M_i + C_i}, \tag{60}$$

which coincides with the modal constant (38). The error vector can be written as

$$\boldsymbol{e}_n = \boldsymbol{p} - 2\mathrm{Re}\left[\sum_{n=1}^{n_c} \lambda_n (s_n \boldsymbol{M} \boldsymbol{\psi}_n + \boldsymbol{C} \boldsymbol{\psi}_n)\right] - \sum_{j=1}^{m_c} \lambda_j (s_j \boldsymbol{M} \boldsymbol{\psi}_j + \boldsymbol{C} \boldsymbol{\psi}_j) \tag{61}$$

where n_c and m_c are, respectively, the number of complex conjugate and real modal contributions included in the analysis. Even in this case the error norm can be expressed in the form of Eq. (58).

4.3.2 Base shear

The maximum shear force at the base is the most significant parameter in order to represent the overall dynamic stresses on the structure. For this reason a new error norm has been introduced taking into account the maximum shear force at the base due to the impulsive loading

$$\boldsymbol{f}(t) = \boldsymbol{p}\delta(t). \tag{62}$$

This error norm can be expressed in the following form

$$E_n = \frac{\left| V_{\max} - \sum_{i=1}^{n} V_{\max}^{(i)} \right|}{V_{\max}} \tag{63}$$

where $V_{\max}^{(i)}$ is the maximum base shear due to the contribution of the i-th Ritz vector, $V_{\max}$ is the maximum base shear due to the contributions of all Ritz vectors and n is the number of contributions included in the analysis.

4.4 Parametric analysis

A parametric analysis has been carried out in order to compare the rate of convergence of the previously described procedures with respect to the overall amount of damping, the extent of non-classical damping and the distribution of loading.

Therefore three of the dimensionless parameters have been taken constant, that is $N = 10$, $\alpha = 1$, $\gamma = 1$, which correspond to a 10-storey frame with constant mass and stiffness, while the other two, namely η and δ, have been varied. Four cases of damping have been considered, corresponding to $\eta = 0.2$ or $\eta = 0.5$ and $\delta = 2$ or $\delta = 5$. Increasing η leads to a more damped system, while increasing δ leads to a greater extent of non-classical damping.

The characteristics of the structural system in terms of frequency moduli and damping ratios, evaluated solving the eigenvalue-eigenvector problem (23), have been reported in Table 1 for the values of damping parameters considered. It is worth noticing that for $\eta = 0.5$ and $\delta = 5$ one of the complex conjugate pairs of eigenvalues appears as two different real negative numbers, equal to -0.8 and -2.0 respectively. The corresponding frequency modulus and damping ratio, evaluated by Eq. (45), are $|\hat{\omega}| = 1.281$ and $\xi = 110.1\%$ and the modal contribution is, therefore, overcritically-damped.

Furthermore two different loading distributions have been considered which, in the following, will be termed *loading condition 1* and *2*. The first loading vector is of the form

$$\boldsymbol{p}^T = [1 \;\; 0 \;\; 0 \;\; \dots \;\; 0], \tag{64}$$

which corresponds to the excitation induced by a rotating machinery acting at the first floor of the building frame, while the second is

$$\boldsymbol{p}^T = [1 \;\; 1 \;\; 1 \;\; \dots \;\; 1], \tag{65}$$

which is typical of a seismic excitation.

4.4.1 Loading condition 1

The error norm of the spatial distribution of forces, given by Eq. (58), is shown in Fig. 2 as a function of the number of modal contributions.

