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Transient linear elastodynamic analysis in time domain based on the integro-differential equations

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Abstract. A finite element formulation for the time-domain analysis of linear transient elastodynamic problems is presented based on the weak form obtained by applying the Galerkin's method to the integro-differential equations which contain the initial conditions implicitly and does not include the inertia terms. The weak form is extended temporally under the assumptions of the constant and linear time variations of field variables, since the time-stepping algorithms such as the Newmark method and the Wilson θ-method are not necessary, obtaining two kinds of implicit finite element equations which are tested for numerical accuracy and convergency. Three classical examples having finite and infinite domains are solved and numerical results are compared with the other analytical and numerical solutions to show the versatility and accuracy of the presented formulation.

Key words: transient; elastodynamic; time-integral formulation; wave diffraction; finite element.

1. Introduction

Elastodynamics is one of the main subjects which attract a number of researchers in engineering applications. To analyze the elastodynamic problems, numerical tools such as the FEM(finite element method) (Bathe 1996, Hughes 1987) and the BEM(boundary element method) (Banerjee 1994, Dominguez 1993) are inevitably necessary. It is said that the BEM is comparatively suited to the problems with infinite and semi-infinite domains and requires the reduced dimensionality, but less versatile in handling the complex geometry, inhomogeneity, and complicated material behaviour than the FEM.

Until now, many works of the elastodynamics by those methods have been reported in numerous literatures (Belytschko and Hughes 1983, Argyris and Mlejnek 1991, Harari *et al*. 1996, Beskos 1997) mainly for the purpose of improving the numerical accuracy and stability, which have been analyzed in the time domain, the Laplace domain, and the Fourier domain (Bedford and Drumheller 1996, Manolis 1983).

In the FE analysis of linear transient elastodynamic problems in time domain, the FE formulations have been traditionally derived based on the principle of virtual displacements or the weak form of the governing differential equations(the Galerkin's method) (Reddy 1993). As an alternative to the

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traditional methods, there are variational principles in time such as the Hamilton's principle (Achenbach 1975, Washizu 1975) and the Gurtin's variational principle (Gurtin 1964, Oden and Reddy 1976, Zienkiewicz 1991).

In this paper, the simplified time-domain FE formulation for linear transient elastodynamic analysis is presented by using the modified Gurtin's variational principle, where the weak(or variational) form is obtained by applying the Galerkin's method to the integro-differential equations which are equivalent to the governing differential equations. The resulting weak form includes only one convolution operator while the variational form obtained by taking the first variation of the Gurtin's variational functional includes two convolution operators. Contrary to the traditional elastodynamic FE formulations, the presented weak form does not include the inertia terms so that the time-stepping algorithms such as the Newmark method and the Wilson θ-method (Bathe 1996) are not necessary. The weak form is approximated temporally using the globally constant and linear time interpolation functions on the discretized time axis (Israil and Banerjee 1990, Wang *et al*. 1997), resulting in the implicit FE equations taking account of the past dynamic history. But the time-stepping procedures of former elastodynamic applications by the Gurtin's variational principle used the quadratic interpolation (Nickell and Sackman 1968) and the Hermitian interpolation in time (Atluri 1973) within a (local) time step, resulting in the explicit FE equations where the current results are used as the initial values of next time step.

To show the accuracy and versatility of the proposed method, classical three examples about plane wave, cylindrical wave, and wave diffraction with finite and infinite domains are solved and numerical results are compared with the analytical and numerical solutions by other researchers.

2. Weak formulation

The governing equations for the linear elastodynamic analysis of an elastic body are given as follows:

i) Equations of motion

$$\sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i \tag{1}$$

where σ_{ij} is the stress tensor, ρ the mass density, f_i the body force vector per unit mass, u_i the displacement vector, x_i the position vector, t the time variable, $\sigma_{ij,j} \equiv \partial\sigma_{ij}/\partial x_j$ the partial derivative with respect to the spatial variable, and $\ddot{u}_i \equiv \partial^2 u/\partial t^2$ the acceleration.

ii) Strain-displacement relations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

where ε_{ij} is the small strain tensor.

iii) Stress-strain relations

$$\sigma_{ij} = D_{ijkl}\varepsilon_{kl} \tag{3}$$

where the material is assumed homogeneous and isotropic and D_{ijkl} is the elasticity tensor.

Boundary and initial conditions are given by

$$u_i = \hat{u}_i \text{ on } \Gamma_u, \quad t_i = \hat{t}_i \text{ on } \Gamma_t$$
$$u_i = u_{0i}, \quad \dot{u}_i = \dot{u}_{0i} \quad \text{at } t = 0 \tag{4}$$

where t_i is the traction which satisfies the Cauchy's stress formula $t_i \equiv \sigma_{ij} n_j$, Γ_u and Γ_t are the portions of the boundary ($\Gamma = \Gamma_u + \Gamma_t$) where the displacement and traction are specified as $\hat{u}_i$ and $\hat{t}_i$, respectively, and u_{0i} and $\dot{u}_{0i}$ are the prescribed initial values.

Application of the Laplace transform to the equations of motion (1) yields

$$\bar{\sigma}_{ij,j} + \rho \bar{f}_i = \rho\{-\dot{u}_i(0) - s u_i(0) + s^2 \bar{u}_i\} \tag{5}$$

where s is the Laplace transform parameter, $\dot{u}_i(0)$ the initial velocity and $u_i(0)$ the initial displacement, and the Laplace transform of a function $u(\boldsymbol{x}, t)$ is defined by

$$L(u) = \bar{u}(\boldsymbol{x}, s) = \int_0^\infty u(\boldsymbol{x}, t) e^{-st} dt \tag{6}$$

where $\boldsymbol{x}$ is the position vector, and $u(\boldsymbol{x}, t) = 0$ for $t < 0$.

