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Dynamic response analysis for structures with interval parameters

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Abstract. In this paper, a new method to solve the dynamic response problem for structures with interval parameters is presented. It is difficult to obtain all possible solutions with sharp bounds even an optimum scheme is adopted when there are many interval structural parameters. With the interval algorithm, the expressions of the interval stiffness matrix, damping matrix and mass matrices are developed. Based on the matrix perturbation theory and interval extension of function, the upper and lower bounds of dynamic response are obtained, while the sharp bounds are guaranteed by the interval operations. A numerical example, dynamic response analysis of a box cantilever beam, is given to illustrate the validity of the present method.

Key words. interval extension of function; interval characters matrices; matrix perturbation theory; interval of dynamic response.

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1. Introduction

In engineering design it is important to calculate the response quantities such as the displacement, stress, and vibration frequencies, etc. to assess the integrity of a proposed structure, which are usually functions of design parameters. However the structure may have uncertainty caused by such as manufacture errors and errors in observation. On the basis of probability, the random analysis method has been developed in structural analysis early. The uncertainty of the structural parameters can be described by a random model. However, probabilistic modeling is not the only way to describe the uncertainty, and uncertainty is not equal to randomness. Indeed, the probabilistic approaches are not able to deliver the reliable results at the required precision without sufficient experimental data to validate made regarding the joint probability densities of the random variables or functions involved. At least the assumption of stochastic nature of uncertainty can not be accepted in following two cases: (1) when the volume of a priori experimental data on the nature of the uncertain factors is so lack that it does not allow the conclusion to be drawn on the availability of stable statistic

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characteristics; (2) when it is known a priori that the uncertainty basically can not be considered to be produced by some probabilistic mechanism.

The interval model is derived from the interval mathematics, in which the number is treated as interval variable with lower and upper bounds. The principal advantage of the interval analysis over the deterministic analysis is that it provides at once both an approximation to the solution and error bounds of the approximate solution. Recently, interval analysis method has been employed in structural analysis (Chen and Qiu 1994, Qiu *et al.* 1995, Chen and Yang 2000). Chen, Qiu and etc have used an interval set model in the studies of the static displacement and dynamic eigenvalue problems for structures with bounded uncertain parameters. So far, few works have been done on the response analysis of structures with interval parameters. In reference (Dimarogonas 1994) the author used the interval method on rotor dynamic response analysis. For a singular DOF system, the response can be obtained by replacing corresponding parameters with interval parameters in the original response equations. The interval evaluation is correct only if the function is monotonic with respect to the interval parameters, which hold back its extensive application in engineering. Further more, for a complex structure with more variables, it is impossible to use the direct extension method. Thus it is necessary to develop a new method for computing the response of structures with interval parameters.

In this paper, we will start with a brief review of the interval mathematics, and then give the interval expressions of characteristic matrices of the structures for dynamic response analysis. With the matrix perturbation theory of dynamic response and interval extension of function, the algorithm for interval response analysis is developed. A numerical example, dynamic response analysis of a box cantilever beam, is given to illustrate the validity of the present method.

2. Mathematical backgrounds

In this section, we will give a brief review on the interval analysis (Moore 1966 & 1979, Alefeld and Herzberger 1983, Deif 1991).

2.1 The definitions of the interval and interval operations

Assume that $I(\boldsymbol{R})$, $I(\boldsymbol{R}^n)$ and $I(\boldsymbol{R}^{n\times n})$ denote the sets of all closed real interval numbers, n-dimensional real interval vectors and $n\times n$ real interval matrices, respectively, and $\boldsymbol{R}$ is the set of all real numbers. $X^I = [\underline{x}, \overline{x}]$ is a member of $I(\boldsymbol{R})$ and X^I can be usually written in the following form:

$$X^I = [X^c - \Delta X, X^c + \Delta X] \tag{1}$$

in which X^c and ΔX denote the mean (or midpoint) value of X^I and the uncertainty (or the maximum width) in X^I, respectively. It follows that

$$X^c = \frac{\underline{x} + \overline{x}}{2}$$
$$\Delta X = \frac{\overline{x} - \underline{x}}{2}. \tag{2}$$

In terms of the interval addition, Eq. (1) can be put into the more useful form:

$$X^I = X^c + \Delta X^I, \tag{3}$$

where $\Delta X^I = [-\Delta X, \Delta X]$.

An n-dimensional real interval vector $\boldsymbol{X}^I \in \boldsymbol{I}(\boldsymbol{R}^n)$ can be written as

$$\boldsymbol{X}^I = (X_1^I, X_2^I, \ldots, X_n^I)^T. \tag{4}$$

The mean value and uncertainty of $\boldsymbol{X}^I$ are

$$\begin{aligned} \boldsymbol{X}^c &= (X_1^c, X_2^c, \ldots, X_n^c)^T \\ \Delta\boldsymbol{X} &= (\Delta X_1, \Delta X_2, \ldots, \Delta X_n)^T \end{aligned} \tag{5}$$

Similar expressions exist for an $n\times n$ interval matrix $\boldsymbol{A}^I = [\underline{\boldsymbol{A}}, \overline{\boldsymbol{A}}] \in I(\boldsymbol{R}^{n\times n})$

$$\boldsymbol{A}^I = \boldsymbol{A}^c + \boldsymbol{\Delta A}^I, \tag{6}$$

where $\Delta\boldsymbol{A}^I = [-\Delta\boldsymbol{A}, \Delta\boldsymbol{A}]$, $\boldsymbol{A}^c$ and $\Delta\boldsymbol{A}$ denote the mean matrix of A^I and the uncertain (or the maximum width) matrix of $\boldsymbol{A}^I$ respectively. It follows that

$$\begin{aligned} \boldsymbol{A}^C &= \frac{(\overline{\boldsymbol{A}} + \underline{\boldsymbol{A}})}{2} \quad \text{or} \quad a_{ij}^C = \frac{(\overline{a}_{ij} + \underline{a}_{ij})}{2} \\ \boldsymbol{\Delta A} &= \frac{(\overline{\boldsymbol{A}} - \underline{\boldsymbol{A}})}{2} \quad \text{or} \quad \Delta a_{ij} = \frac{(\overline{a}_{ij} - \underline{a}_{ij})}{2} \end{aligned} \tag{7}$$

where $\boldsymbol{A}^C = (a_{ij}^C)$ and $\boldsymbol{\Delta A} = (\Delta a_{ij})$.

