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Scaling laws for vibration response of anti-symmetrically laminated plates

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Abstract. The scaling laws for vibration response of anti-symmetrically laminated plates are derived by applying the similitude transformation to the governing differential equations directly. With this approach, a closed-form solution of the governing equations is not required. This is a significant advantage over the method employed by other researchers where similitude transformation is applied to the closed-form solution. The scaling laws are tested by comparing the similitude fundamental frequencies to the theoretical fundamental frequencies determined from the available closed-form solutions. In case of complete similitude, similitude solutions from the scaling laws exactly agree with the theoretical solutions. Sometimes, it may not be feasible to select the model which obeys the similarity requirement completely, therefore partial similitude is theoretically investigated and approximate scaling laws are recommended. The distorted models in stacking sequences and laminated material properties demonstrate reasonable accuracy. On the contrary, a model with distortion in fiber angle is not recommended. The derived scaling laws are very useful to determine the vibration response of complex prototypes by performing the experiment on a model with required similarities.

Key words: similitude; laminate; plates; vibration; anti-symmetric laminate; scaling law; frequency invariant.

1. Introduction

Composite laminated plates are widely used in mechanical and aerospace engineering applications because of their high stiffness-to-weight and strength-to-weight ratios. Besides buckling phenomenon, vibration is another interesting issue when composites are designed in form of a thin plate. Analytical or approximate solutions are available for vibration of laminated plates with specific configurations and boundary conditions only. For example, closed-form solutions can be obtained only if the laminates are rectangular, symmetric, and simply supported. Solutions of unsymmetrical or irregular-shaped composite plates, i.e., triangular or elliptical plates, require numerical approach. Also, mathematical model does not include the imperfections of plates, load, boundary conditions, and composite structure. Consequently, the analytical or numerical solutions are not exactly matched the results from experiments on the real structures. Tuttle *et al.* (1999) performed buckling experiments on the symmetrically laminated plates subjected to biaxial loading. They found large

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discrepancies in buckling loads between measurement and theoretical prediction, in some cases up to 35%. Imperfections of the specimens as mentioned are probably the sources of the discrepancy. Thus, it would be more efficient if these imperfections were included in the model as they are in the experiment. This is where the similitude method appears as an indispensable tool in establishing the scaling laws to relate the behaviors of the model to those of the real structure or the prototype. If similarity conditions can be found among parameters of the model and prototype, then the scaled replica can be built to duplicate the behaviors of the full-scaled system and the results from the model experiments can be employed to predict the behaviors of the prototype.

Simitses *et al.* (1993, 2000) and Rezaeepazhand *et al.* (1995) have published several papers on symmetrically laminated plates that deal with the establishment of the similarity conditions between the model and the prototype. However, they have applied the similitude transformation to the solutions of the governing differential equations instead of to the governing differential equations directly. This procedure puts serious limitation on the applicability of the concept of similitude theory because some forms of exact or approximate analytical solutions must be obtained before they can apply the similitude transformation. Singhatanadgid and Ungbhakorn (2002) proposed a new similitude approach which applies the similitude transformation to the buckling governing differential equations directly. They showed that the buckling loads of the model and the prototype are related via the scaling law if the model-prototype pair has complete similarity. The main objective of the present study is to derive the frequency invariants for vibration response of the anti-symmetric cross-ply and angle-ply laminated plates by applying the similitude transformation to the governing differential equations directly. In real situation, due to the complexity of the scaling laws or the need to economize the costly experiments, it may not be feasible to construct the model obeying the scaling laws completely, therefore partial similitude is theoretically investigated and approximate scaling laws are recommended by considering the distorted models. In absence of the test model data, the validity of the scaling laws is theoretically verified by substituting the theoretical fundamental frequencies of the model into the scaling laws to predict the fundamental frequencies of the prototype which are then compared to the fundamental frequencies of the prototype calculated from the available analytical solutions.

2. Conditions for complete similitude

Similitude theory is extensively described in a few textbooks (Kline 1965, Skoglund 1967, and Szücs 1980). Only a brief summary which is relevant to this study will be presented as follows. The essence of similitude theorem relevant to our research can be stated as: *the sufficient and necessary condition of similitude between two systems is that the mathematical model of the one be related by a bi-unique transformation to that of the other* (Szücs 1980). Considering all variables, geometric and physical, of the prototype and the model denoted by X_{pi} and X_{mi} respectively, where $i = 1, 2, \ldots, n$. The two systems or phenomena are similar if

$$\boldsymbol{X_p} = \boldsymbol{C}\,\boldsymbol{X_m}$$

and

$$\boldsymbol{X_m} = \boldsymbol{C}^{-1}\,\boldsymbol{X_p}$$

Since the mathematical models of similar systems are invariable under similitude transformation, hence, the differential equations of any two similar systems must be the same, therefore

$$L(X_{mi}) = L(X_{pi}) \tag{1}$$

Let the model and prototype variables be related to each other by the equations:

$$X_{pi} = C_i X_{mi} \tag{2}$$

Substitute Eq. (2) into (1),

$$L(X_{mi}) = L(C_i X_{mi})$$

From the above theorem, it is necessary that

$$L(X_{mi}) = \varphi(C_i)\, L(X_{mi})$$

where $\varphi(C_i)$ is the functional relationship among the transformation parameters. Therefore, for the mathematical models of the two phenomena or systems to be similar the function linking the transformation parameters must equal to unity, hence the following conditional equation is obtained.

$$\varphi(C_i) = 1$$

From this function corresponding similitude invariant can be derived. The vibration similitude invariant between prototypes and models will be derived by considering the governing equations of both systems. When the similitude invariant is derived, scaling law which relates the vibration behavior of both systems is consequently obtained.

3. Vibration of anti-symmetrically cross-ply laminated plates

The general class of anti-symmetric laminates has an even number of layers with adjacent laminae having alternating signs of the principal material property directions with respect to the laminate axes. Two types of anti-symmetric laminates considered in this study include cross-ply and angle-ply plates. For anti-symmetric cross-ply laminates the principal material directions alternate at 0 and 90 degrees to the laminate axes with an even number of layers, for example, $[0/90]_n$. Such laminates do not have A_{16}, A_{26}, D_{16}, and D_{26} but do have coupling between bending and extension. Consider plates made of regular anti-symmetric cross-ply laminates, their other non-zero stiffnesses are (Jones 1975):

Extensional stiffnesses: A_{11}, A_{12}, $A_{22} = A_{11}$, A_{66}
Coupling stiffnesses: B_{11}, and $B_{22} = -B_{11}$
Bending stiffnesses: D_{11}, D_{12}, $D_{22} = D_{11}$, D_{66}

Because of this bending-extension coupling, the governing differential equations for classical vibration analysis are coupled. Dropping the variational symbol δ for the middle surface displacements, the coupled equations for vibration response of anti-symmetric cross-ply laminated plates are as follows (Jones 1975).