It should be noted that the curves corresponding to the Lanczos algorithm and to the standard modal analysis (denoted by “Lanczos” and “Real” in Fig. 2 respectively) do not depend on the

Table 1 Frequency moduli and damping ratios of the structural system

	$\eta = 0.2$				$\eta = 0.5$			
	$\delta = 2$		$\delta = 5$		$\delta = 2$		$\delta = 5$	
	$\lvert\hat{\omega}\rvert$	ξ (%)	$\lvert\hat{\omega}\rvert$	ξ (%)	$\lvert\hat{\omega}\rvert$	ξ (%)	$\lvert\hat{\omega}\rvert$	ξ (%)
1	0.150	1.78	0.150	2.62	0.150	4.44	0.151	6.39
2	0.445	5.25	0.450	7.44	0.447	13.06	0.462	16.27
3	0.732	8.50	0.743	11.56	0.737	21.06	0.775	23.48
4	1.001	11.42	1.024	15.22	1.008	28.33	1.069	29.79
5	1.248	13.93	1.283	19.04	1.256	34.81	1.281	110.1
6	1.466	15.98	1.451	24.15	1.470	40.35	1.334	35.50
7	1.651	17.55	1.591	20.12	1.641	44.54	1.563	40.46
8	1.800	18.68	1.767	19.38	1.780	46.91	1.749	44.52
9	1.909	19.42	1.895	19.60	1.897	48.46	1.887	47.53
10	1.977	19.86	1.974	19.89	1.974	49.59	1.972	49.38

values of η and δ, because the derivation of the corresponding vectors does not take into account the damping matrix. The error is always smaller in the case of Lanczos vectors because they are derived from the load-dependent algorithm and, therefore, include the static correction concept. On the contrary, the very first natural modes of vibration look very different from the static deformation due to the force distribution of loading condition 1. Therefore, in this case, the rate of convergence of the standard mode superposition method is very slow.

The curve corresponding to the complex modal analysis (denoted by “Complex” in Fig. 2), is obviously dependent on the values of η and δ, because the complex eigenvectors are derived solving the eigenproblem (23) which includes the damping matrix. When non-classical damping has a small extent, i.e., $\delta = 2$, the curve shows a rate of convergence more or less similar to that of the standard modal analysis, becoming slower when η increases, that is when the overall damping of the system increases. On the other hand, when non-classical damping has a great extent, i.e., $\delta = 5$, the curve shows an unusual trend. The error is greater than one when only a few modal contributions are considered and decreases abruptly with the inclusion of the fifth modal