Let $X^I, Y^I \in I(\boldsymbol{R})$, $X^I = [\underline{x}, \overline{x}]$, $Y^I = [\underline{y}, \overline{y}]$, then the operations for $X^I + Y^I$, $X^I - Y^I$, $X^I \times Y^I$ and X^I/Y^I are

$$X^I + Y^I = [\underline{x}, \overline{x}] + [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}] \tag{8}$$

$$X^I - Y^I = [\underline{x}, \overline{x}] - [\underline{y}, \overline{y}] = [\underline{x} - \overline{y}, \overline{x} - \underline{y}] \tag{9}$$

$$X^I \times Y^I = [\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}] = [\min(\underline{x}\cdot\underline{y}, \underline{x}\cdot\overline{y}, \overline{x}\cdot\underline{y}, \overline{x}\cdot\overline{y}), \max(\underline{x}\cdot\underline{y}, \underline{x}\cdot\overline{y}, \overline{x}\cdot\underline{y}, \overline{x}\cdot\overline{y})] \tag{10}$$

$$\frac{X^I}{Y^I} = \frac{[\underline{x}, \overline{x}]}{[\underline{y}, \overline{y}]} = [\underline{x}, \overline{x}] \times \left[\frac{1}{\overline{y}}, \frac{1}{\underline{y}}\right] \tag{11}$$

2.2 Interval function extension

An interval function is an interval-valued function of one or more interval arguments. Assume that $\boldsymbol{F}(\boldsymbol{X}^I) = \boldsymbol{F}(X_1^I, X_2^I, \ldots, X_n^I)$ is the interval value function of interval variable $\boldsymbol{X}^I = (X_1^I, X_2^I, \ldots, X_n^I)^T$, if $X_i^I \subseteq Y_i^I$, $i = 1, 2, \ldots, n$, one has

$$\boldsymbol{F}(X_1^I, X_2^I, \ldots, X_n^I) \subseteq \boldsymbol{F}(Y_1^I, Y_2^I, \ldots, Y_n^I) \tag{12}$$

We say that the interval value function $\boldsymbol{F}(\boldsymbol{X}^I)$ of the interval variables $X_1^I, X_2^I, \ldots, X_n^I$ is inclusion monotonic. If $\boldsymbol{f}$ is the real function of n real variables $x_1, x_2, \ldots, x_n$ and the interval value function $\boldsymbol{F}$ of n interval variables $X_1^I, X_2^I, \ldots, X_n^I$ satisfy

$$\boldsymbol{F}(x_1, x_2, \ldots, x_n) = \boldsymbol{f}(x_1, x_2, \ldots, x_n), \qquad x_i \in X_i^I,\ i = 1, 2, \ldots, n \tag{13}$$

$\boldsymbol{F}$ is known as the interval extension of $\boldsymbol{f}$.

Real rational functions of n real variables may have natural extensions. Given a rational expression in real variables, we can replace the real variables by corresponding interval variables and replace the real arithmetic operations by the corresponding interval arithmetic operations to obtain a rational interval function, which is called natural extension of the real rational function. The extensions of the real rational function are inclusion monotonic and they can be calculated through finite interval arithmetic operations.

The interval extensions of a given function $\boldsymbol{f}$ are not unique. For example, two expressions for the function $\boldsymbol{g}$ are given by

$$\boldsymbol{g}^{(1)}(x, a) = \frac{ax}{1-x},\ x \neq 1,\ x \neq 0, \tag{14}$$

$$\boldsymbol{g}^{(2)}(x, a) = \frac{a}{\frac{1}{x}-1},\ x \neq 1,\ x \neq 0, \tag{15}$$

Using $A^I = [0, 1]$ and $X^I = [2, 3]$ replace a and x, two possible evaluations can be obtained:

$$\boldsymbol{g}^{(1)}([2, 3], [0, 1]) = \frac{[0, 1][2, 3]}{1-[2, 3]} = [-3, 0]$$

$$\boldsymbol{g}^{(2)}([2, 3], [0, 1]) = \frac{[0, 1]}{\frac{1}{[2, 3]}-1} = [-2, 0] \neq \boldsymbol{g}^{(1)}([2, 3], [0, 1])$$

Both interval results contain the exact result of $\boldsymbol{f}$ for $x \in [2, 3]$ and $a \in [0, 1]$, which is $[-2, 0]$. The result for $\boldsymbol{g}^{(2)}$ is precisely the range of $\boldsymbol{g}$ over the given sets, because X and A occur only once in the expression in $\boldsymbol{g}^{(2)}$. It shows one important rule in interval calculation, that is, the least times the interval parameters appear, the sharper the interval is, which is important in interval calculations.

3. Interval characteristic matrices for structures with interval parameters

Assume that the interval parameters of the structures are denoted by $\boldsymbol{b}$

$$\boldsymbol{b} = (b_1, b_2, \ldots, b_m)^T \in \boldsymbol{b}^I = (b_1^I, b_2^I, \ldots, b_m^I)^T \tag{16}$$

The mean-value of the vector $\boldsymbol{b}^c$ is

$$\boldsymbol{b}^c = (b_1^c, b_2^c, \ldots, b_m^c)^T \tag{17}$$

For each component of the vector, $b_j \in b_j^I = [\underline{b}_j, \overline{b}_j] = b_j^c + \Delta b_j e_j$, where $\Delta b_j = \frac{\overline{b}_j - \underline{b}_j}{2}$, $e_j = [-1, 1]$ and $j = 1, 2, \ldots, m$, m is the number of all parameters. The following discussions will be limited to the case where the changes of parameters do not lead to the change of the shape of the element.