$$A_{11}u,_{xx} + A_{66}u,_{yy} + (A_{12} + A_{66})v,_{xy} - B_{11}w,_{xxx} = 0 \tag{3}$$

$$(A_{12}+A_{66})u,_{xy}+A_{66}v,_{xx}+A_{11}v,_{yy}+B_{11}w,_{yyy} = 0 \tag{4}$$

$$D_{11}(w,_{xxxx}+w,_{yyyy})+2(D_{12}+2D_{66})w,_{xxyy}-B_{11}(u,_{xxx}-v,_{yyy})+\rho w,_{tt} = 0 \tag{5}$$

Let the variations in displacement for vibration response of the plates be represented by the equations:

$$u(x, y, t) = U(x, y)e^{i\omega t}$$
$$v(x, y, t) = V(x, y)e^{i\omega t}$$
$$w(x, y, t) = W(x, y)e^{i\omega t}$$

and make substitution into Eqs. (3-5), the following time-independent equations for vibration analysis are obtained:

$$A_{11}U,_{xx}+A_{66}U,_{yy}+(A_{12}+A_{66})V,_{xy}-B_{11}W,_{xxx} = 0 \tag{6}$$

$$(A_{12}+A_{66})U,_{xy}+A_{66}V,_{xx}+A_{11}V,_{yy}+B_{11}W,_{yyy} = 0 \tag{7}$$

$$D_{11}(W,_{xxxx}+W,_{yyyy})+2(D_{12}+2D_{66})W,_{xxyy}-B_{11}(U,_{xxx}-V,_{yyy})-\rho\omega^2 W = 0 \tag{8}$$

Let the variables of the prototype be related to those of the model through the similitude scaling factors as follows.

$$x_p = C_x x_m,\quad y_p = C_y y_m,\quad U_p = C_U U_m,\quad V_p = C_V V_m,\quad W_p = C_W W_m,\quad b_p = C_b b_m$$
$$(A_{ij})_p = C_{Aij}(A_{ij})_m,\quad (B_{ij})_p = C_{Bij}(B_{ij})_m,\quad (D_{ij})_p = C_{Dij}(D_{ij})_m,$$
$$\rho_p = C_\rho \rho_m,\quad \omega_p = C_\omega \omega_m$$

Now, Eqs. (6-8) are employed for the model and the prototype. The governing equations for the model can be represented as Eqs. (6-8) using the variables subscripted by "*m*." By substituting the appropriate variables in Eqs. (6-8), the governing equations for the prototype can be written as

$$C_{A11}(A_{11})_m\frac{C_U}{C_x^2}(U,_{xx})_m+C_{A66}(A_{66})_m\frac{C_U}{C_y^2}(U,_{yy})_m+\{C_{A12}(A_{12})_m+C_{A66}(A_{66})_m\}$$
$$\times\frac{C_V}{C_xC_y}(V,_{xy})_m-C_{B11}(B_{11})_m\frac{C_W}{C_x^3}(W,_{xxx})_m = 0 \tag{9a}$$

$$\{C_{A12}(A_{12})_m+C_{A66}(A_{66})_m\}\frac{C_U}{C_xC_y}(U,_{xy})_m+C_{A66}(A_{66})_m\frac{C_V}{C_x^2}(V,_{xx})_m$$
$$+C_{A11}(A_{11})_m\frac{C_V}{C_y^2}(V,_{yy})_m+C_{B11}(B_{11})_m\frac{C_W}{C_y^3}(W,_{yyy})_m = 0 \tag{9b}$$

$$C_{D11}(D_{11})_m\left\{\frac{C_W}{C_x^4}(W_{,xxxx})_m+\frac{C_W}{C_y^4}(W_{,yyyy})_m\right\}+2\{C_{D12}(D_{12})_m+2C_{D66}(D_{66})_m\}$$
$$\times\frac{C_W}{C_x^2C_y^2}(W_{,xxyy})_m-C_{B11}(B_{11})_m\left\{\frac{C_U}{C_x^3}(U_{,xxx})_m-\frac{C_V}{C_y^3}(V_{,yyy})_m\right\}-C_\rho C_\omega^2C_W\rho_m\omega_m^2W_m = 0 \quad (9c)$$

To achieve the similarity between the prototype and the model, all of the similitude scaling factor groups shown in each of Eq. (9) must be equal. Therefore, the following requirements are obtained:

$$\frac{C_{A11}C_U}{C_x^2}=\frac{C_{A66}C_U}{C_y^2}=\frac{C_{A12}C_V}{C_xC_y}=\frac{C_{A66}C_V}{C_xC_y}=\frac{C_{B11}C_W}{C_x^3} \quad (10a)$$

$$\frac{C_{A12}C_U}{C_xC_y}=\frac{C_{A66}C_U}{C_xC_y}=\frac{C_{A66}C_V}{C_x^2}=\frac{C_{A11}C_V}{C_y^2}=\frac{C_{B11}C_W}{C_y^3} \quad (10b)$$

$$\frac{C_{D11}C_W}{C_x^4}=\frac{C_{D11}C_W}{C_y^4}=\frac{C_{D12}C_W}{C_x^2C_y^2}=\frac{C_{D66}C_W}{C_x^2C_y^2}=\frac{C_{B11}C_U}{C_x^3}=\frac{C_{B11}C_V}{C_y^3}=C_\rho C_\omega^2C_W \quad (10b)$$

Considering that the model-prototype pair has a complete geometric similarity, i.e., $C_x = C_y = C_U = C_V = C_W = C_b$, then the following conditional equations for the model to behave exactly as the prototype are derived:

$$C_{A11}=C_{A66}=C_{A12}=\frac{C_{B11}}{C_b} \quad (11a)$$

$$C_{A12}=C_{A66}=C_{A11}=\frac{C_{B11}}{C_b} \quad (11b)$$

$$C_{D11}=C_{D12}=C_{D66}=C_{B11}C_b=C_\rho C_\omega^2C_b^4 \quad (11c)$$

which can be combined into the following equation:

$$C_{A11}C_b^2=C_{A12}C_b^2=C_{A66}C_b^2=C_{B11}C_b=C_{D11}=C_{D12}=C_{D66}=C_\rho C_\omega^2C_b^4 \quad (12)$$

In conclusion, the complete similarity between the model and the prototype requires (1) a complete geometric similarity between two systems, and (2) scaling factors that satisfy the conditional equations, Eq. (12) which can be rewritten as:

$$C_{A11}=C_{A12}=C_{A66} \quad (13)$$

$$C_{D11}=C_{D12}=C_{D66} \quad (14)$$

$$C_{Aij}C_b^2=C_{B11}C_b=C_{Dij} \quad (15)$$

By representing each term in Eq. (15) with C_{stiff}, the conditional equation in Eq. (12) is rearranged and the similitude invariant (frequency invariant) for vibration response of anti-symmetric cross-ply laminated plates can be written as

$$\frac{C_\rho C_\omega^2 C_b^4}{C_{stiff}} = 1 \tag{16}$$

The scaling law is then derived as

$$\omega_p^2 = \omega_m^2 C_{stiff} \frac{(b^4\rho)_m}{(b^4\rho)_p} \tag{17}$$

Recall that:

$$A_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k (z_k - z_{k-1})$$

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{N} (\bar{Q}_{ij})_k (z_k^2 - z_{k-1}^2)$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} (\bar{Q}_{ij})_k (z_k^3 - z_{k-1}^3)$$

Therefore, two laminated plates with the same material properties and stacking sequences but different number of ply, for example, $[0/90]_n$ and $[0_m/90_m]_n$ will satisfy conditions in Eqs. (13-14) identically, or equivalently, they have different ply thickness. Also, conditions in Eq. (15) can be satisfied by selecting the appropriate geometric scaling factors.