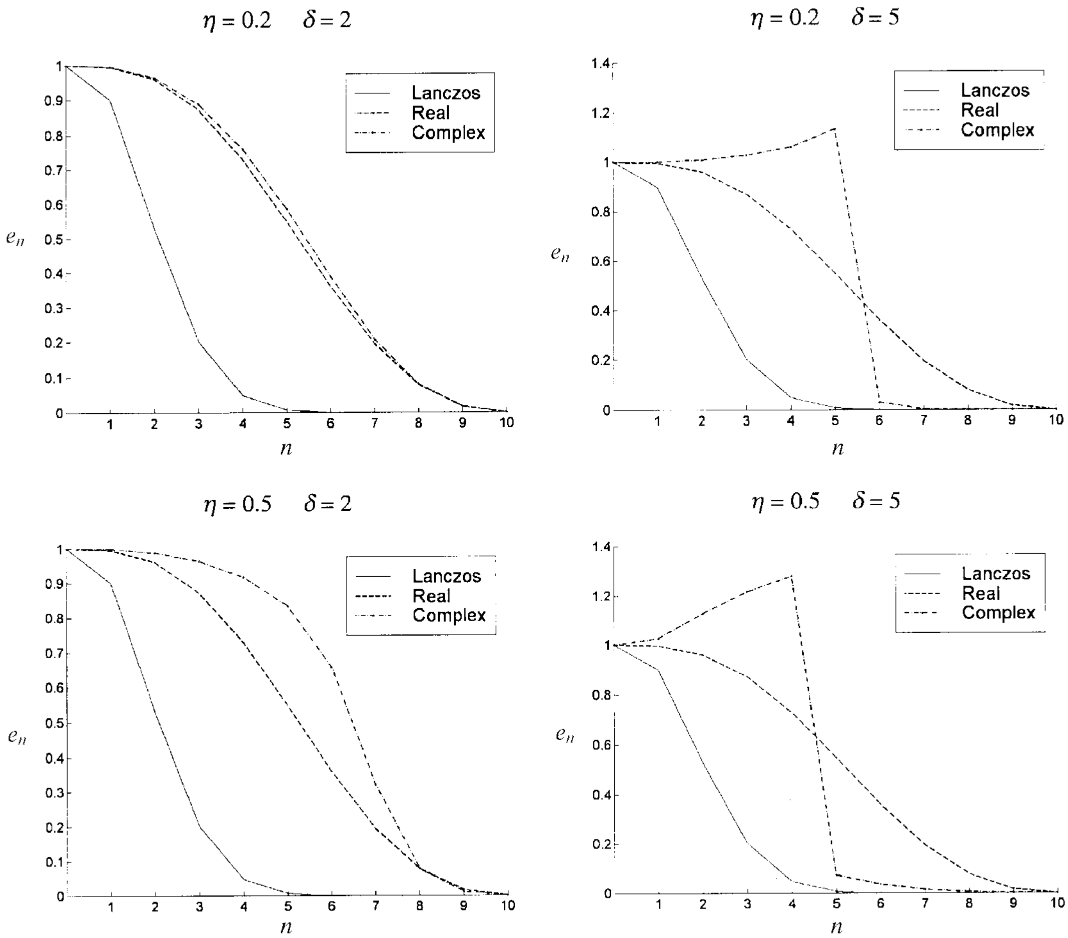

Fig. 2 Errors in spatial distribution of forces (loading condition 1)

contribution when η = 0.2, or the inclusion of the sixth modal contribution when η = 0.5.

The error norm of the base shear, evaluated by Eq. (63), is reported in Fig. 3. In this case all the curves depend on the value of η and δ, that is on the amount of damping and on its distribution along the system. It can be noted that the Lanczos algorithm gives the best result for all the cases of damping considered. Here its rate of convergence is even faster than that of the spatial distribution of forces, shown in Fig. 2.

Furthermore the standard and the complex mode superposition methods show the same trend, corresponding to quite similar error curves.

4.4.2 Loading condition 2

The curves of the error norm of the spatial distribution of forces are reported in Fig. 4. In this case all the methods present a very fast rate of convergence. This is because the first natural and complex modes of vibration are similar to the static deformation due to the force distribution of this