For any $\boldsymbol{b} \in \boldsymbol{b}^I$, the characteristic matrices of the element can be expressed as

$$\begin{aligned} \boldsymbol{K}_i^e(\boldsymbol{b}) &= \boldsymbol{K}_i^e(\boldsymbol{b}^c) + \sum_{j=1}^{m}\left(\frac{\partial \boldsymbol{K}_i^e(\boldsymbol{b})}{\partial b_j}\right)_{\boldsymbol{b}=\boldsymbol{b}^c}(b_j - b_j^c) \\ \boldsymbol{M}_i^e(\boldsymbol{b}) &= \boldsymbol{M}_i^e(\boldsymbol{b}^c) + \sum_{j=1}^{m}\left(\frac{\partial \boldsymbol{M}_i^e(\boldsymbol{b})}{\partial b_j}\right)_{\boldsymbol{b}=\boldsymbol{b}^c}(b_j - b_j^c) \\ \boldsymbol{C}_i^e(\boldsymbol{b}) &= \boldsymbol{C}_i^e(\boldsymbol{b}^c) + \sum_{j=1}^{m}\left(\frac{\partial \boldsymbol{C}_i^e(\boldsymbol{b})}{\partial b_j}\right)_{\boldsymbol{b}=\boldsymbol{b}^c}(b_j - b_j^c) \end{aligned} \tag{18}$$

In general, it is difficult to express the stiffness, damping and mass matrices coefficients as explicit functions of design variables. To carry out the calculations of $\left(\frac{\partial \boldsymbol{K}_i^e(\boldsymbol{b})}{\partial b_j}\right)_{\boldsymbol{b}=\boldsymbol{b}^c}$, $\left(\frac{\partial \boldsymbol{M}_i^e(\boldsymbol{b})}{\partial b_j}\right)_{\boldsymbol{b}=\boldsymbol{b}^c}$ and $\left(\frac{\partial \boldsymbol{C}_i^e(\boldsymbol{b})}{\partial b_j}\right)_{\boldsymbol{b}=\boldsymbol{b}^c}$ by directly using the differential method is inconvenient. It is desirable to transform the differential approach into finite element perturbation. Let $\Delta K_{ij}^e, \Delta C_{ij}^e$ and ΔM_{ij}^e be the increments of the stiffness, damping and mass matrices of the ith element resulting from the changes of the structural parameter ΔB_j, i.e.,

$$\begin{aligned} \Delta\boldsymbol{K}_{ij}^e &= \boldsymbol{K}_i^e(b_1^c, \ldots b_j^c + \Delta B_j, \ldots, b_n^c) - \boldsymbol{K}_i^e(b_1^c, \ldots, b_j^c, \ldots, b_m^c) \\ \Delta\boldsymbol{M}_{ij}^e &= \boldsymbol{M}_i^e(b_1^c, \ldots b_j^c + \Delta B_j, \ldots, b_n^c) - \boldsymbol{M}_i^e(b_1^c, \ldots, b_j^c, \ldots, b_m^c) \\ \Delta\boldsymbol{C}_{ij}^e &= \boldsymbol{C}_i^e(b_1^c, \ldots b_j^c + \Delta B_j, \ldots, b_n^c) - \boldsymbol{C}_i^e(b_1^c, \ldots, b_j^c, \ldots, b_m^c) \end{aligned} \tag{19}$$

Then $\boldsymbol{K}_{i,j}^c, \boldsymbol{M}_{i,j}^c$ and $\boldsymbol{C}_{i,j}^c$, the approximation of $\left(\frac{\partial \boldsymbol{K}_i^e(\boldsymbol{b})}{\partial b_j}\right)_{\boldsymbol{b}=\boldsymbol{b}^c}$, $\left(\frac{\partial \boldsymbol{M}_i^e(\boldsymbol{b})}{\partial b_j}\right)_{\boldsymbol{b}=\boldsymbol{b}^c}$ and $\left(\frac{\partial \boldsymbol{C}_i^e(\boldsymbol{b})}{\partial b_j}\right)_{\boldsymbol{b}=\boldsymbol{b}^c}$ are as follows

$$\boldsymbol{K}_{i,j}^c = \frac{\Delta\boldsymbol{K}_{ij}^e}{\Delta B_j} \quad \boldsymbol{M}_{i,j}^c = \frac{\Delta\boldsymbol{M}_{ij}^e}{\Delta B_j} \quad \boldsymbol{C}_{i,j}^c = \frac{\Delta\boldsymbol{C}_{ij}^e}{\Delta B_j} \tag{20}$$

Using the natural interval extension of function to Eq. (18), one can obtain the interval characteristic matrices

$$\begin{aligned} \boldsymbol{K}_i^e(\boldsymbol{b}^I) &= \boldsymbol{K}_i^e(\boldsymbol{b}^c) + \sum_{j=1}^{m}\boldsymbol{K}_{i,j}^c(b_j^I - b_j^c) \\ \boldsymbol{M}_i^e(\boldsymbol{b}^I) &= \boldsymbol{M}_i^e(\boldsymbol{b}^c) + \sum_{j=1}^{m}\boldsymbol{M}_{i,j}^c(b_j^I - b_j^c) \\ \boldsymbol{C}_i^e(\boldsymbol{b}^I) &= \boldsymbol{C}_i^e(\boldsymbol{b}^c) + \sum_{j=1}^{m}\boldsymbol{C}_{i,j}^c(b_j^I - b_j^c) \end{aligned} \tag{21}$$

The global stiffness and mass matrices are assembled by using the element matrices

$$\boldsymbol{K}(b) = \sum_{i=1}^{n}\boldsymbol{K}_i(\boldsymbol{b}) = \boldsymbol{K}(\boldsymbol{b}^c) + \Delta\boldsymbol{K}(\boldsymbol{b})$$

$$\boldsymbol{M}(b) = \sum_{i=1}^{n} \boldsymbol{M}_i(\boldsymbol{b}) = \boldsymbol{M}(\boldsymbol{b}^c) + \Delta\boldsymbol{M}(\boldsymbol{b})$$
$$\boldsymbol{C}(b) = \sum_{i=1}^{n} \boldsymbol{C}_i(\boldsymbol{b}) = \boldsymbol{C}(\boldsymbol{b}^c) + \Delta\boldsymbol{C}(\boldsymbol{b}) \tag{22}$$

where

$$\boldsymbol{K}(b^c) = \sum_{i=1}^{n} \boldsymbol{K}_i(\boldsymbol{b}^c) \qquad \Delta\boldsymbol{K}(\boldsymbol{b}) = \sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{K}_{i,j}^c \Delta b_j$$
$$\boldsymbol{M}(b^c) = \sum_{i=1}^{n} \boldsymbol{M}_i(\boldsymbol{b}^c) \qquad \Delta\boldsymbol{M}(\boldsymbol{b}) = \sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{M}_{i,j}^c \Delta b_j$$
$$\boldsymbol{C}(b^c) = \sum_{i=1}^{n} \boldsymbol{C}_i(\boldsymbol{b}^c) \qquad \Delta\boldsymbol{C}(\boldsymbol{b}) = \sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{C}_{i,j}^c \Delta b_j \tag{23}$$

where n is the total number of the elements. It should be pointed out that in Eq. (22), the element characteristic matrices should be expanded by FEM rules before forming the global matrices. Appling the natural interval extension of function to Eq. (22), one can obtain the interval matrices as follows.