4. Vibration of anti-symmetric angle-ply laminated plates

An anti-symmetric angle-ply laminate has an even number of layers with adjacent laminae having alternating signs of the principal material property at $\pm\theta$ degrees to the laminate axes, for example, $[+\theta/-\theta]_n$. Such laminates exhibit a different characteristic of coupling between bending and extension than that of the anti-symmetric cross-ply laminates. The non-zero stiffnesses are:

Extensional stiffnesses: A_{11}, A_{12}, A_{22}, and A_{66}
Coupling stiffnesses: B_{16} and B_{26}
Bending stiffnesses: D_{11}, D_{12}, D_{22}, and D_{66}

Dropping the variational symbol δ of the middle surface displacements, the coupled differential equations for vibration response of anti-symmetric angle-ply laminated plates are (Jones 1975):

$$A_{11}u,_{xx} + A_{66}u,_{yy} + (A_{12} + A_{66})v,_{xy} - 3B_{16}w,_{xxy} - B_{26}w,_{yyy} = 0 \tag{18}$$

$$(A_{12}+A_{66})u,_{xy}+A_{66}v,_{xx}+A_{22}v,_{yy}-B_{16}w,_{xxx}-3B_{26}w,_{xyy} = 0 \tag{19}$$

$$D_{11}w,_{xxxx}+2(D_{12}+2D_{66})w,_{xxyy}+D_{22}w,_{yyyy}-B_{16}(3u,_{xxy}+v,_{xxx})-B_{26}(u,_{yyy}+3v,_{xyy})+\rho w,_{tt}=0 \tag{20}$$

Assuming the variations in displacement for vibration response of the plates as in the former case and making substitution into Eqs. (18-20) the time-independent equations for vibration analysis are as follows.

$$A_{11}U,_{xx}+A_{66}U,_{yy}+(A_{12}+A_{66})V,_{xy}-3B_{16}W,_{xxy}-B_{26}W,_{yyy} = 0 \tag{21}$$

$$(A_{12}+A_{66})U,_{xy}+A_{66}V,_{xx}+A_{22}V,_{yy}-B_{16}W,_{xxx}-3B_{26}W,_{xyy} = 0 \tag{22}$$

$$\begin{aligned} &D_{11}W,_{xxxx}+2(D_{12}+2D_{66})W,_{xxyy}+D_{22}W,_{yyyy}-B_{16}(3U,_{xxy}+V,_{xxx}) \\ &\quad -B_{26}(U,_{yyy}+3V,_{xyy})-\rho\omega^2 W = 0 \end{aligned} \tag{23}$$

Applying the similitude transformation to Eqs. (21-23), with the consideration of complete geometric similarity, the conditional equation for similarity among the prototypes and models is obtained as

$$\begin{aligned} C_{A11}C_b^2 &= C_{A12}C_b^2 = C_{A22}C_b^2 = C_{A66}C_b^2 = C_{B16}C_b = C_{B26}C_b \\ &= C_{D11} = C_{D12} = C_{D22} = C_{D66} = C_\rho C_\omega^2 C_b^4 \end{aligned} \tag{24}$$

Similar to cross-ply laminates, this equation leads to the conditions for complete similitude requirements as

$$C_{A11} = C_{A12} = C_{A22} = C_{A66} \tag{25}$$

$$C_{D11} = C_{D12} = C_{D22} = C_{D66} \tag{26}$$

$$C_{Aij}C_b^2 = C_{Bij}C_b = C_{Dij} \tag{27}$$

Again, the model can satisfy Eqs. (25-27) by the same reasoning as the previous case. Thus, the similitude invariant and scaling law for anti-symmetric angle-ply laminated plate can be written in the same form as Eqs. (16-17) as follows.

$$\frac{C_\rho C_\omega^2 C_b^4}{C_{stiff}} = 1 \tag{28}$$

$$\omega_p^2 = \omega_m^2 C_{stiff}\frac{(b^4\rho)_m}{(b^4\rho)_p} \tag{29}$$

where C_{stiff} is either one of $C_{Aij}C_b^2$, $C_{Bij}C_b$, or C_{Dij} in Eq. (27).

5. Theoretical solutions for frequencies of vibration

In order to be able to test the recommended scaling laws, the closed-form solution of vibration response for each case is needed. For the case of anti-symmetric cross-ply laminated plates, the boundary condition which allows the closed-form solution is the simply-supported edge S2 (Jones 1975). Dropping the variational symbol δ for the displacements and stress and moment resultants, they are:

Along $x = 0, a$:

$$w = 0, \quad M_{xx} = B_{11}u,_x - D_{11}w,_{xx} - D_{12}w,_{yy} = 0,$$

$$v = 0, \quad N_{xx} = A_{11}u,_x + A_{12}v,_y - B_{11}w,_{xx} = 0$$

Along $y = 0, b$:

$$w = 0, \quad M_{yy} = -B_{11}v,_y - D_{12}w,_{xx} - D_{11}w,_{yy} = 0,$$

$$u = 0, \quad N_{yy} = A_{12}u,_x + A_{11}v,_y + B_{11}w,_{yy} = 0$$

The closed-form solution for free vibration response is given in (Jones 1975) as follows.

$$\omega^2 = \frac{\pi^4}{\rho}\left\{T_{33} + \frac{2T_{12}T_{23}T_{13} - T_{22}T_{13}^2 - T_{11}T_{23}^2}{T_{11}T_{22} - T_{12}^2}\right\} \tag{30}$$

where,

$$T_{11} = A_{11}\left(\frac{i}{a}\right)^2 + A_{66}\left(\frac{j}{b}\right)^2$$

$$T_{12} = (A_{12} + A_{66})\left(\frac{i}{a}\right)\left(\frac{j}{b}\right)$$

$$T_{13} = -B_{11}\left(\frac{i}{a}\right)^3$$

$$T_{22} = A_{11}\left(\frac{j}{b}\right)^2 + A_{66}\left(\frac{i}{a}\right)^2$$

$$T_{23} = B_{11}\left(\frac{j}{b}\right)^3$$

$$T_{33} = D_{11}\left[\left(\frac{i}{a}\right)^4 + \left(\frac{j}{b}\right)^4\right] + 2(D_{12} + 2D_{66})\left(\frac{i}{a}\right)^2\left(\frac{j}{b}\right)^2$$

For the case of anti-symmetric angle-ply laminated plates the only available closed-form solution is the simply-supported boundary condition S3.