Division of Eq. (5) by s^2 yields

$$\frac{1}{s^2}\bar{\sigma}_{ij,j} + \frac{1}{s^2}\rho \bar{f}_i = \rho\left\{-\frac{1}{s^2}\dot{u}_i(0) - \frac{1}{s}u_i(0) + \bar{u}_i\right\} \tag{7}$$

By the inverse transformation of Eq. (7) into the time domain, we obtain

$$t * \sigma_{ij,j} + t * \rho f_i = \rho\{-t\dot{u}_i(0) - u_i(0) + u_i\} \tag{8a}$$

or

$$g * \sigma_{ij,j} + g * \rho f_i - \rho\{-t\dot{u}_i(0) - u_i(0) + u_i\} = 0 \tag{8b}$$

where $g = g(t) = t$, and the operator $*$ means the convolution of two functions of space and time u and v which is defined by

$$u * v = \int_0^t u(\boldsymbol{x}, t - \tau) v(\boldsymbol{x}, \tau) d\tau \tag{9}$$

The integro-differential equations Eq. (8) which contain the initial conditions implicitly and no inertia terms are equivalent to the original differential equations Eq. (1), and known as the Euler equations of the Gurtin's variational functional (Gurtin 1964, Oden and Reddy 1976).

In this paper, the weak formulation for a transient time-domain finite element analysis is constructed by applying the Galerkin's method to Eq. (8) instead of Eq. (1) traditionally adopted. Then,

$$\int_\Omega [g * \sigma_{ij,j} + \rho\{g * f_i + (t\dot{u}_{0i} + u_{0i})\} - \rho u_i] \delta u_i d\Omega = 0 \tag{10}$$

where Ω represents the spatial domain.

By the application of the Gauss' theorem and Cauchy's stress formula and employing the relations $\sigma_{ij}\delta u_{i,j} = \sigma_{ij}\delta\varepsilon_{ij}$ and arranging, the first term of Eq. (10) can be written as

$$\int_\Omega g * \sigma_{ij,j} \delta u_i d\Omega = \int_\Gamma g * t_i \delta u_i d\Gamma - \int_\Omega g * \sigma_{ij} \delta\varepsilon_{ij} d\Omega \tag{11}$$

Substituting Eq. (11) into Eq. (10) and rearranging, the weak form for a time-domain finite element analysis of transient linear elastodynamic problems is obtained as

$$\int_\Omega g*\sigma_{ij}\delta\varepsilon_{ij}d\Omega + \int_\Omega \rho u_i\delta u_i d\Omega - \int_\Omega \rho(t\dot{u}_{0i} + u_{0i})\delta u_i d\Omega$$
$$= \int_\Gamma g*t_i\delta u_i d\Gamma + \int_\Omega g*\rho f_i\delta u_i d\Omega \tag{12}$$

For reference, the traditional weak formulation (Achenbach 1975) based on the equations of motion Eq. (1) is given by

$$\int_\Omega \sigma_{ij}\delta\varepsilon_{ij}d\Omega + \int_V \rho\ddot{u}_i\delta u_i d\Omega = \int_\Gamma t_i\delta u_i d\Gamma + \int_\Omega \rho f_i\delta u_i d\Omega \tag{13}$$

Note that inertia terms are disappeared in Eq. (12) in contrast to Eq. (13), so that the time-stepping algorithms such as the Wilson θ-method and the Newmark method (Bathe 1996) to approximate the acceleration and velocity are not necessary, instead only the time interpolation functions to approximate the dependent variables(displacement and stress) in the discretized time axis are required, resulting in simplified numerical procedure comparing to the traditional formulation. Also Eq. (12) is an implicit time-integral formulation since the displacements at a time t are calculated by taking account of the past dynamic history up to and including the time t (see Eq. (24)). But formerly they (Nickell and Sackman 1968, Atluri 1972) solved it using the time-stepping procedure where the current results are used as the initial values of next time step.

3. Finite element approximation

The finite element equations for the transient linear elastodynamics are derived based on Eq. (12) under the assumptions of homogeneous initial conditions and no body forces. Then,

$$\int_\Omega g*\sigma_{ij}\delta\varepsilon_{ij}d\Omega + \int_\Omega \rho u_i\delta u_i d\Omega = \int_\Gamma g*t_i\delta u_i d\Gamma \tag{14}$$

Now, Eq. (14) is extended temporally under two kinds of assumptions, i.e., the constant and linear time variations, on the discretized time axis, which method is commonly adopted in the elastodynamic BEM formulations (Israil and Banerjee 1990, Wang *et al.* 1997).

3.1 Constant time variation

In the development of Eq. (14), it is first assumed that the time axis is divided equally and the dependent variables remain constant during a time step. Then, the stresses may be approximated by linear combinations of the spatial and time functions as follows.

$$\sigma_{ij}(\boldsymbol{x}, t) = \sum_{n=1}^{N}\Phi_n(t)\sigma_{ij}^n(\boldsymbol{x}) \tag{15}$$

where n is the arbitrary time node, N the current time node, $\sigma_{ij}^n(\boldsymbol{x})$ the spatial distributions of the stress at an arbitrary time t_n, and $\Phi_n(t)$ the global time interpolation functions defined by

$$\begin{aligned} \Phi_n(t) &= 1, \quad t_{n-1} \le t \le t_n \\ \Phi_n(t) &= 0, \quad \text{otherwise} \end{aligned} \tag{16}$$

where $t_n = n\Delta t$ ($n = 1, 2, 3, ..., N$).