$$\boldsymbol{K}(\boldsymbol{b}^I) = \boldsymbol{K}(\boldsymbol{b}^c) + \Delta\boldsymbol{K}(\boldsymbol{b}^I)$$
$$\boldsymbol{M}(\boldsymbol{b}^I) = \boldsymbol{M}(\boldsymbol{b}^c) + \Delta\boldsymbol{M}(\boldsymbol{b}^I)$$
$$\boldsymbol{C}(\boldsymbol{b}^I) = \boldsymbol{C}(\boldsymbol{b}^c) + \Delta\boldsymbol{C}(\boldsymbol{b}^I) \tag{24}$$

where

$$\Delta\boldsymbol{K}(\boldsymbol{b}^I) = \sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{K}_{i,j}^e (b_j^c - b_j^I)$$
$$\Delta\boldsymbol{M}(\boldsymbol{b}^I) = \sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{M}_{i,j}^e (b_j^c - b_j^I)$$
$$\Delta\boldsymbol{C}(\boldsymbol{b}^I) = \sum_{i=1}^{n}\sum_{j=1}^{m} \boldsymbol{C}_{i,j}^e (b_j^c - b_j^I) \tag{25}$$

The damping coefficients are taken for the Rayleigh damping, i.e., $\boldsymbol{C} = \alpha\boldsymbol{K} + \beta\boldsymbol{M}$, where α and β are the coefficients, which can be taken as the structural parameters.

4. Dynamic response analysis of structures with interval parameters

4.1 Perturbation analysis of the dynamic response of deterministic system

The vibration equation of n-degree of freedom systems can be given as follows:

$$\boldsymbol{M}(\boldsymbol{b})\ddot{\boldsymbol{x}} + \boldsymbol{C}(\boldsymbol{b})\dot{\boldsymbol{x}} + \boldsymbol{K}(\boldsymbol{b})\boldsymbol{x} = \boldsymbol{P}(\boldsymbol{t}) \tag{26}$$

If the design variables have some perturbations $\varepsilon\Delta\boldsymbol{b}$, i.e., $\boldsymbol{b} = \boldsymbol{b}^c + \varepsilon\Delta\boldsymbol{b} \in \boldsymbol{b}^I$, then the characteristic matrices can be written as (Chen 1999).

$$\boldsymbol{M}(\boldsymbol{b}) = \boldsymbol{M}(\boldsymbol{b}^c) + \varepsilon\boldsymbol{M}_1 \quad \boldsymbol{C}(\boldsymbol{b}) = \boldsymbol{C}(\boldsymbol{b}^c) + \varepsilon\boldsymbol{C}_1 \quad \boldsymbol{K}(\boldsymbol{b}) = \boldsymbol{K}(\boldsymbol{b}^c) + \varepsilon\boldsymbol{K}_1 \tag{27}$$

and the responses are

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{x}_0 + \varepsilon\boldsymbol{x}_1 + \varepsilon^2\boldsymbol{x}_2 + \dots \\ \dot{\boldsymbol{x}} &= \dot{\boldsymbol{x}}_0 + \varepsilon\dot{\boldsymbol{x}}_1 + \varepsilon^2\dot{\boldsymbol{x}}_2 + \dots \\ \ddot{\boldsymbol{x}} &= \ddot{\boldsymbol{x}}_0 + \varepsilon\ddot{\boldsymbol{x}}_1 + \varepsilon^2\ddot{\boldsymbol{x}}_2 + \dots \end{aligned} \tag{28}$$

Then Eq. (26) changes into

$$\begin{aligned} (\boldsymbol{M}(\boldsymbol{b}^c) + \varepsilon\boldsymbol{M}_1)(\ddot{\boldsymbol{x}}_0 + \varepsilon\ddot{\boldsymbol{x}}_1 + \dots) + (\boldsymbol{C}(\boldsymbol{b}^c) + \varepsilon\boldsymbol{C}_1)(\dot{\boldsymbol{x}}_0 + \varepsilon\dot{\boldsymbol{x}}_1 + \dots) \\ +(\boldsymbol{K}(\boldsymbol{b}^c) + \varepsilon\boldsymbol{K}_1)(\boldsymbol{x}_0 + \varepsilon\boldsymbol{x}_1 + \dots) = \boldsymbol{P}(t) \end{aligned} \tag{29}$$

Expanding and equating the coefficients with the same power in Eq. (29), one can obtain

$$\varepsilon^0: \quad \boldsymbol{M}(\boldsymbol{b}^c)\ddot{\boldsymbol{x}}_0 + \boldsymbol{C}(\boldsymbol{b}^c)\dot{\boldsymbol{x}}_0 + \boldsymbol{K}(\boldsymbol{b}^c)\boldsymbol{x}_0 = \boldsymbol{P}(t) \tag{30}$$

$$\varepsilon^1: \quad \boldsymbol{M}(\boldsymbol{b}^c)\ddot{\boldsymbol{x}}_1 + \boldsymbol{C}(\boldsymbol{b}^c)\dot{\boldsymbol{x}}_1 + \boldsymbol{K}(\boldsymbol{b}^c)\boldsymbol{x}_1 = -(\boldsymbol{M}_1\ddot{\boldsymbol{x}}_0 + \boldsymbol{C}_1\dot{\boldsymbol{x}}_0 + \boldsymbol{K}_1\boldsymbol{x}_0) \tag{31}$$