Along $x = 0, a$:

$$w = 0, \quad M_{xx} = B_{16}(v,_x + u,_y) - D_{11}w,_{xx} - D_{12}w,_{yy} = 0,$$

$$u = 0, \quad N_{xy} = A_{66}(v,_x + u,_y) - B_{16}w,_{xx} - B_{26}w,_{yy} = 0$$

Along $y = 0, b$:

$$w = 0, \quad M_{yy} = B_{26}(v,_x + u,_y) - D_{12}w,_{xx} - D_{22}w,_{yy} = 0,$$

$$v = 0, \quad N_{xy} = A_{66}(v,_x + u,_y) - B_{16}w,_{xx} - B_{26}w,_{yy} = 0$$

Note that this boundary condition differs significantly from S2 but it will not affect the principle of similitude presented herein. The expression of vibration response, just for the purpose of demonstrating the derived scaling law, is as follows (Jones 1975).

$$\omega^2 = \frac{\pi^4}{\rho}\left\{T_{33} + \frac{2T_{12}T_{23}T_{13} - T_{22}T_{13}^2 - T_{11}T_{23}^2}{T_{11}T_{22} - T_{12}^2}\right\} \tag{31}$$

where,

$$T_{11} = A_{11}\left(\frac{i}{a}\right)^2 + A_{66}\left(\frac{j}{b}\right)^2$$

$$T_{12} = (A_{12} + A_{66})\left(\frac{i}{a}\right)\left(\frac{j}{b}\right)$$

$$T_{13} = -\left[3B_{16}\left(\frac{i}{a}\right)^2 + B_{26}\left(\frac{j}{b}\right)^2\right]\left(\frac{j}{b}\right)$$

$$T_{22} = A_{22}\left(\frac{j}{b}\right)^2 + A_{66}\left(\frac{i}{a}\right)^2$$

$$T_{23} = -\left[B_{16}\left(\frac{i}{a}\right)^2 + 3B_{26}\left(\frac{j}{b}\right)^2\right]\left(\frac{i}{a}\right)$$

$$T_{33} = D_{11}\left(\frac{i}{a}\right)^4 + 2(D_{12} + 2D_{66})\left(\frac{i}{a}\right)^2\left(\frac{j}{b}\right)^2 + D_{22}\left(\frac{j}{b}\right)^4$$

The fundamental frequencies of free vibration can be calculated from Eqs. (30-31) by minimizing ω^2 with respect to the integer i and j.

6. Complete similitude results

Although the boundary conditions of the two cases are quite different and may not be realistic for actual application but there are no problems in verifying the scaling laws because any boundary condition effects will be included in the results of model experiment. If proven correctly, then one can design experiments to simulate actual application environment rather than design them to fit the theoretical boundary conditions as has been done in the past. Mechanical properties of three types of materials used in this numerical study are shown in Table 1.

6.1 Cross-ply case

The scaling law in Eq. (17) can be verified by comparing the theoretical solutions to the results from the scaling law. In case of complete similitude, a pair of model and prototype plates has to satisfy all of the similitude requirements. As mention previously, $[0/90]_n$ and $[0_m/90_m]_n$ laminated plates satisfy the similitude requirement in Eqs. (13-15) if an appropriate geometric scaling factor is selected. Thus, $[0/90]_4$ and $[0_2/90_2]_4$ laminates are selected as models and prototypes for the study shown in Table 2. This model-prototype pair has the stiffness scaling factors as: $C_{Aij} = 2$, $C_{B11} = 4$, and $C_{Dij} = 8$. Thus, the complete similitude is achieved if C_b is selected as 2. The 3rd column of Table 2 shows the theoretical fundamental frequencies of 200 mm-width model plates with various

Table 1 Ply properties of composites used in this study

	E_{11} (GPa)	E_{22} (GPa)	G_{12} (GPa)	v_{12}	Ply Thickness (mm)	Density (g/cm^3)
Graphite/Epoxy	132	10.8	5.65	0.24	0.127	1.54
Kevlar/Epoxy	76.8	5.50	2.07	0.34	0.127	1.38
E-Glass/Epoxy	38.6	8.27	4.14	0.26	0.127	1.80

Table 2 Fundamental frequencies of anti-symmetric cross-ply plates determined from theory and similitude

Configuration		Model, $[0/90]_4$ (b_m = 200 mm)	Prototype, $[0_2/90_2]_4$ (b_p = 400 mm, C_b = 2)		Prototype, $[0_2/90_2]_4$ (b_p = 200 mm, C_b = 1)		Prototype, $[0_2/90_2]_4$ (b_p = 600 mm, C_b = 3)	
Materials	Aspect Ratio	Theory (rad/s)	Theory (rad/s)	Similitude (rad/s)	Theory (rad/s)	Similitude (rad/s)	Theory (rad/s)	Similitude (rad/s)
Graphite/Epoxy	1	737.7	368.8	368.8	1475	1475	163.9	163.9
	2	513.5	256.7	256.7	1027	1027	114.1	114.1
	3	489.6	244.8	244.8	979.2	979.2	108.8	108.8
Kevlar/Epoxy	1	590.8	295.4	295.4	1182	1182	131.3	131.3
	2	415.6	207.8	207.8	831.2	831.2	92.35	92.35
	3	398.5	199.3	199.3	797.0	797.0	88.56	88.56
E-glass/Epoxy	1	443.0	221.5	221.5	886.0	886.0	98.45	98.45
	2	295.3	147.7	147.7	590.6	590.6	65.63	65.63
	3	274.6	137.3	137.3	549.2	549.2	61.02	61.02

aspect ratios and material properties. These theoretical solutions of models are substituted, as ω_m, in the scaling law to determine the similitude fundamental frequencies ω_p of prototypes which are shown in the 5th column. They are exactly matched the theoretical solutions presented in the 4th column. Therefore, the correctness of the scaling law is verified in the case of complete similitude.

A further numerical study is performed by selecting prototype laminates with different geometric scaling factors, i.e., $C_b = 1$ and 3. With these different geometric scaling factors, the similarity requirement in Eq. (15) is not satisfied. However, when C_{Dij} is used as C_{stiff} in the scaling law, the similitude and theoretical solutions are identical. These two cases may not be considered as a "complete" similitude cases because one of the similarity requirements is not fulfilled, however, the scaling law gives exact solutions compared to theoretical solutions.