Then, the convolution in the first term on the left-hand side of Eq. (14) can be integrated analytically as follows:

$$\begin{aligned} g*\sigma_{ij} &= \int_0^t g(t-\tau)\sigma_{ij}(\tau)d\tau = \int_0^t (t-\tau)\sum_{n=1}^{N}\Phi_n(\tau)\sigma_{ij}^n(\boldsymbol{x})d\tau \\ &= \sum_{n=1}^{N}\sigma_{ij}^n(\boldsymbol{x})\int_0^t (t-\tau)\Phi_n(\tau)d\tau \\ &= \sum_{n=1}^{N}\sigma_{ij}^n(\boldsymbol{x})\int_{t_{n-1}}^{t_n}(t-\tau)d\tau \\ &= \sum_{n=1}^{N}\sigma_{ij}^n(\boldsymbol{x})\left(N-n+\frac{1}{2}\right)\Delta t^2 \end{aligned} \tag{17}$$

Substitution of Eq. (17) into Eq. (14) yields

$$\sum_{n=1}^{N}\left(N-n+\frac{1}{2}\right)\Delta t^2\int_\Omega \sigma_{ij}^n(\boldsymbol{x})\delta\varepsilon_{ij}d\Omega + \int_\Omega \rho u_i\delta u_i d\Omega = \int_\Gamma g*t_i\delta u_i d\Gamma \tag{18}$$

The convolution on the right-hand side of Eq. (18) can be also integrated analytically if the tractions are expressed or approximated in terms of the time variable. Here we assume that tractions are applied suddenly and kept constant in time. i.e.,

$$t_i(\boldsymbol{x}, t) = \hat{t}_i(\boldsymbol{x})H(t) \tag{19}$$

where $H(t)$ is a Heaviside step function.
Then, the right-hand side of Eq. (18) becomes

$$\int_\Gamma g*t_i\delta u_i d\Gamma = \frac{t^2}{2}\int_\Gamma \hat{t}_i(\boldsymbol{x})\delta u_i d\Gamma \tag{20}$$

Substituting Eq. (20) into Eq. (18), the final system of linear algebraic equations is obtained as

$$\begin{aligned} &\frac{1}{2}\Delta t^2\int_\Omega [B]^T[D][B]d\Omega\{u_i\}^N + \int_\Omega \rho[N]^T[N]d\Omega\{u_i\}^N \\ &= \frac{t^2}{2}\int_\Gamma [N]^T\hat{t}_i(\boldsymbol{x})d\Gamma - \Delta t^2\sum_{n=1}^{N-1}\left(N-n+\frac{1}{2}\right)\int_\Omega [B]^T[D][B]d\Omega\{u_i\}^n \end{aligned} \tag{21}$$

where $[N]$ is the shape function, $[B]$ the strain-displacement and $[D]$ the elasticity matrix, and these notations may be referred in Zienkiewicz (1991).

After simplifying Eq. (21), the FE equations for the case of the constant time variation are obtained as follows:

$$\left[\frac{1}{2}\Delta t^2[K] + [M]\right]\{u_i\}^N = \frac{t^2}{2}\{F\} - \frac{1}{2}\Delta t^2[K]\sum_{n=1}^{N-1}(2N-2n+1)\{u_i\}^n \tag{22}$$

where $[K]$ is the stiffness matrix, $[M]$ the mass matrix, $\{F\}$ and the force vector:

$$[K] = \int_\Omega [B]^T[D][B]d\Omega, \quad [M] = \int_\Omega \rho[N]^T[N]d\Omega, \quad \{F\} = \int_\Gamma [N]^T\hat{t}_i(\boldsymbol{x})d\Gamma \tag{23}$$

Eq. (22) can be expressed in abbreviated form as

$$[\bar{K}]\{u_i\}^N = \{\bar{F}(t)\} + \{\bar{R}\}^{N-1} \tag{24}$$

where $\{\bar{R}\}^{N-1}$ represents the effect of past dynamic history on the current time node, and

$$[\bar{K}] = \left[\frac{1}{2}\Delta t^2[K] + [M]\right], \quad \{\bar{F}(t)\} = \frac{t^2}{2}\{F\}, \quad \{\bar{R}\}^{N-1} = -\frac{1}{2}\Delta t^2[K]\sum_{n=1}^{N-1}(2N-2n+1)\{u_i\}^n$$

In Eq. (24), the coefficient matrix of the displacement vector $[\bar{K}]$ is constant so that the computational work at every time step is just computing the traction load $\{\bar{F}(t)\}$ with $t = N\Delta t$ and updating the hereditary effect $\{\bar{R}\}^{N-1}$.

After solving Eq. (24) for the displacements, the stresses are obtained by

$$\{\sigma\} = [D]\{\varepsilon\} = [D][B]\{\delta\} \tag{25}$$

where $\{\delta\}$ is the nodal displacement vector.