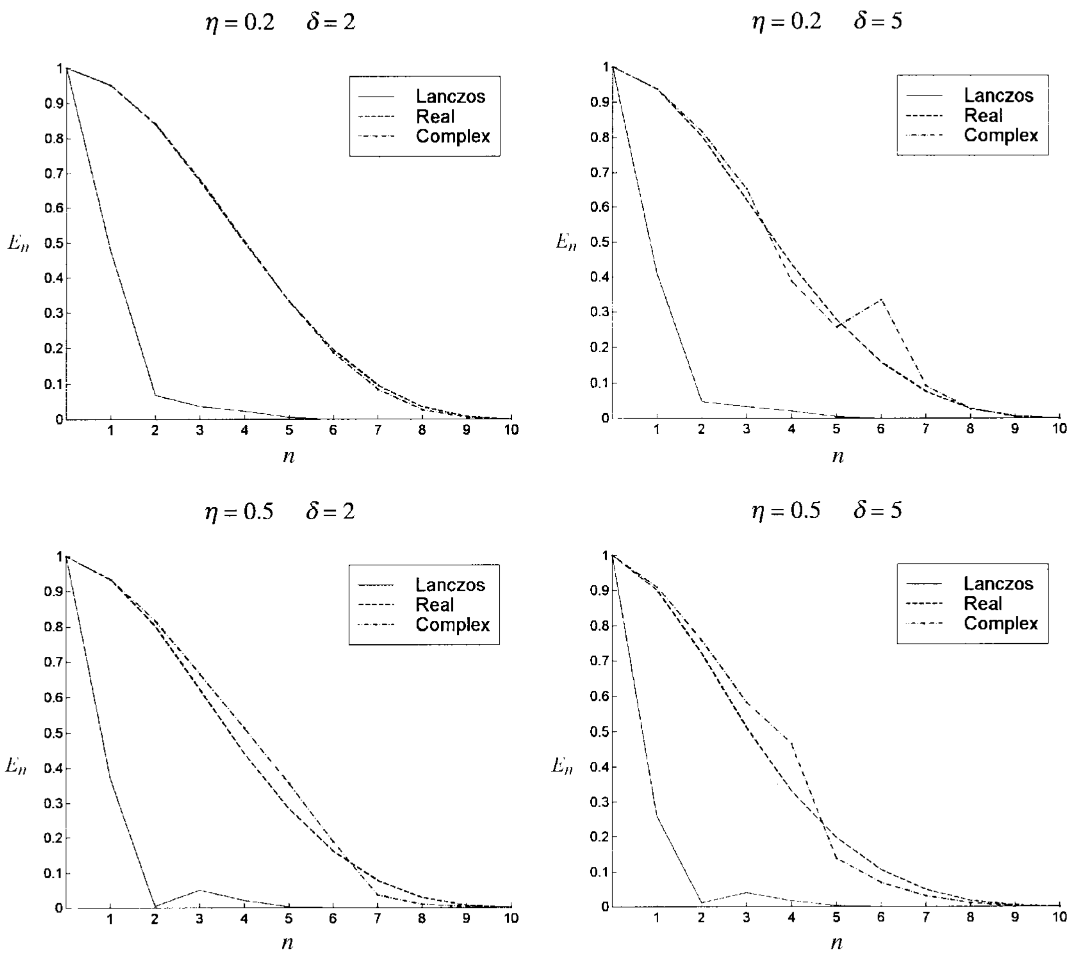

Fig. 3 Errors in base shear (loading condition 1)