6.2 Angle-ply case

A similar study is performed on the anti-symmetric angle-ply laminates as shown in Table 3. Theoretical solutions of $[\pm 45]_4$ laminated model plates are employed in the scaling law to predict the fundamental frequency of the prototype plates which have similar stacking sequences but different ply thickness comparing to the models. It can be shown that the prototype A and C are selected such that all similarity requirements, both geometric similarity and scaling factor requirement in Eqs. (25-27), are satisfied but prototype B has the scaling factor that violates the requirement in Eq. (27). Again, the scaling law works perfectly for all prototypes. The similitude solutions are identical to the closed-form solutions. Like the cross-ply case, although the scaling factor of prototype B does not satisfy the conditional equation, the similitude results agree exactly with the closed-form solutions when C_{Dij} is used as C_{stiff}. Therefore, the derived scaling laws for both anti-symmetric cross-ply and angle-ply laminates yield exact solutions for a model-prototype pair with complete similarity. If the scaling factor C_b is selected such that it does not satisfy the similarity requirement in Eq. (15) or Eq. (27), the scaling laws still give perfect solutions if C_{Dij} is used as C_{stiff}. This is true only if there is an existing C_b which can be selected to satisfy the

Table 3 Fundamental frequencies of anti-symmetric angle-ply plates modeling with complete similitude

Materials	Aspect Ratio	Model, $[\pm 45]_4$ (b = 200 mm)	Prototype A, $[45_2/-45_2]_4$ (b = 400 mm, C_b = 2)		Prototype B, $[45_2/-45_2]_4$ (b = 800 mm, C_b = 4)		Prototype C, $[45_4/-45_4]_4$ (b = 800 mm, C_b = 4)	
		Theory (rad/s)	Theory (rad/s)	Similitude (rad/s)	Theory (rad/s)	Similitude (rad/s)	Theory (rad/s)	Similitude (rad/s)
Graphite/ Epoxy	1	952.6	476.3	476.3	119.1	68.85	238.2	238.2
	2	550.8	275.4	275.4	68.85	57.05	137.7	137.7
	3	456.4	228.2	228.2	57.05	97.44	114.1	114.1
Kevlar/ Epoxy	1	779.6	389.8	389.8	97.44	56.04	194.9	194.9
	2	448.3	224.2	224.2	56.04	46.18	112.1	112.1
	3	369.4	184.7	184.7	46.18	65.11	92.35	92.35
E-glass/ Epoxy	1	520.9	260.4	260.4	65.11	38.61	130.2	130.2
	2	308.9	154.5	154.5	38.61	32.81	77.23	77.23
	3	262.5	131.3	131.3	32.81	68.85	65.63	65.63

similarity requirement. It is different from the partial similitude where no C_b can be found to satisfy the conditional equations, as shown in the next section. The partial similitude transformation will not result in a perfect solution for any selected C_b.

7. Partial similitude results

For some cases, it is impossible to get a complete similarity between the model and the prototype. This does not mean that the scaling law is useless. If some similarity conditions are ignored, it is still possible to apply the scaling laws to obtain results with reasonable accuracy by using an appropriate C_{stiff}. This similarity condition is called “partial similitude”. However, the accuracy of the similitude theory is expected to be compromised. If the error is estimated and found to be sufficiently low, the concept of partial similarity will give much more flexibility to design experiment to fulfill the objective.

7.1 Cross-ply case

7.1.1 Distortion in stacking sequences

Occasionally, it is impractical to choose a model which has complete similarity for a particular prototype. The partial similitude is applied in this case by adopting the most appropriate model and relaxing some similarity requirements. Numerical studies of the partial similitude are shown in Table 4 where various $[0/90]_8$ prototypes are modeled by $[0/90]_4$ models. The stiffness scaling factors for this pair of stacking sequences are $C_{Aij} = C_{B11} = 2$ and $C_{Dij} = 8$. It is not possible to fulfill the similarity requirement in Eq. (15), thus, it will be ignored and C_{Dij} is used as C_{stiff} in the scaling law instead. Results from the scaling law are not exactly conformed to the theoretical solutions. The similitude results deviate from the closed-form solution by less than 1.5% as shown in the “%Disc”

Table 4 Fundamental frequencies of anti-symmetric cross-ply plates with partial similitude in stacking sequences

Materials	Aspect Ratio	Model, $[0/90]_4$ (b = 200 mm)	Prototype, $[0/90]_8$ (b = 200 mm, C_b = 1)			Prototype, $[0/90]_8$ (b = 400 mm, C_b = 2)			Prototype, $[0/90]_8$ (b = 600 mm, C_b = 3)		
		Theory (rad/s)	Theory (rad/s)	Similitude (rad/s)	% Disc	Theory (rad/s)	Similitude (rad/s)	% Disc	Theory (rad/s)	Similitude (rad/s)	% Disc
Graphite/ Epoxy	1	737.7	1492	1475	–1.11	373.0	368.8	–1.11	165.8	163.9	–1.11
	2	513.5	1039	1027	–1.20	259.9	256.7	–1.20	115.5	114.1	–1.20
	3	489.6	991.6	979.2	–1.24	247. 9	244.8	–1.24	110.2	108.8	–1.24
Kevlar/ Epoxy	1	590.8	1196	1182	–1.22	299.1	295.4	–1.22	132.9	131.3	–1.22
	2	415.6	842.0	831.2	–1.28	210.5	207.8	–1.28	93.55	92.35	–1.28
	3	398.5	807.6	797.0	–1.31	201.9	199.3	–1.31	89.74	88.56	–1.31
E-glass/ Epoxy	1	443.0	891.1	886.0	–0.566	222.8	221.5	–0.566	99.01	98.45	–0.566
	2	295.3	594.5	590.6	–0.645	148.6	147.7	–0.645	66.05	65.63	–0.645
	3	274.6	553.0	549.1	–0.693	138.2	137.3	–0.693	61.44	61.02	–0.693

columns. The percent discrepancy is calculated from:

$$\frac{Similitude - theory}{Theory} \times 100\%$$

It should be noted that the discrepancy percentage is dependent on the type of material and aspect ratio but is independent on geometric scaling factor.

Another study is shown in Table 5 where the fundamental frequencies of the $[0/90]_{40}$ prototypes are predicted using models with different number of plies. The stiffness scaling factors for each model-prototype pair are shown in the 2nd column. It is noticed that the stiffness scaling factors are separated into a group of C_{Aij} and C_{Bij} and a group of C_{Dij}. The next column is the theoretical fundamental frequency of each model. They are used to model the frequency of the $[0/90]_{40}$ prototype and the similitude fundamental frequency is shown in the 4th column. When C_{Dij} is used as C_{stiff}, the similitude fundamental frequency converges to the theoretical solution as numbers of plies approach n = 40. The discrepancies shown in column 5th are lower when number of plies of the model approaches the number of plies of the prototype. The percent discrepancy is less than 2% if the model has at least 8 plies (n = 4). The percent discrepancies are plotted versus values of n and shown in Fig. 1. They converge to zero as the thickness of the model approaches thickness of the prototype.