3.2 Linear time variation

When the dependent variables are assumed to vary linearly during a time step, the stresses may be approximated as follows.

$$\sigma_{ij}(\boldsymbol{x}, t) = \sum_{n=1}^{N}\{\phi_1(t)\sigma_{ij}^{n-1}(\boldsymbol{x}) + \phi_2(t)\sigma_{ij}^{n}(\boldsymbol{x})\}(H(t-t_{n-1}) - H(t-t_n)) \tag{26}$$

where $\phi_1(t)$ and $\phi_2(t)$ are the local time interpolation functions at an arbitrary time step and defined by

$$\phi_1(t) = \frac{t_n - t}{\Delta t}, \quad \phi_2(t) = \frac{t - t_{n-1}}{\Delta t} \quad \text{for} \quad t_{n-1} \le t \le t_n$$

$$\phi_1(t) = 0, \quad \phi_2(t) = 0 \qquad \text{otherwise} \tag{27}$$

Then, the convolution in the first term on the left-hand side of Eq. (14) can be integrated analytically using Eq. (26) as before:

$$
\begin{aligned}
g*\sigma_{ij} &= \int_0^t g(t-\tau)\sigma_{ij}(\tau)d\tau \\
&= \int_0^t (t-\tau)\sum_{n=1}^{N}\{\phi_1(t)\sigma_{ij}^{n-1}(\boldsymbol{x}) + \phi_2(t)\sigma_{ij}^{n}(\boldsymbol{x})\}(H(t-t_{n-1})-H(t-t_n))d\tau \\
&= \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(t-\tau)\left\{\frac{t_n-\tau}{\Delta t}\sigma_{ij}^{n-1}(\boldsymbol{x}) + \frac{\tau-t_{n-1}}{\Delta t}\sigma_{ij}^{n}(\boldsymbol{x})\right\}d\tau \\
&= \frac{1}{\Delta t}\sum_{n=1}^{N}\left\{\sigma_{ij}^{n-1}(\boldsymbol{x})\int_{t_{n-1}}^{t_n}(t-\tau)(t_n-\tau)d\tau + \sigma_{ij}^{n}(\boldsymbol{x})\int_{t_{n-1}}^{t_n}(t-\tau)(\tau-t_{n-1})d\tau\right\} \\
&= \Delta t^2\sum_{n=1}^{N}\{\sigma_{ij}^{n-1}(\boldsymbol{x})A(n,N) + \sigma_{ij}^{n}(\boldsymbol{x})B(n,N)\}
\end{aligned}
\tag{28}
$$

where $A(n, N) = \frac{1}{6}[3N-3n+2]$, $B(n, N) = \frac{1}{6}[3N-3n+1]$.

Substituting Eqs. (28) and (20) into Eq. (14), the FE equations for the case of the linear time variation are obtained as follows:

$$\left[\frac{1}{6}\Delta t^2[K] + [M]\right]\{u_i\}^N = \frac{t^2}{2}\{F\} - \Delta t^2[K]\sum_{n=1}^{N-1}(N-n)\{u_i\}^n \tag{29}$$

Note that two equations for dynamic analysis (22) and (29) are similar to the FE equations for static analysis except updating the past dynamic history. Thus, The FE program for dynamic analysis can be constructed only with small modifications of the FE code for static analysis.

4. Numerical examples

In order to show the accuracy and versatility of the presented method three examples with finite and infinite domains are solved, and computed results at some positions are compared with the other numerical results by the FEM or BEM and with the analytical solutions. In discussing the numerical results, the normalized time steps β_i are used and defined by

$$\beta_i = \frac{c_i\Delta t}{l_i} \tag{30}$$

where c_i is the dilatational or P-wave velocity, Δt the time step and l_i the characteristic length.

4.1 Elastic bar subjected to a sudden uniform load

A rectangular elastic bar with one end fixed is subject to a sudden uniform load $\sigma_0 H(t)$ at the free end. For numerical analysis, the bar($L/a = 4$) is discretized by three-dimensional isoparametric linear finite elements as shown in Fig. 1.

Numerical implementations were performed, respectively, for six kinds of meshes which consist of 2, 4, 8, 16, 32 and 64 equally divided finite elements and for three time steps such as $\beta_1 = c_1\Delta t/l_1 = 0.25, 0.5,$ and 1.0($c_1 = \sqrt{E/\rho}$, l_1 = element length, the Poisson's ratio $\nu = 0$) to