From Eq. (30), one can obtain the dynamic response of the original system. However, it is difficult to acquire the perturbation solutions from Eq. (31). If the structural parameters are small, one can expand $\boldsymbol{M}_1, \boldsymbol{C}_1, \boldsymbol{K}_1, \boldsymbol{x}_1, \dot{\boldsymbol{x}}_1$ and $\ddot{\boldsymbol{x}}_1$ around the mean value of the parameters, that is,

$$\boldsymbol{x}_1 = \sum_{j=1}^{m}\boldsymbol{x}_{0,j}\Delta b_j \quad \dot{\boldsymbol{x}}_1 = \sum_{j=1}^{m}\ddot{\boldsymbol{x}}_{0,j}\Delta b_j \quad \ddot{\boldsymbol{x}}_1 = \sum_{j=1}^{m}\ddot{\boldsymbol{x}}_{0,j}\Delta b_j$$

$$\boldsymbol{M}_1 = \sum_{j=1}^{m}\boldsymbol{M}_{0,j}(b_j - b^c) \quad \boldsymbol{C}_1 = \sum_{j=1}^{m}\boldsymbol{C}_{0,j}(b_j - b^c) \quad \boldsymbol{K}_1 = \sum_{j=1}^{m}\boldsymbol{K}_{0,j}(b_j - b^c) \tag{32}$$

in which

$$\boldsymbol{x}_{0,j} = \frac{\partial\boldsymbol{x}_0}{\partial b_j} \quad \dot{\boldsymbol{x}}_{0,j} = \frac{\partial\dot{\boldsymbol{x}}_0}{\partial b_j} \quad \ddot{\boldsymbol{x}}_{0,j} = \frac{\partial\ddot{\boldsymbol{x}}_0}{\partial b_j}$$

$$\boldsymbol{M}_{0,j} = \frac{\partial M(\boldsymbol{b}^c)}{\partial b_j} \quad \boldsymbol{C}_{0,j} = \frac{\partial C(\boldsymbol{b}^c)}{\partial b_j} \quad \boldsymbol{K}_{0,j} = \frac{\partial K(\boldsymbol{b}^c)}{\partial b_j}$$

Substituting Eq. (32) into (31), one can obtain

$$\sum_{j=1}^{m}(\boldsymbol{M}(\boldsymbol{b}^c)\ddot{\boldsymbol{x}}_{0,j} + \boldsymbol{C}(\boldsymbol{b}^c)\dot{\boldsymbol{x}}_{0,j} + \boldsymbol{K}(\boldsymbol{b}^c)\boldsymbol{x}_{0,j})\Delta b_j = -\sum_{j=1}^{m}(\boldsymbol{M}_{0,j}\ddot{\boldsymbol{x}}_0 + \boldsymbol{C}_{0,j}\dot{\boldsymbol{x}}_0 + \boldsymbol{K}_{0,j}\boldsymbol{x}_0)\Delta b_j \tag{33}$$

From Eq. (33), one can get

$$\boldsymbol{M}(b^c)\ddot{\boldsymbol{x}}_{0,j} + \boldsymbol{C}(b^c)\dot{\boldsymbol{x}}_{0,j} + \boldsymbol{K}(b^c)\boldsymbol{x}_{0,j} = -(\boldsymbol{M}_{0,j}\ddot{\boldsymbol{x}}_0 + \boldsymbol{C}_{0,j}\dot{\boldsymbol{x}}_0 + \boldsymbol{K}_{0,j}\boldsymbol{x}_0) \quad (34)$$

It is easy to get the solutions of those equations by the normal numerical integral methods as Willson-θ and Newmark etc. substituting the solutions into Eq. (28), the response perturbation part is obtained, and then the response solution of the perturbed system is

$$\boldsymbol{x} = \boldsymbol{x}_0 + \varepsilon\boldsymbol{x}_1 \quad (35)$$

4.2 Interval dynamic response of structures with interval parameters

Using the interval extension of function to Eq. (26), one can have

$$\boldsymbol{M}(\boldsymbol{b}^I)\ddot{\boldsymbol{x}} + \boldsymbol{C}(\boldsymbol{b}^I)\dot{\boldsymbol{x}} + \boldsymbol{K}(\boldsymbol{b}^I)\boldsymbol{x} = \boldsymbol{P}(t) \quad (36)$$

where

$$\begin{aligned} \boldsymbol{M}(\boldsymbol{b}^I) &= \{\boldsymbol{M}(\boldsymbol{b}) \mid \underline{\boldsymbol{b}} \le \boldsymbol{b} \le \overline{\boldsymbol{b}}\} \\ \boldsymbol{C}(\boldsymbol{b}^I) &= \{\boldsymbol{C}(\boldsymbol{b}) \mid \underline{\boldsymbol{b}} \le \boldsymbol{b} \le \overline{\boldsymbol{b}}\} \\ \boldsymbol{K}(\boldsymbol{b}^I) &= \{\boldsymbol{K}(\boldsymbol{b}) \mid \underline{\boldsymbol{b}} \le \boldsymbol{b} \le \overline{\boldsymbol{b}}\} \end{aligned} \quad (37)$$

It is the basic problem for given interval characteristic matrices, $\boldsymbol{M}(\boldsymbol{b}^I), \boldsymbol{C}(\boldsymbol{b}^I), \boldsymbol{K}(\boldsymbol{b}^I)$ and $\boldsymbol{P}(t)$, to find all possible $\boldsymbol{x}$ satisfying Eq. (26), that is, to obtain $\boldsymbol{x} \in \boldsymbol{x}^I = [\underline{\boldsymbol{x}}, \overline{\boldsymbol{x}}]$ where

$$\begin{aligned} \underline{\boldsymbol{x}} &= \min\{\boldsymbol{x} \mid \boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{P}(t), \boldsymbol{M} \in \boldsymbol{M}(\boldsymbol{b}^I), \boldsymbol{C} \in \boldsymbol{C}(\boldsymbol{b}^I), \boldsymbol{K} \in \boldsymbol{K}(\boldsymbol{b}^I)\} \\ \overline{\boldsymbol{x}} &= \max\{\boldsymbol{x} \mid \boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{P}(t), \boldsymbol{M} \in \boldsymbol{M}(\boldsymbol{b}^I), \boldsymbol{C} \in \boldsymbol{C}(\boldsymbol{b}^I), \boldsymbol{K} \in \boldsymbol{K}(\boldsymbol{b}^I)\} \end{aligned}$$

The centered form of Eq. (36) is

$$(\boldsymbol{M}(\boldsymbol{b}^c) + \Delta\boldsymbol{M}(\boldsymbol{b}^I))\ddot{\boldsymbol{x}} + (\boldsymbol{C}(\boldsymbol{b}^c) + \Delta\boldsymbol{C}(\boldsymbol{b}^I))\dot{\boldsymbol{x}} + (\boldsymbol{K}(\boldsymbol{b}^c) + \Delta\boldsymbol{K}(\boldsymbol{b}^I))\boldsymbol{x} = \boldsymbol{P}(t) \quad (38)$$