Therefore, although some model-prototype pairs may not have stacking sequences that entirely satisfy the similitude requirements, some similitude requirements can be relaxed to allow the implementation of the scaling law. The discrepancy induced is considered low for the model-

Table 5 Modeling of $[0/90]_{40}$ graphite/epoxy laminates from $[0/90]_n$ laminates (400 × 400 mm^2 plates)

Value of n in $[0/90]_n$	Stiffness Scaling Factor C_{Aij} and C_{B11} : C_{Dij}	Theoretical fundamental frequencies of $[0/90]_n$ (rad/s)	Similitude fundamental frequencies of $[0/90]_{40}$ using $[0/90]_n$ as model and $C_{stiff} = C_{Dij}$	% Disc. Comparing to Theoretical Sol. of 1871.6 rad/s
1	40 : 64000	34.08	1363	–27.2
2	20 : 8000	87.93	1759	–6.04
3	13.33 : 2370	136.7	1822	–2.63
4	10 : 1000	184.4	1844	–1.46
5	8 : 512	231.8	1854	–0.929
6	6.67 : 296.3	278.9	1860	–0.640
7	5.71 : 186.6	326.0	1863	–0.466
8	5 : 125	373.0	1865	–0.353
9	4.44 : 87.79	420.0	1866	–0.276
10	4 : 64	466.9	1867	–0.220
15	2.67 : 18.96	701.2	1870	–0.0897
20	2 : 8	935.4	1871	–0.0440
25	1.6 : 4.096	1169	1871.2	–0.0229
30	1.33 : 2.370	1404	1871.4	–0.0114
35	1.14 : 1.493	1638	1871.5	–0.0045
40 (Prototype)	-	1871.6	-	-

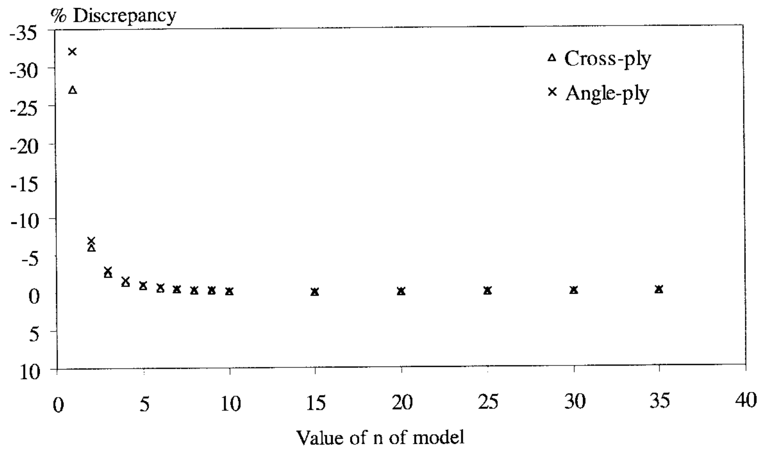

Fig. 1 Percent discrepancies in prediction of $[0/90]_{40}$ plates from Table 5 and $[45/-45]_{40}$ plates from Table 9

prototype pairs in this study. Models with low numbers of ply also facilitate the design and economics of experiments.

7.1.2 Distortion in material properties

To economize the experimental cost, a cheaper material can be used to model the expensive prototype. For example, graphite fiber or Kevlar fiber composites are modeled by E-glass/epoxy laminate which is cheaper. To study the effect of the material property distortion, stacking sequences of the model and prototype are selected such that the complete similitude is achieved if they were made from the same materials. Table 6 shows the stiffness scaling factors of the prototypes using $[0/90]_4$ E-glass/epoxy as a model. The prototypes are $[0_2/90_2]_4$ plates with $b = 400$ mm and $[0/90]_4$ plates with $b = 200$ mm. The discrepancies between the results from similitude theory and the closed-form solution are shown in Table 7. They are in the range of –12% to –20% which are high as compared to results from the previous partial similitude model. The relatively high discrepancy is

Table 6 Stiffness scaling factors (C_{Aij}, C_{Bij}, C_{Dij}) of various prototypes

	Stiffness of E-glass/ epoxy model, $[0/90]_4$	Stiffness scaling factors (C_{Aij}, C_{Bij}, C_{Dij}) of prototype			
		Graphite/epoxy, $[0_2/90_2]_4$	Graphite/epoxy, $[0/90]_4$	Kevlar/epoxy, $[0_2/90_2]_4$	Kevlar/epoxy, $[0/90]_4$
A_{11}	24.16	6.034	3.017	3.490	1.745
A_{12}	2.217	2.387	1.194	1.728	0.8642
A_{66}	4.206	2.729	1.365	1.000	0.5000
B_{11}	–0.9928	15.83	3.957	9.344	2.336
D_{11}	2.078	24.13	3.017	13.96	1.745
D_{12}	0.1907	9.549	1.194	6.914	0.8642
D_{66}	0.3618	10.92	1.365	4.000	0.5000

Note : A_{ij} in GPa-mm, B_{11} in GPa-mm^2, D_{ij} in GPa-mm^3

probably the result of the non-uniform stiffness scaling factors. Unlike studies in Tables 4 and 5, each flexural stiffness scaling factor C_{Dij} are not identical. Some of them are three times as much as the others, e.g., C_{D11} and C_{D66} of the Kevlar/epoxy prototype. The equality of the stiffness scaling factors is required for similarity condition. So, the distortion in material properties in the partial similitude model is applicable if the degree of non-uniformity of the stiffness scaling factors is kept at minimum.

7.2 Angle-ply case

7.2.1 Distortion in stacking sequences

A similar study to that of shown in Table 4 is performed for angle-ply laminated and presented in Table 8. $[\pm 45]_8$ laminates with different dimensions are modeled by $[\pm 45]_4$ laminated plates. The stiffness scaling factors of this laminate pair are $C_{Aij} = 2$, $C_{Bij} = 2$ and $C_{Dij} = 8$. Like the cross-ply

Table 7 Modeling of $[0_2/90_2]_4$ graphite/epoxy and Kevlar/epoxy laminates from E-glass/epoxy laminates

Aspect Ratio	Theoretical frequencies in rad/s of $[0/90]_4$ E-glass/epoxy (b = 200 mm)	%Discrepancy of similitude fundamental frequency compared to closed-form solutions			
		Graphite/epoxy $[0_2/90_2]_4$ (b = 400 mm)	Graphite/epoxy $[0/90]_4$ (b = 200 mm)	Kevlar/epoxy $[0_2/90_2]_4$ (b = 400 mm)	Kevlar/epoxy $[0/90]_4$ (b = 200 mm)
1	443.0	–13.2	–13.2	–12.8	–12.8
1.5	328.9	–14.9	–14.9	–14.9	–14.9
2	295.3	–16.8	–16.8	–17.4	–17.4
2.5	281.5	–18.1	–18.1	–18.9	–18.9
3	274.6	–18.9	–18.9	–19.9	–19.9