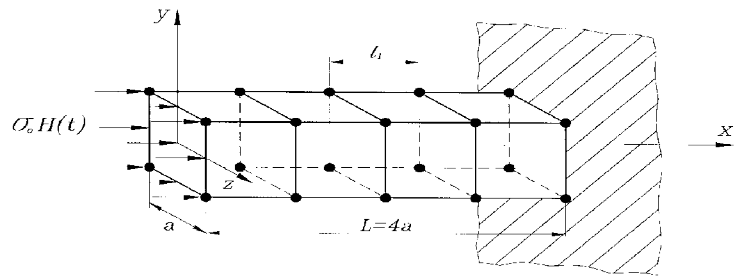

Fig. 1 Elastic rectangular bar subjected to sudden uniform load

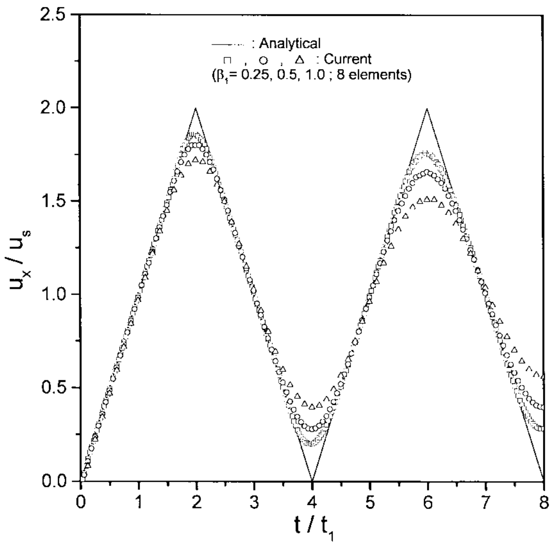

Fig. 2 Axial displacements at the free end for constant time variation

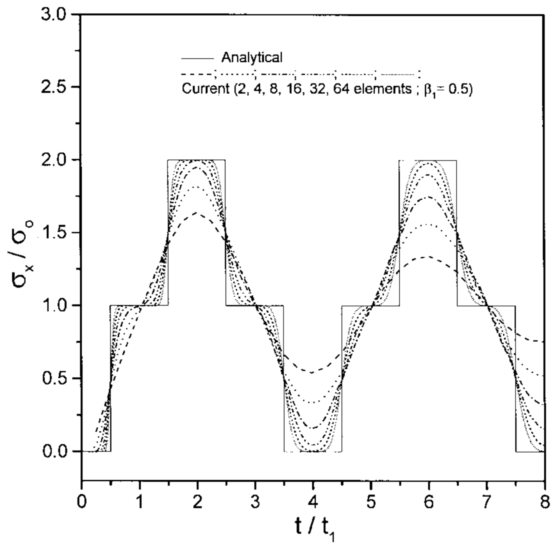

Fig. 3 Normal stresses at the mid-length for constant time variation

examine the numerical accuracy and convergence of the presented formulation under the assumptions of the constant and linear time variations.

In Figs. 2 and 3, the axial displacements at the free end and the normal stresses at the mid-length are depicted normalized with respect to the static solution ($u_s = \sigma_0 L/E$) and uniform external load (σ_0), respectively, and compared with the analytical solution (Miles 1961) for the case of the constant time variation, where the characteristic time t_1 is defined as $t_1 = L/c_1$. It is shown that numerical results always converge closely and consistently to the analytical solution as the time step goes smaller and/or the mesh becomes finer. Numerical results of Fig. 2 were obtained using the mesh with 16 finite elements and those of Fig. 3 were obtained using the time step $\beta_1 = 0.5$, and so more accurate results can be obtainable using the finer mesh and the smaller time step, respectively. For the large time step or the coarse mesh, the numerical results approach to the static solution after a long time elapses because of the cumulative errors due to numerical damping.

The same arguments as above were recomputed for the case of the linear time variation and best results were obtained for the time step $\beta_1 = 1.0$ and meshes with 2, 4 and 8 finite elements. But the numerical divergence was observed for some other time steps and this tendency is found also in

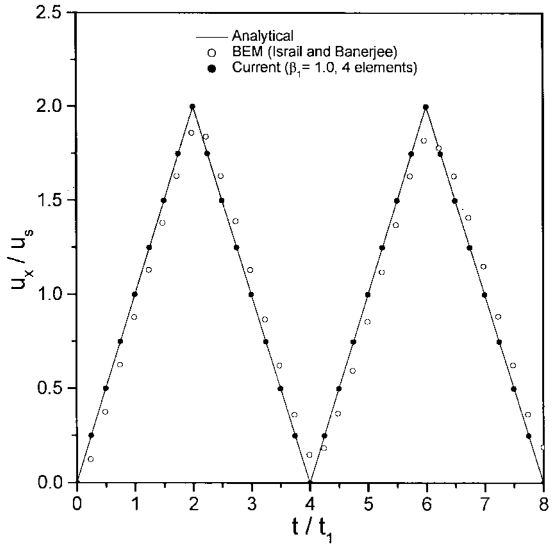

Fig. 4 Axial displacements at the free end for linear time variation

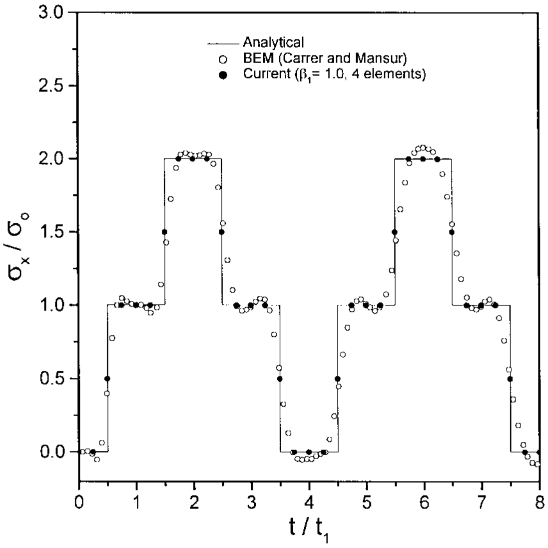

Fig. 5 Normal stresses at the mid-length for linear time variation

next examples. Numerical results shown in Figs. 4 and 5 were obtained using the mesh with 4 finite elements.

This kind of example has been treated by many researchers, but here the numerical results analyzed using the BEM, respectively, by Israil and Banerjee (1990) ($L/a = 2$, 12 quadratic boundary elements and $c_1\Delta t/l = 1.0$) and by Carrer and Mansur (1999) ($L/a = 2$, 24 linear boundary elements, $c_1\Delta t/l = 0.6$) are adopted for the purpose of comparisons.