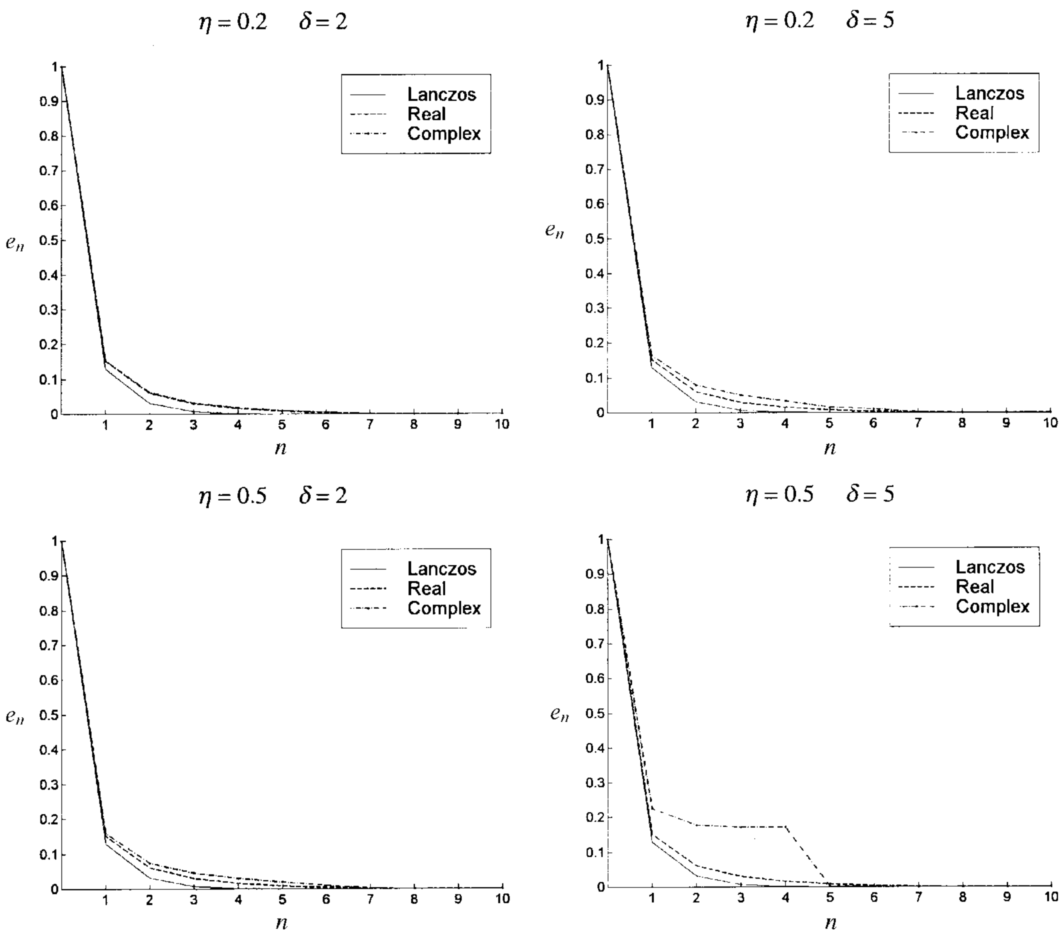

Fig. 4 Errors in spatial distribution of forces (loading condition 2)

loading condition. However, when η = 0.5 and δ = 5, the complex mode superposition method shows, at the beginning, a slight lower rate of convergence and a relative high importance of the fifth real overcritically-damped modal contribution.

The error norm of the base shear is shown in Fig. 5. Even in this case all the methods present a fast rate of convergence, but, unlike the loading condition 1, the Lanczos algorithm gives slightly worse results.

5. Conclusions

In this paper some techniques for the dynamic analysis of a non-classically damped linear system have been reviewed and compared. All the methods lead to a reduction of the number of unknowns obtained by transforming the equations of motion through the use of an appropriate vector basis.

Two of these methods are based on a real basis of Ritz vectors. The first is the standard mode superposition analysis, which uses the normal undamped modes of vibration. These are given by the