Table 8 Fundamental frequencies of anti-symmetric angle-ply laminates modeled with partial similitude in stacking sequences

Materials	Aspect Ratio	Model, $[\pm 45]_4$ (b = 200 mm)	Prototype, $[\pm 45]_8$ (b = 200 mm, C_b = 1)			Prototype, $[\pm 45]_8$ (b = 400 mm, C_b = 2)			Prototype, $[\pm 45]_8$ (b = 600 mm, C_b = 3)		
		Theory (rad/s)	Theory (rad/s)	Similitude (rad/s)	% Disc	Theory (rad/s)	Similitude (rad/s)	% Disc	Theory (rad/s)	Similitude (rad/s)	% Disc
Graphite/ Epoxy	1	952.6	1930	1905	–1.29	482.5	476.3	–1.29	214.5	211.7	–1.29
	2	550.8	1116	1102	–1.24	278.9	275.4	–1.24	123.9	122.4	–1.24
	3	456.4	923.9	912.9	–1.20	231.0	228.2	–1.20	102.7	101.4	–1.20
Kevlar/ Epoxy	1	779.6	1580	1559	–1.34	395.1	389.8	–1.34	175.6	173.2	–1.34
	2	448.3	908.6	896.7	–1.31	227.2	224.2	–1.31	101.0	99.63	–1.31
	3	369.4	748.4	738.8	–1.28	187.1	184.7	–1.28	83.16	82.09	–1.28
E-glass/ Epoxy	1	520.9	1049.5	1042	–0.742	262.4	260.4	–0.742	116.6	115.7	–0.742
	2	308.9	622.1	617.8	–0.693	155.5	154.5	–0.693	69.12	68.65	–0.693
	3	262.5	528.4	525.0	–0.645	132.1	131.3	–0.640	58.71	58.34	–0.645

Table 9 Modeling of $[\pm45]_{40}$ graphite/epoxy laminates from $[\pm45]_n$ laminates (400×400 mm^2 plates)

Value of n In $[\pm45]_n$	Stiffness Scaling Factor C_{Aij} and C_{Bij} : C_{Dij}	Theoretical fundamental frequencies of $[\pm45]_n$ (rad/s)	Similitude fundamental frequencies of $[\pm45]_{40}$ using $[\pm45]_n$ as model and $C_{stiff} = C_{Dij}$	% Disc. Comparing to Theoretical Sol. of 2422.4 rad/s
1	40 : 64000	41.07	1643	–32.2
2	20 : 8000	112.7	2253	–6.98
3	13.33 : 2370	176.2	2349	–3.03
4	10 : 1000	238.2	2382	–1.69
5	8 : 512	299.6	2397	–1.07
6	6.67 : 296.3	360.7	2405	–0.736
7	5.71 : 186.6	421.7	2409	–0.536
8	5 : 125	482.5	2413	–0.406
9	4.44 : 87.79	543.3	2415	–0.317
10	4 : 64	604.1	2416	–0.254
15	2.67 : 18.96	907.5	2420	–0.103
20	2 : 8	1211	2421	–0.051
25	1.6 : 4.096	1514	2422	–0.026
30	1.33 : 2.370	1817	2422	–0.013
35	1.14 : 1.493	2120	2422.3	–0.0052
40 (Prototype)	-	2422.4	-	-

laminates, this group of stiffness scaling factors does not yield a unique C_b that satisfies the similarity conditions in Eq. (27). The concept of partial similitude is adopted by relaxing the requirement and selecting C_{Dij} as C_{stiff}. The scaling law yields the similitude fundamental frequencies which are less than 1.5% different from the theoretical solutions. Once again, the percent discrepancies are independent of the geometric scaling factors, C_b. Thus, only small discrepancies are induced for angle-ply model-prototype pairs which satisfy the similarity requirements in Eqs. (25-26), but does not satisfy the requirement in Eq. (27).

Table 9 presents a study for the angle-ply laminates similar to those of the cross-ply laminates in Table 5. Several $[\pm45]_n$ laminated plates are employed as a model to simulate the $[\pm45]_{40}$ plate. It is notice that, like the case of cross-ply plate, the stiffness scaling factors are separated into two groups. The percent discrepancy is less than 2% if the model is at least eight plies. The plot of the percent discrepancy, which is similar to that of the cross-ply case, is shown and compared to the cross-ply case in Fig. 1.

7.2.2 Distortion in material properties

The effects of distortion in material properties for angle-ply laminates are presented in Tables 10 and 11. Table 10 shows the stiffness scaling factors of each model-prototype pair. It is noticed that the degree of non-uniformity of the C_{Dij} is less pronounced comparing to the case of cross-ply plates. Because of the more uniform flexural stiffness scaling factors, the partial similitude model in this case yields lower percentage of discrepancy. Therefore, a model with different material properties can be used to simulate the behavior of the prototype. The degree of discrepancy can be

Table 10 Stiffness scaling factors (C_{Aij}, C_{Bij}, C_{Dij}) of various prototypes

	Stiffness of model	Stiffness scaling factors (C_{Aij}, C_{Bij}, C_{Dij}) of prototype			
	E-glass/epoxy, $[45/-45]_4$	Graphite/epoxy, $[45_2/-45_2]_4$	Graphite/epoxy, $[45/-45]_4$	Kevlar/epoxy, $[45_2/-45_2]_4$	Kevlar/epoxy, $[45/-45]_4$
A_{11}	17.39	5.002	2.501	2.776	1.388
A_{12}	8.982	7.131	3.566	4.439	2.219
A_{22}	17.39	5.002	2.501	2.776	1.388
A_{66}	10.97	6.402	3.201	3.668	1.834
B_{16}	–0.4964	15.83	3.957	9.344	2.336
B_{26}	–0.4964	15.83	3.957	9.344	2.336
D_{11}	1.496	20.01	2.501	11.10	1.388
D_{12}	0.7726	28.52	3.566	17.75	2.219
D_{22}	1.496	20.01	2.501	11.10	1.388
D_{66}	0.9438	25.61	3.201	14.67	1.834

Note : A_{ij} in GPa-mm, B_{11} in GPa-mm^2, D_{ij} in GPa-mm^3

Table 11 Modeling of angle-ply graphite/epoxy and Kevlar/epoxy laminates from E-glass/epoxy laminates

Aspect Ratio	Fundamental frequency in rad/s of $[45/-45]_4$, E-glass/epoxy (b = 200 mm)	%Discrepancy of similitude fundamental frequency compared to closed-form solutions			
		Graphite/epoxy $[45_2/-45_2]_4$ (b = 400 mm)	Graphite/epoxy $[45/-45]_4$ (b = 200 mm)	Kevlar/epoxy $[45_2/-45_2]_4$ (b = 400 mm)	Kevlar/epoxy $[45/-45]_4$ (b = 200 mm)
1	520.9	–0.525	–0.525	–0.294	–0.294
1.5	368.4	0.438	0.438	0.872	0.872
2	308.9	2.03	2.03	2.82	2.82
2.5	279.3	3.48	3.48	4.60	4.60
3	262.5	4.64	4.64	6.04	6.04

kept at minimum by selecting a model-prototype pair with the most uniform flexural stiffness scaling factors.