For any $\boldsymbol{b} = \boldsymbol{b}^c + \delta\boldsymbol{b} \in \boldsymbol{b}^I$, there is a group of $\delta\boldsymbol{M}$, $\delta\boldsymbol{C}$, and $\delta\boldsymbol{K}$, which satisfy

$$\underline{\Delta\boldsymbol{M}} \le \delta\boldsymbol{M} \le \overline{\Delta\boldsymbol{M}} \qquad \underline{\Delta\boldsymbol{C}} \le \delta\boldsymbol{C} \le \overline{\Delta\boldsymbol{C}} \qquad \underline{\Delta\boldsymbol{K}} \le \delta\boldsymbol{K} \le \overline{\Delta\boldsymbol{K}} \quad (39)$$

and the vibration equation is

$$(\boldsymbol{M}(\boldsymbol{b}^c) + \delta\boldsymbol{M})\ddot{\boldsymbol{x}} + (\boldsymbol{C}(\boldsymbol{b}^c) + \delta\boldsymbol{C})\dot{\boldsymbol{x}} + (\boldsymbol{K}(\boldsymbol{b}^c) + \delta\boldsymbol{K})\boldsymbol{x} = \boldsymbol{P}(t) \quad (40)$$

By neglecting the higher order terms, from Eq. (35), one can obtain

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{x}_0 + \delta\boldsymbol{x} \\ \delta\boldsymbol{x} &= \sum_{j=1}^{m} \boldsymbol{x}_{0,j}(b_j - b^c) \end{aligned} \quad (41)$$

in which $\boldsymbol{x}_0$ and $x_{0,j}$ are obtained by solving Eqs. (26) and (34). Eq. (38) is equivalent to Eq. (40)

under the constrains Eq. (39), therefore using the interval extension of function to Eq. (41), one can have

$$x^I = x_0 + \Delta x^I$$
$$\Delta x^I = \sum_{j=1}^{m} x_{0,j}(b_j^I - b^c) = \sum_{j=1}^{m} x_{0,j}\Delta b_j e_j$$
$$= \sum_{j=1}^{m} |x_{0,j}\Delta b_j|[-1, 1] = \Delta x[-1, 1] \tag{42}$$

where $\Delta x = \sum_{j=1}^{m} |x_{0,j}\Delta b_j|$, and the upper and lower bounds of the dynamic responses will be

$$\overline{x} = x_0 + \Delta x$$
$$\underline{x} = x_0 - \Delta x \tag{43}$$

From Eq. (43), one can obtain the interval responses which are symmetrical about the mean value x_0, and the intervals of dynamic responses are sharp, because the interval parameters are used at the least times to calculate $x_{0,j}$ in Eq. (34).

5. Numerical example

In order to demonstrate the applications of the present method and estimate the effect of the interval parameters on the analysis results, An example is considered. The results are listed in Tables, in which $\underline{x_{i,t}}$ and $\overline{x_{i,t}}$ are used to show the lower and upper bounds of response for the 3rd DOF on node 86 at time t. $\Delta x_{i,t}/\Delta b$ is also listed in the Tables to show the relative ratio between the intervals of response and intervals of variables, which gives the response interval when the parameters have 1% error.

Consider a box cantilever beam shown in Fig. 1. The finite element model of the given structure consists of 88 nodes and 80 plate elements. The structural parameters are given as follows: the thickness δ^c of the plate is 1 cm; the Young's modules of the elements is $E^c = 2.1\text{E}11\text{N/m}^2$; The

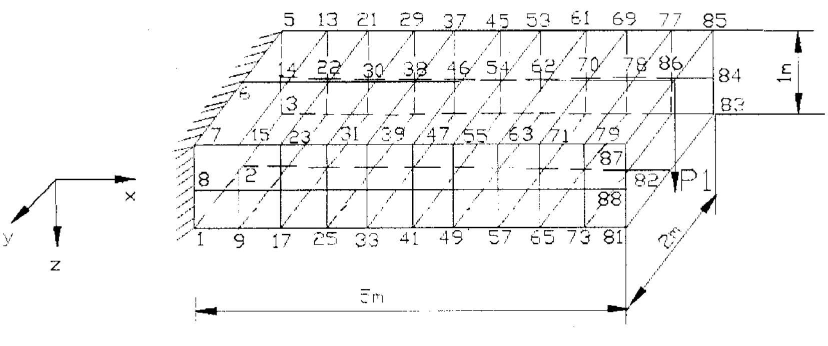

Fig. 1 A box cantilever beam structure subjected to excitation $P1$

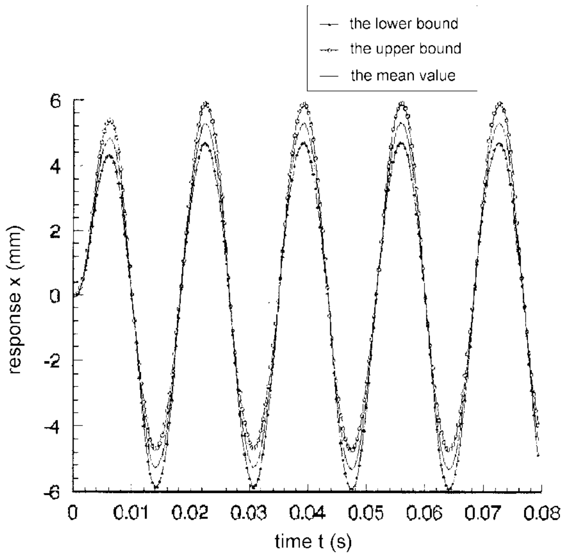

Fig. 2 The interval response shape at $\Delta\delta = \frac{10}{100}\delta^c$ and f = 60 Hz

mass density for elements is ρ^c = 7.8E3kg/m^3; the proportion damping coefficients α^c = 206.5 and β^c = 7.7e − 4; the sine excitation at node 86 is along the z positive direction, the initial phase is zero, and the amplitude of the load is 1200N; the first order natural frequency of the original structure is 80 Hz; the initial conditions are $\boldsymbol{x} = 0$ and $\dot{\boldsymbol{x}} = 0$.