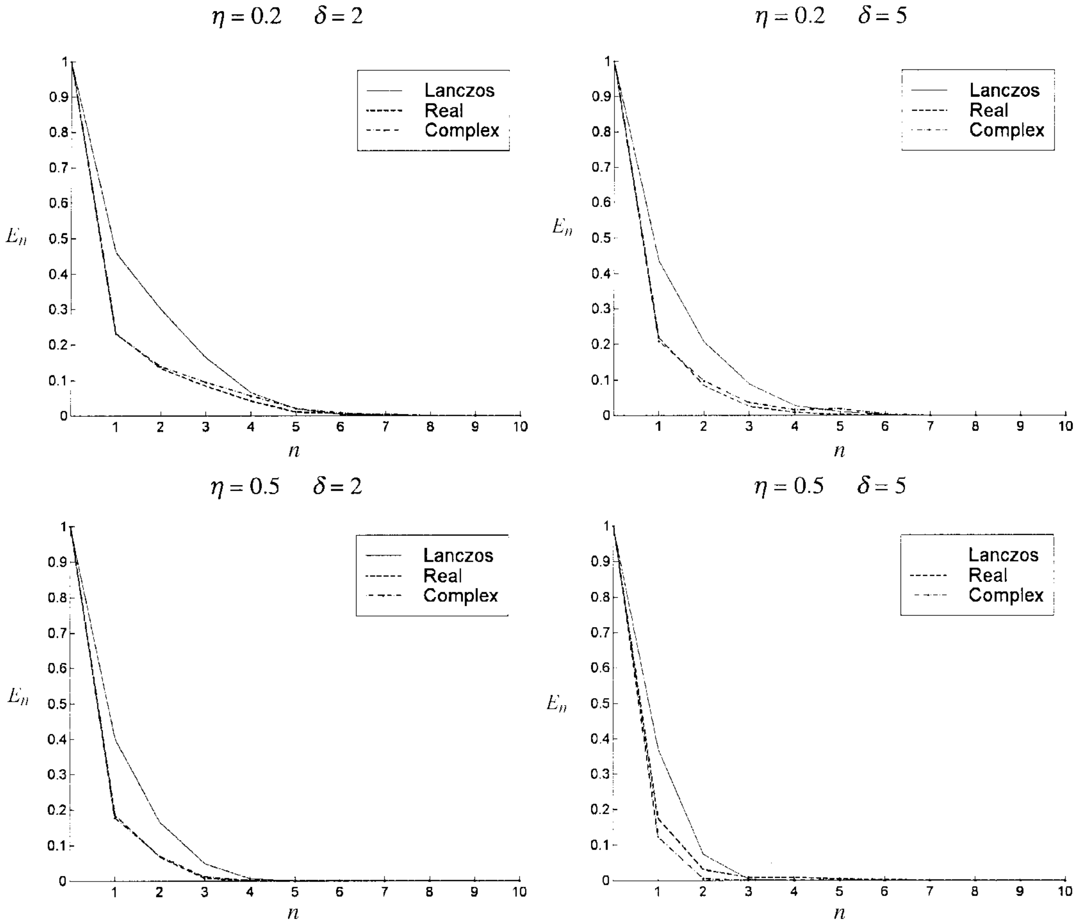

Fig. 5 Errors in base shear (loading condition 2)

solution of the real eigenproblem, the order of which is equal to the number of degrees of freedom. The second is based on a set of Lanczos vectors obtained by the load dependent vector algorithm, which includes the static correction concept. The third method is the so-called complex mode superposition analysis, in which the vector basis is constituted by the true complex conjugate eigenvectors. These are obtained by solving the complex eigenproblem, the order of which is twice the number of degrees of freedom, considering the non-classical damping matrix of the structural system.

Both the first two methods lead to a truncated set of transformed coupled equations of motion which have to be solved simultaneously, while the third decouples the equations of motion exactly.

The rate of convergence of the three considered methods has been evaluated with reference to a parametric structural system. To this aim two error norms have been taken into account. The first is based on the spatial distribution of the loading, while the other is based on the shear force at the base due to impulsive loading. Four cases of damping have been considered, varying the amount of the overall damping and the extent of non-classical damping within the structural system.

It has been shown that, in general, the rate of convergence is strongly influenced by the spatial distribution of the applied forces. When the load vector is non-uniform the method based on the Lanczos vectors gives the best result, although, as observed by Chopra (1995), the standard mode superposition method has the advantage to provide uncoupled modal equations. On the contrary in the case of uniform load vectors all methods almost present the same rate of convergence.

However it should be emphasised that the method which uses the Lanczos vectors should be recommended because it is much less expensive, from a computational point of view, than the other two. In fact, although it leads to a coupled reduced system of transformed equations of motion, it avoids the solution of the eigenproblem that, especially in the case of complex modal analysis, requires a great computational effort.

References

Bielak, J. (1976), "Modal analysis for building-soil interaction", *J. Eng. Mech. Div.*, ASCE, **102**, 771-786.

Caughey, T.K. (1960), "Classical normal modes in damped linear dynamic systems", *J. Appl. Mech.*, ASME, **27**, 269-271.

Caughey, T.K. and O'Kelly, M.E.J. (1965), "Classical normal modes in damped linear dynamic systems", *J. Appl. Mech.*, ASME, **32**, 583-588.

Chopra, A.K. (1995), *Dynamics of Structures, Theory and Applications to Earthquake Engineering*, Prentice Hall, Upper Saddle River, New Jersey.

Clough, R.W. and Penzien, J. (1993), *Dynamic of Structures*, Second Edition, McGraw-Hill.