7.2.3 Distortion in fiber angles

The possibility of using the model with different fiber angles from that of the prototype is studied and shown in Tables 12 and 13. Laminated plates with $[\pm 45]_4$ stacking sequence are employed to model $[\pm\theta]_4$ laminates, where $\theta = 15°$, 30°, or 60°. Similar to the previous studies, Table 12 shows the stiffness scaling factors for all model-prototype pairs. It is observed that C_{D11} of the $[\pm 15]_4$ and $[\pm 30]_4$ prototypes and C_{D22} of the $[\pm 60]_4$ prototypes are relatively higher than other flexural stiffness scaling factors. From the previous case experience, this highly non-uniform stiffness scaling factors suggest that this distorted model should not yield good results. The hypothesis is confirmed in the studied as shown in Table 13. The range of similitude discrepancies are between –19.3% and 67.9% in all cases of study. So, the partial similitude with distortion in fiber angle is not recommended.

Table 12 Stiffness scaling factors (C_{Aij}, C_{Bij}, C_{Dij}) of graphite plates with different fiber angles

	Stiffness of model, $[\pm 45]_4$	Stiffness scaling factors (C_{Aij}, C_{Bij}, C_{Dij})		
		$[\pm 15]_4$	$[\pm 30]_4$	$[\pm 60]_4$
A_{11}	43.51	2.738	1.880	0.4579
A_{12}	32.0	0.3120	0.7707	0.7707
A_{22}	43.51	0.2751	0.4579	1.880
A_{66}	35.12	0.3726	0.7909	0.7909
B_{16}	–1.964	0.9113	1.277	0.4547
B_{26}	–1.964	0.0887	0.4547	1.277
D_{11}	3.742	2.738	1.880	0.4579
D_{12}	2.755	0.3120	0.7707	0.7707
D_{22}	3.742	0.2751	0.4579	1.880
D_{66}	3.021	0.3726	0.7909	0 .7909

Note : A_{ij} in GPa-mm, B_{11} in GPa-mm^2, D_{ij} in GPa-mm^3

Table 13 Modeling of $[\pm\theta]_4$ graphite/epoxy plates from $[\pm 45]_4$ plates (b = 200 mm)

Aspect Ratio	Theoretical Solution of $[\pm 45]_4$ Model (rad/s)	Prototype, $[\pm 15]_4$			Prototype, $[\pm 30]_4$			Prototype, $[\pm 60]_4$		
		Theory (rad/s)	Similitude (rad/s)	% Disc	Theory (rad/s)	Similitude (rad/s)	% Disc	Theory (rad/s)	Similitude (rad/s)	% Disc
1	952.6	802.1	915.9	14.2	904.5	940.6	3.99	904.5	940.6	3.99
1.5	667.3	462.6	641.6	38.7	579.1	658.9	13.8	700.0	658.9	–5.88
2	550.8	344.6	529.6	53.7	451.5	543.8	20.5	619.8	543.8	–12.3
2.5	491.1	491.1	472.2	–3.86	386.7	484.9	25.4	580.3	484.9	–16.4
3	456.4	261.4	438.8	67.9	348.9	450.6	29.1	558.1	450.6	–19.3

8. Conclusions

The authors have employed the similitude transformation to the governing equations to establish the similitude invariants and the scaling laws for the vibration response of the anti-symmetrically laminated plates. For all cases of complete similitude, the predicted fundamental frequencies of the prototypes using the data of the models from theory substitute into the scaling laws have shown exact agreement with the theoretical results of the prototypes.

In certain situation, it is not feasible to fulfill complete similarity, then one might relax certain conditions in the scaling laws to enable the construction of the test model or to economize the costly experiment. However, employing partial similitude is allowed, providing that the complete set of similitude criteria is known on the basis of the mathematical model and the error caused by disregarding certain criterion can be assessed theoretically beforehand. Otherwise, complete similitude must be fulfilled. By using distortion model in stacking sequences, the results show very good agreement when the number of plies is equal to or greater than eight. The distorted models in laminate material properties also show the capability of predicting the frequencies of the prototypes. This means that the scaling laws allow the flexibility in choosing cheaper materials for the models, if selected properly. However, the distortion model in fiber angle does not yield good agreement between similitude and theoretical solutions because of the highly non-uniform stiffness scaling factors. Thus, the model with different fiber angle from the prototype should not be used in the experiments.

During the process of deriving the scaling law, it is noticed that the boundary condition is not limited to a particular condition. This implies that the derived scaling law is applicable to all boundary conditions because the effects of the boundary conditions will be included in the test data of the models. Thus, it is able to design an experiment to simulate actual application on a model and predict the behavior of the prototype from the scaling laws rather than to simulate the theoretically available boundary conditions. The merit of the method is very beneficial to other types of structural geometry whose available solutions are not capable of predicting the test results accurately enough. For example, by conjecturing that the imperfection effect is the same for both prototype and model, one can simulate the sensitivity of imperfection effect of cylindrical shells subjected to axial compression. The scaling law can also be applied to verify the accuracy of the numerical methods such as the finite element and boundary element methods.

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Notation

a : plate length
A_{ij} : laminate extensional stiffnesses
b : plate width
B_{ij} : laminate coupling stiffnesses
$\boldsymbol{C}$: similitude scaling factor matrix

C_i	: similitude scaling factors
D_{ij}	: laminate flexural stiffnesses
E_{ij}	: Young's moduli of elasticity
h	: total laminate thickness
$L(..)$	: differential or algebraic operator
M_{ij}	: variation of moment resultant
N_{ij}	: variation of stress resultant
m	: model
i, j	: number of half waves
p	: prototype
$\overline{Q}_{ij}$	: transformed reduced stiffness coefficient
R	: plate aspect ratio, $a/b \geq 1$
ρ	: mass density
ω	: natural frequencies of free vibration
u, v, w	: variation of middle surface displacements
$\boldsymbol{X_m}$	: vector of model variables
$\boldsymbol{X_p}$	: vector of prototype variables
x, y	: rectangular coordinates
z_k	: z-coordinate of the bottom of k^{th} layer