4.2 Cylindrical cavity subjected to a sudden internal pressure

A cylindrical cavity with radius r_0 is imbedded in an infinitely extending medium, and subjected to a suddenly applied uniform pressure $p_0H(t)$. The FE model for numerical analysis is shown in Fig. 6, where capitalizing on the symmetry of the problem an infinite domain is approximated by one-quarter of the circular cylinder which consists of an artificial outer boundary $b/r_0 = 11$ and two-dimensional 200 isoparametric quadratic quadrilateral finite elements.

Chou and Keonig (1966) solved this problem using the method of characteristics under the condition of plane stress and there the material data of $\nu = 0.3$, $E = 205 \times 10^6$ kN/m^2 and $\rho = 7.85 \times 10^3$ kg/m^3 were used, which are also adopted in this computation.

Figs. 7 and 8 show the tangential and radial stresses at three radial points $r/r_0 = 1.0$, 2.0, and 3.4 with $\beta_2 = 0.25$ for the case of the constant time variation, where the cylindrical wave velocity is defined by $c_2 = [E/\rho(1-\nu^2)]^{1/2}$, $l_2 = r_0$ and the characteristic time $t_2 = l_2/c_2$. Good agreements are observed between two solutions as a whole, and it is shown also that the numerical results approach asymptotically to the static solution as the time goes further. But the peak stresses due to the arrival of the wave front are estimated a little lower than the analytical solution. Through the thorough examination on the convergence of numerical results due to the variations of the time step and mesh size, it was found that the presented formulation with the constant time variation brings on

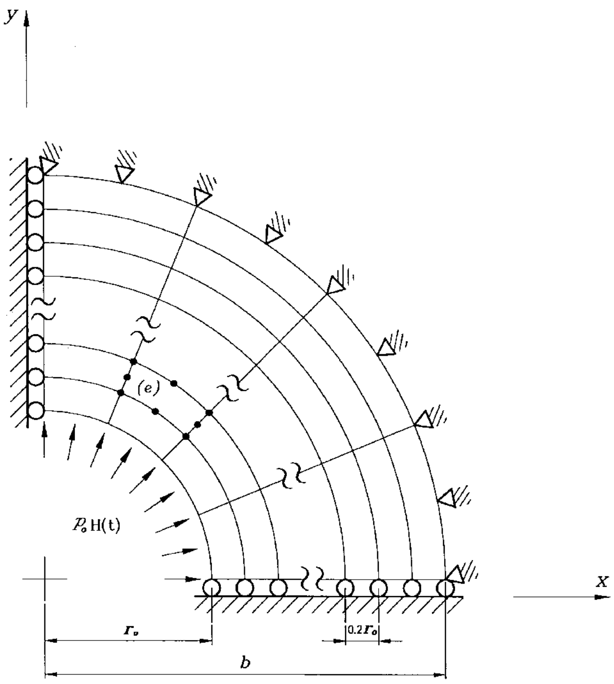

Fig. 6 Finite element mesh for cylindrical cavity subjected to sudden internal pressure

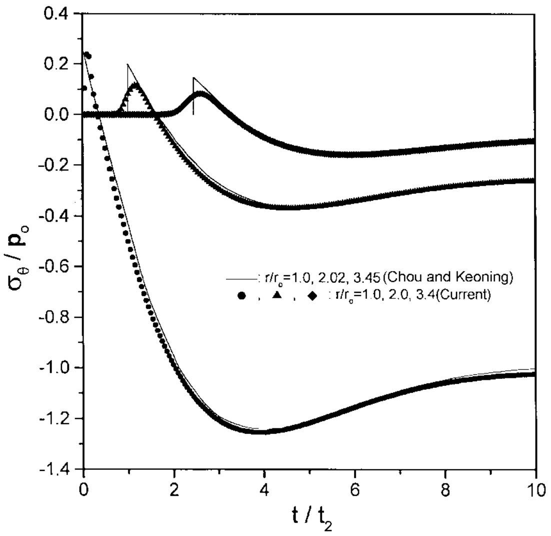

Fig. 7 Tangential stresses under sudden internal pressure for constant time variation

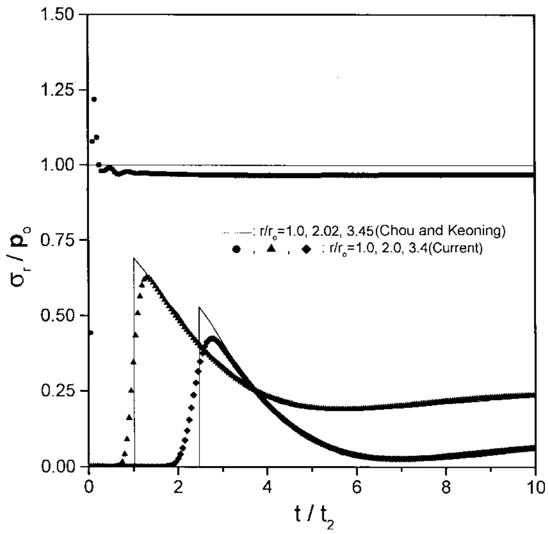

Fig. 8 Radial stresses under sudden internal pressure for constant time variation

unconditionally stable numerical results if too large time steps or coarse meshes are not used.