When the thickness δ of the plate on the top side of the box and the coefficients α and β are interval parameters, the interval response are obtained for two different frequencies of excitation. The interval response shape at $\Delta\delta = 10\%\delta^c$ and f = 60 Hz is given in Fig. 2, which is similar for other interval parameters. Data in more details are given in Tables 1-6. From the Tables, one can know not only the original response values but also its upper and lower bounds. As can be seen in Table 1, the maximum and minimum value of the displacement at t = 0.0064s are 0.5336E − 2, and 0.4256E − 2, which bound all possible responses when the thickness are intervals. The thickness has more influence on the response because max($\Delta x_{i,t}/\Delta\delta$) = 0.12E − 3 in Table 1 is larger than max($\Delta x_{i,t}/\Delta\alpha$) = 0.9E − 5 in Table 2 and max($\Delta x_{i,t}/\Delta\alpha$) =0.156E − 4 in Table 3. However, α and β can't be neglected because they often have greater uncertainties than δ. At the same frequency excitation, α is less important than β which can be seen from that max($\Delta x_{i,t}/\Delta\alpha$) in Table 1 is less than mas($\Delta x_{i,t}/\Delta\beta$) in Table 2. For different excitation frequencies, t has no more effect because $\max\left(\frac{\Delta x_{i,t}}{\Delta\delta}_{f=60\text{Hz}}\right) = \max\left(\frac{\Delta x_{i,t}}{\Delta\delta}_{f=75\text{Hz}}\right)$ = 0.12E − 3. For α and β, the closer the excitation frequency to the first natural frequency, the larger the change is, which can be seen from that $\max\left(\frac{\Delta x_{i,t}}{\Delta\alpha}_{f=60}\right)$ = 0.9E − 5 is less than $\max\left(\frac{\Delta x_{i,t}}{\Delta\alpha}_{f=75}\right)$ = 0.98E − 5 and $\max\left(\frac{\Delta x_{i,t}}{\Delta\beta}_{f=60}\right)$ = 0.156E − 4 is less than $\max\left(\frac{\Delta x_{i,t}}{\Delta\beta}_{f=75}\right)$ = 0.178E − 4 too.

Table 1 The lower and upper bounds of dynamic response of node 86 at z direction $((\overline{\delta}_i - \underline{\delta}_i) = 10\delta_i^c/100$, for the elements on top side and $f = 60$ Hz)

$t(s)$	$\underline{x}_{i,t}$ (E – 2m)	$x_{i,t}^c$ (E – 2m)	$\overline{x}_{i,t}$ (E – 2m)	$\frac{\Delta x_{i,t}}{\Delta t}$ (E – 3)
0.0008	0.006748	0.007208	0.007668	0.00092
0.0016	0.04463	0.04813	0.05163	0.007
0.0024	0.1192	0.1292	0.1392	0.020
0.0032	0.2142	0.2352	0.2562	0.042
0.0040	0.3093	0.3423	0.3753	0.066
0.0048	0.3843	0.4283	0.4723	0.088
0.0056	0.4260	0.4770	0.5280	0.102
0.0064	0.4256	0.4796	0.5336	0.108
0.0072	0.3823	0.4343	0.4863	0.104
0.0080	0.3021	0.3451	0.3881	0.086
0.0088	0.1900	0.2210	0.2520	0.062
0.0096	0.05891	0.07381	0.08871	0.0298
0.0104	–0.08489	–0.08239	–0.07989	0.005
0.0112	–0.252	–0.2327	–0.2127	0.040
0.0120	–0.3989	–0.3629	–0.3269	0.072
0.0128	–0.5086	–0.4606	–0.4126	0.096
0.0136	–0.5725	–0.5165	–0.4605	0.112
0.0144	–0.5854	–0.5254	–0.4654	0.120
0.0152	–0.5443	–0.4863	–0.4283	0.116
0.0160	–0.4527	–0.4027	–0.3527	0.100

Table 2 The lower and upper bounds of dynamic response of node 86 at z direction $((\overline{\alpha} - \underline{\alpha}) = 20\alpha^c/100$ and $f = 60$ Hz)

$t(s)$	$\underline{x}_{i,t}$ (E – 2m)	$x_{i,t}^c$ (E – 2m)	$\overline{x}_{i,t}$ (E-2m)	$\frac{\Delta x_{i,t}}{\Delta t}$ (E – 5)
0.0008	0.007172	0.007208	0.007244	0.0036
0.0016	0.04779	0.04813	0.04847	0.034
0.0024	0.1282	0.1292	0.1302	0.10
0.0032	0.2328	0.2352	0.2376	0.24
0.0040	0.3385	0.3423	0.3461	0.38
0.0048	0.4237	0.4283	0.4329	0.46
0.0056	0.4720	0.4770	0.4820	0.50
0.0064	0.4752	0.4796	0.4840	0.44
0.0072	0.4311	0.4343	0.4375	0.32
0.0080	0.3441	0.3451	0.3461	0.10
0.0088	0.2198	0.2210	0.2222	0.12
0.0096	0.07007	0.07381	0.07755	0.37
0.0104	–0.08841	–0.08239	–0.07637	0.60
0.0112	–0.2405	–0.2327	–0.2249	0.78
0.0120	–0.3717	–0.3629	–0.3541	0.88
0.0128	–0.4696	–0.4606	–0.4516	0.90
0.0136	–0.5249	–0.5165	–0.5081	0.84
0.0144	–0.5324	–0.5254	–0.5184	0.70
0.0152	–0.4909	–0.4863	–0.4817	0.46
0.0160	–0.4045	–0.4027	–0.4009	0.18

Table 3 The lower and upper bounds of dynamic response of node 86 at z direction $((\overline{\beta} - \underline{\beta}) = 20\beta^c/100$ and $f = 60$ Hz)

$t(s)$	$\underline{x}_{i,t}$ (E – 2m)	$x^c_{i,t}$ (E – 2m)	$\overline{x}_{i,t}$ (E – 2m)	$\frac{\Delta x_{i,t}}{\Delta t}$ (E – 4)
0.0008	0.007100	0.007208	0.007316	0.00108
0.0016	0.04719	0.04813	0.04907	0.0094
0.0024	0.1264	0.1292	0.1320	0.028
0.0032	0.2294	0.2352	0.2410	0.058
0.0040	0.3339	0.3423	0.3507	0.084
0.0048	0.4187	0.4283	0.4379	0.096
0.0056	0.4680	0.4770	0.4860	0.090
0.0064	0.4728	0.4796	0.4864	0.068
0.0072	0.4313	0.4343	0.4373	0.030
0.0080	0.3437	0.3451	0.3465	0.014
0.0088	0.2150	0.2210	0.2270	0.060
0.0096	0.06379	0.07381	0.08383	0.1002
0.0104	–0.09561	–0.08239	–0.06917	0.1322
0.0112	–0.2477	–0.2327	–0.2177	0.150
0.0120	–0.3785	–0.3629	–0.3473	0.156
0.0128	–0.4754	–0.4606	–0.4458	0.148
0.0136	–0.5287	–0.5165	–0.5043	0.122
0.0144	–0.5342	–0.5254	–0.5166	0.088
0.0152	–0.4905	–0.4863	–0.4821	0.042
0.0160	–0.4035	–0.4027	–0.4019	0.008