Clough, R.W. and Mojtahedi, S. (1976), "Earthquake response analysis considering non-proportional damping", *Earthq. Eng. Struct. Dyn.*, **4**, 489-496.

Cornwell, R.E. Craig, R.R. and Johnston, C.P. (1983), "On the application of the mode acceleration method to structural dynamics problems", *Earthq. Eng. Struct. Dyn.*, **11**, 679-688.

Cronin, D.L. (1976), "Approximation for determining harmonically excited response of nonclassically damped systems", *J. Eng. for Industry*, **98**, 43-47.

Duncan, P.E. and Eatock Taylor, R. (1979), "A note on the dynamic analysis of non-proportionally damped systems", *Earthq. Eng. Struct. Dyn.*, **7**, 99-105.

Foss, K.A. (1958), "Coordinates which uncouple the equations of motion of damped linear dynamic systems", *J. Appl. Mech.*, ASME, **25**, 361-364.

Hansteen, O.E. and Bell, K. (1979), "On the accuracy of mode superposition analysis in structural dynamics", *Earthq. Eng. Struct. Dyn.*, **7**, 405-411.

Hasselman, T.K. (1976), "Modal coupling in lightly damped structures", *American Institute of Aeronautics and Astronautics J.*, **14**, 1627-1628.

Hwang, J.H. and Ma, F. (1993), "On the approximate solution of nonclassically damped linear systems", *J. Appl. Mech.*, ASME, **60**, 695-701.

Ibrahimbegovic, A., Chen, H.C., Wilson, E.L. and Taylor, R.L. (1990), "Ritz method for dynamic analysis of large discrete linear systems with non-proportional damping", *Earthq. Eng. Struct. Dyn.*, **19**, 877-889.

Inman, D.J. and Andry jr., A.N. (1980), "Some results on the nature of eigenvalues of discrete damped linear systems", *J. Appl. Mech.*, ASME, **47**, 927-930.

Nair, S.S. and Singh, R. (1986), "Examination of the validity of proportional damping approximations with two further numerical indices", *J. Sound Vib.*, **104**, 348-350.

Nour-Humid, B. and Clough, R.W. (1984), "Dynamic analysis of structures using Lanczos co-ordinates", *Earthq. Eng. Struct. Dyn.*, **12**, 365-377.

Prater, G. and Singh, R. (1986), "Quantification of the extent of non-proportional viscous damping in discrete vibratory systems", *J. Sound Vib.*, **104**, 109-125.

Roesset, J., Whitman, R.V. and Dobry, R. (1973), "Modal analysis for structures with foundation interaction", *J. Struct. Div.*, ASCE, **99**, 399-416.

Shahruz, S.M. and Ma, F. (1988), "Approximate decoupling of the equations of motion of linear underdamped

systems", *J. Appl. Mech.*, ASME, **55**, 716-720.
Veletsos, A.S. and Ventura, C.E. (1986), "Modal analysis of non-classically damped linear systems", *Earthq. Eng. Struct. Dyn.*, **14**, 217-243.
Warburton, G.B. and Soni, S.R. (1977), "Errors in response calculations for non-classically damped structures", *Earthq. Eng. Struct. Dyn.*, **5**, 365-376.
Warburton, G.B. (1978), "Soil-structure interaction for tower structures", *Earthq. Eng. Struct. Dyn.*, **6**, 535-556.
Wilson, E.L. and Penzien, J. (1972), "Evaluation of orthogonal damping matrices", *Int. J. Numer. Meth. Engng.*, **4**, 5-10.
Wilson, E.L., Yuan, M. and Dickens, J.M. (1982), "Dynamic analysis by direct superposition of Ritz vectors", *Earthq. Eng. Struct. Dyn.*, **10**, 813-821.
Xu, K. and Igusa, T. (1991), "Dynamic characteristics of non-classically damped structures", *Earthq. Eng. Struct. Dyn.*, **20**, 1127-1144.