Numerical implementations were also performed for the case of the linear time variation, and it was observed that numerical results are converged only within the limits of $\beta_2 = 0.1 \sim 0.45$ for the

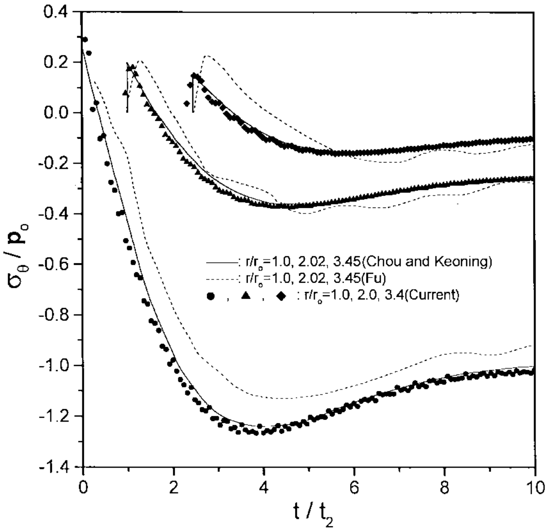

Fig. 9 Tangential stresses under sudden internal pressure for linear time variation

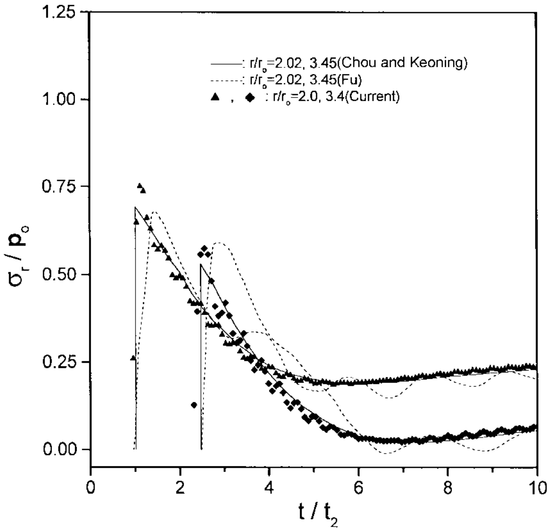

Fig. 10 Radial stresses under sudden internal pressure for linear time variation

mesh of Fig. 6. In general, the convergence depends on the time step, mesh size, element property, and the dimensionality of the problem. The numerical results shown in Figs. 9 and 10 were obtained using the time step of $\beta_2 = 0.4$ and are compared with the FEM results by Fu (1970), where the peak stresses are described better comparing to the case of the constant time variation.

The analytical solution for the tangential stress at the inner boundary decreases to a value of –1.24 and then increase to its static value of –1.0, while the current results –1.253 and –1.024, respectively.

4.3 Wave diffraction by a cylindrical cavity

A long cylindrical cavity in an infinite elastic medium is impinged upon by a compressional *P*-wave whose front is parallel to the axis of cavity. This problem is solved first analytically by Baron and Matthews (1961) and later Pao and Mow (1972), and numerically using the BEM by Manolis and Beskos (1981), Kobayashi (1987), etc.

For the FEM analysis, this problem is modeled as in Fig. 11, where only a half of an infinite domain is discretized by 238 isoparametric quadratic finite elements since the dynamic problem is symmetric about the *x*-axis only, and the artificial boundary is constructed far away from the circular cavity to avoid the undesirable reflection. In BEM analysis, only the boundary is discretized, and Manolis and Beskos (1981) found the stress distribution by superimposing the stress field produced by the *P*-wave in the medium without the hole to the stress field produced by the applications of corrective tractions on the boundary of the cavity in order to render the cavity surface traction free. But, in this paper the FE equations are solved directly without resorting to the superposition.

In Figs. 12, 13, and 14, the dynamic stress concentration factors σ_θ/σ_0 at the boundary of the cylindrical cavity are depicted for the polar angles of $\theta = 90^\circ$, 0°, and 180°, respectively, where the material data of $\nu = 0.25$, $E = 205 \times 10^6$ kN/m^2, and $\rho = 7.85 \times 10^3$ kg/m^3 are used in the computation. It appears that the BEM results follow the analytical solution by Pao and Mow (1972), while the