Table 4 The lower and upper bounds of dynamic response of node 86 at z direction $((\overline{\delta}_i - \underline{\delta}_i) = 10\delta^c_i/100$, for the elements on top side and $f = 75$ Hz)

$t(s)$	$\underline{x}_{i,t}$ (E – 2m)	$x^c_{i,t}$ (E – 2m)	$\overline{x}_{i,t}$ (E – 2m)	$\frac{\Delta x_{i,t}}{\Delta t}$ (E – 3)
0.0008	0.008408	0.008978	0.009548	0.00114
0.0016	0.5515	0.05935	0.06355	0.00840
0.0024	0.1437	0.1567	0.1697	0.026
0.0032	0.2531	0.2771	0.3011	0.048
0.0040	0.3492	0.3862	0.4232	0.074
0.0048	0.4062	0.4532	0.5002	0.094
0.0056	0.4073	0.4583	0.5093	0.102
0.0064	0.3474	0.3954	0.4434	0.096
0.0072	0.2333	0.2713	0.3093	0.076
0.0080	0.08240	0.1034	0.1244	0.042
0.0088	–0.08417	–0.08377	–0.08337	0.0008
0.0096	–0.2837	–0.2637	–0.2437	0.04
0.0104	–0.4502	–0.4102	–0.3702	0.08
0.0112	–0.5540	–0.5010	–0.4480	0.106
0.0120	–0.5821	–0.5221	–0.4621	0.120
0.0128	–0.5267	–0.4697	–0.4127	0.114
0.0136	–0.3988	–0.3508	–0.3028	0.096
0.0144	–0.2118	–0.1818	–0.1518	0.060
0.0152	0.00398	0.01338	0.02278	0.0188
0.0160	0.1944	0.2074	0.2204	0.026

Table 5 The lower and upper bounds of dynamic response of node 86 at z direction ($(\overline{\alpha}-\underline{\alpha}) = 20\alpha^c/100$ and $f = 75$ Hz)

$t(s)$	$\underline{x_{i,t}}$ (E – 2m)	$x^c_{i,t}$ (E – 2m)	$\overline{x_{i,t}}$ (E – 2m)	$\frac{\Delta x_{i,t}}{\Delta t}$ (E – 5)
0.0008	0.008934	0.008978	0.009022	0.0044
0.0016	0.058930	0.05935	0.05977	0.042
0.0024	0.1553	0.1567	0.1581	0.14
0.0032	0.2745	0.2771	0.2797	0.26
0.0040	0.3820	0.3862	0.3904	0.42
0.0048	0.4482	0.4532	0.4582	0.50
0.0056	0.4535	0.4583	0.4631	0.48
0.0064	0.3920	0.3954	0.3988	0.34
0.0072	0.2701	0.2713	0.2725	0.12
0.0080	0.1016	0.1034	0.1052	0.18
0.0088	–0.08857	–0.08377	–0.07897	0.48
0.0096	–0.2711	–0.2637	–0.2563	0.74
0.0104	–0.4194	–0.4102	–0.4010	0.92
0.0112	–0.5108	–0.5010	–0.4912	0.98
0.0120	–0.5311	–0.5221	–0.5131	0.90
0.0128	–0.4765	–0.4697	–0.4629	0.68
0.0136	–0.3544	–0.3508	–0.3472	0.36
0.0144	–0.1822	–0.1818	–0.1814	0.040
0.0152	0.00926	0.01338	0.0175	0.412
0.0160	0.1998	0.2074	0.2150	0.760

Table 6 The lower and upper bounds of dynamic response of node 86 at z direction ($(\overline{\beta}-\underline{\beta}) = 20\beta^c/100$ and $f = 75$ Hz)

$t(s)$	$\underline{x_{i,t}}$ (E – 2m)	$x^c_{i,t}$ (E – 2m)	$\overline{x_{i,t}}$ (E – 2m)	$\frac{\Delta x_{i,t}}{\Delta t}$ (E – 4)
0.0008	0.008844	0.008978	0.009112	0.00134
0.0016	0.05819	0.05935	0.06051	0.0116
0.0024	0.1531	0.1567	0.1603	0.036
0.0032	0.2703	0.2771	0.2839	0.068
0.0040	0.3768	0.3862	0.3956	0.094
0.0048	0.4432	0.4532	0.4632	0.10
0.0056	0.4503	0.4583	0.4663	0.08
0.0064	0.3916	0.3954	0.3992	0.038
0.0072	0.2693	0.2713	0.2733	0.02
0.0080	0.0954	0.1034	0.1114	0.08
0.0088	–0.0969	–0.08377	–0.07063	0.1314
0.0096	–0.2803	–0.2637	–0.2471	0.166
0.0104	–0.4280	–0.4102	–0.3924	0.178
0.0112	–0.5176	–0.5010	–0.4844	0.166
0.0120	–0.5349	–0.5221	–0.5093	0.128
0.0128	–0.4767	–0.4697	–0.4627	0.07
0.0136	–0.3510	–0.3508	–0.3506	0.002
0.0144	–0.1886	–0.1818	–0.1750	0.068
0.0152	0.000512	0.01338	0.02625	0.1287
0.0160	0.1902	0.2074	0.2246	0.172

6. Conclusions

In this paper, with the interval analysis and the matrix perturbation, the method for evaluating the lower and upper bounds of dynamic response of structures is discussed. The calculations are based on the element level and use the same interval parameters at the least times, thus simplifying the computational effort and sharping the bounds. In addition, the great advantage of interval analysis in engineering is that it gives us not only the approximate values but also the bounds of errors at the same time. Because the method is based on the 1st order perturbation, the computational accuracy is limited if the number and width of the interval parameters are too large. The present procedure may be an attempt to bound the dynamic response when the parameters are intervals, rather than random ones.

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