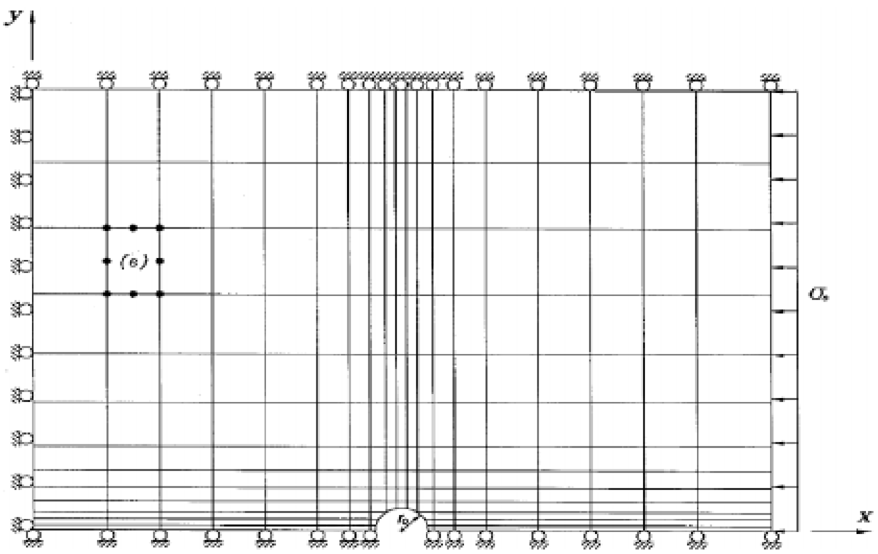

Fig. 11 Finite element mesh for wave diffraction by cylindrical cavity

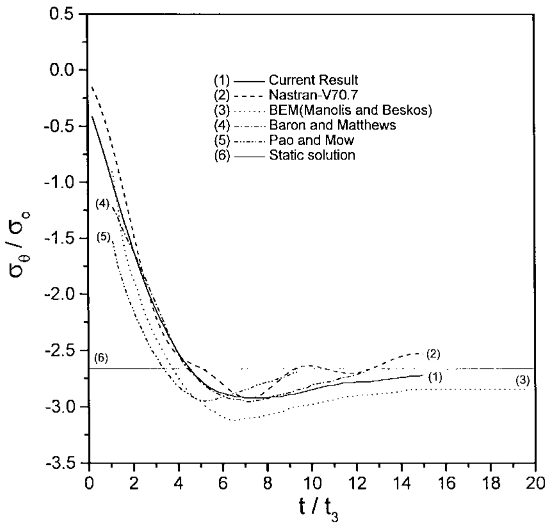

Fig. 12 Tangential stresses by wave diffraction at $\theta = 90^o$

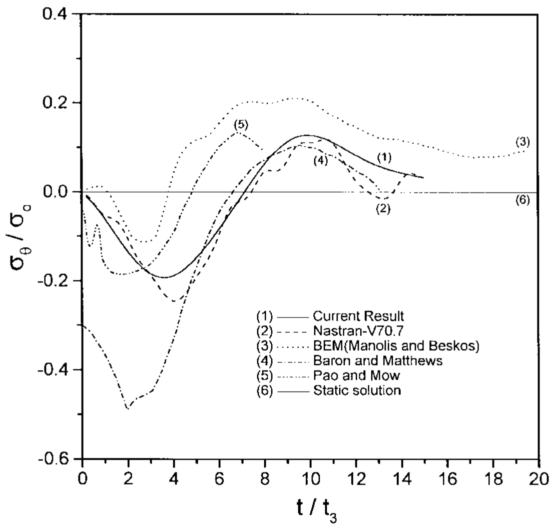

Fig. 13 Tangential stresses by wave diffraction at $\theta = 0^o$

current results follow the analytical solution by Baron and Matthews (1961). The discrepancies between them may be explained as the former was solved under the plane stress condition but the latter under the plane strain condition. Good agreements are shown between the current and Baron and Matthews' results, especially for $\theta = 90^o$, where the current results of the stress concentration factor varies from –2.92 at the bottom to the asymptotic static value –2.72, while those values of the analytical solution varies from –2.92 to –2.667, respectively. Current results shown in Figs. 12, 13, and 14 were obtained under the constant time variation with the parameters $c_3\Delta t / l_3 = 0.2$,

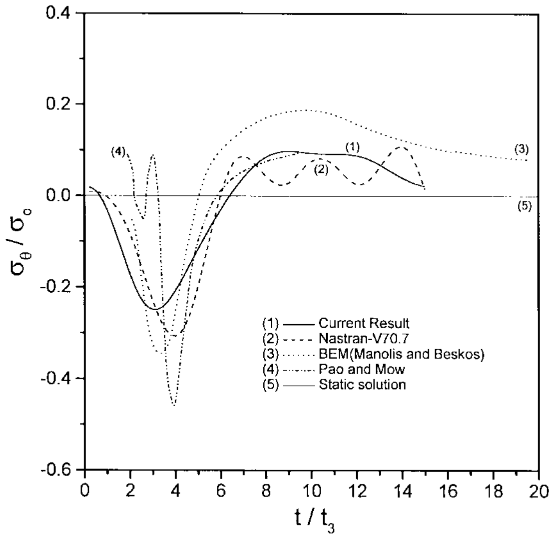

Fig. 14 Tangential stresses by wave diffraction at $\theta = 180^{\circ}$

$c_3 = [E(1-\nu)/\rho(1-\nu-2\nu^2)]^{1/2}$, $l_3 = r_0$, and the characteristic time $t_3 = l_3/c_3$.

The FE analysis by the NASTRAN which is a commercially available package for the general engineering analysis and which finite element formulations are based on the differential equations of motion has also been performed for the same mesh and time step, which results are shown in the previous figures for comparison. It is shown that they are deviated further from the analytical solution by Baron and Matthews (1961) than the current results and show a waving behaviour after some time passes on, but the current results approach stably to the static solution. It is conjectured that these good characteristics are originated from the presented time-integral formulation especially for the constant time variation, because in case of the linear time variation some oscillated results have been observed along those by the constant time variation, which are not plotted in the previous figures for brevity.

5. Conclusions

The linear transient elastodynamic problems have been analyzed in the time domain by using the FEM, for which the simplified FE formulation is presented based on the integro-differential equations which contain the initial conditions implicitly and does not include the inertia terms. It is extended temporally under the assumptions of the constant and linear time variations, and two kinds of implicit FE equations are derived and tested for the numerical accuracy and convergency of the proposed method. It is found that the time-stepping procedure under the constant time variation results in unconditionally stable numerical results while the time-stepping procedure under the linear time variation converges only within certain time step limits even though the latter approximation describes the abrupt jump better. Several examples with finite and infinite domains were solved successfully with good accuracy, being certified that the proposed method may become one of competitive methods for the numerical analysis of the transient linear elastodynamics.

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