Structural Engineering and Mechanics, Vol. 12, No. 6 (2001) 685-698
DOI: http://dx.doi.org/10.12989/sem.2001.12.6.685

Stress path adapting Strut-and-Tie models in cracked and uncracked R.C. elements*

Fabio Biondini†

Department of Structural Engineering, Technical University of Milan, Piazza L. da Vinci 32, 20133 Milan, Italy

Franco Bontempi‡

Department of Structural and Geotechnical Engineering, University of Rome "La Sapienza", Via Eudossiana 18, 00184 Rome, Italy

Pier Giorgio Malerba‡

Department of Structural Engineering, Technical University of Milan, Piazza L. da Vinci 32, 20133 Milan, Italy

Abstract. In this paper, a general method for the automatic search for Strut-and-Tie (S&T) models representative of possible resistant mechanisms in reinforced concrete elements is proposed. The representativeness criterion here adopted is inspired to the principle of minimum strain energy and requires the consistency of the model with a reference stress field. In particular, a highly indeterminate pin-jointed framework of a given layout is generated within the assigned geometry of the concrete element and an optimum truss is found by the minimisation of a suitable objective function. Such a function allows us to search the optimum truss according to a reference stress field deduced through a F.E.A. and assumed as representative of the given continuum. The theoretical principles and the mathematical formulation of the method are firstly explained; the search for a S&T model suitable for the design of a deep beam shows the method capability in handling the reference stress path. Finally, since the analysis may consider the structure as linear-elastic or cracked and non-linear in both the component materials, it is shown how the proposed procedure allows us to verify the possibilities of activation of the design model, oriented to the serviceability condition and deduced in the linear elastic field, by following the evolution of the resistant mechanisms in the cracked non-linear field up to the structural failure.

Key words: Strut-and-Tie models; R.C. analysis and design; structural optimisation.

1. Introduction

When we consider the transition to a highly technological environment, every aspect of human life must be guaranteed by sound safety requirements. As regards buildings and structures, in spite

† Research Associate
‡ Professor

*The earlier version of this paper appears in proceedings of ASEM '99, 23-25 August 1999, Seoul, Korea of a growing knowledge, advanced technology and materials which should have reduced defects over the years, it is still quite common to find both large and small structural deficiencies in structures. However there is no doubt that the final practical product has not always met the engineers' original abstract concept. From a structural analysis point of view, the F.E.M. allows us to handle any type of complex structure whatsoever. In spite of this, the design of the same structure as a Reinforced Concrete (R.C.) element is neither immediate nor unique and we can say that there is no general procedure to pass from a given stress field to a corresponding resistant scheme.

As regards the computation of slender beams subjected to axial force, flexure, shear and torsion, codes and manuals propose simple and reliable as well as refined solutions, for both the serviceability and the ultimate state. On the contrary, structural elements which cannot fit the standard beam theory, such as the diffusion areas, seen both as complete structural elements or as local zones having higher stress gradients, were and are still considered problematical. In fact, over the past two decades fruitful contributions have been given to this field by intensive research.

At present, a general solution to the diffusion problems in R.C. structures may be deduced from the static methods of limit analysis by means of discrete schemes suitable to model the load transfer mechanism and share the carrying functions between concrete and steel reinforcement (Malerba 1999). The most common of these schemes models the R.C. members through elementary stress elements like struts and ties (Marti 1985, Schlaich *et al.* 1987, Schlaich and Schäfer 1991), stringers and shear panels (Blaauwendraad and Hoogenboom 1996).

A Strut-and-Tie (S&T) model is generally formed by a set of prismatic elements working in uni-axial stress state and connected between them by polyhedral nodal regions working in multi-axial stress state. The prismatic elements are usually identified with concrete struts and steel ties, while the polyhedral regions consist of the concrete volume which surrounds the intersection of the axes of such elements and/or of the lines along which the loads and reactions act. If the equilibrium conditions and the limiting strength of the materials are satisfied, according to the lower bound theorem of the limit analysis, the S&T model leads us to a safe evaluation of the ultimate load carrying capacity of the structure.

In reality, the actual behaviour of the concrete is quite different from the ideal hypothesis of perfect plasticity, as assumed by the theory of limit analysis. It follows that the validity of such a fundamental result is subordinated to the verification of some design conditions (Nielsen 1984):

(1) At the *material level*, the concrete strength must be reduced by a suitable efficiency factor.

(2) At the *element level*, (a) the integrity of the concrete struts must be assured by a transversal reinforcement able to support eventual transversal tensile stresses, and (b) the reinforcement used for the ties must be properly anchored beyond the nodal regions.

(3) At the *structural level*, the assumed resistant mechanism must be suitable to be activated before that the limited strain capacity of the materials is reached.

The fulfilment of these statements assures that local collapses don't arise in the nodal zones. This allows us to focus directly on the evaluation of the flow of the forces conveyed into the concrete struts and the steel ties. According to this purpose, the stress field corresponding to the S&T model can be condensed into a truss model whose bars are connected at the intersection of the axes of the prismatic elements and of the lines of the thrust of the applied loads and of the support reactions. Since such a truss usually results statically determinate, the forces in the bars can be computed on the basis of the equilibrium equations only.

The standard approach to define a proper S&T model starts from a reference stress field, usually deduced through an elastic analysis, and lays out the truss elements by modelling the curvilinear paths of the isostatic flow through polygonal lines. As known, such a procedure is not only safe with respect to the ultimate limit states, but insures the serviceability requirements too (Schlaich *et al.* 1987, Schlaich and Schäfer 1991). However, as it is easy to understand, the choice of a proper S&T model is not unique and the definition of a reliable equilibrated load path, even if implicitly safe from the theoretical point of view, is neither easy nor immediate. Therefore, even if a wide literature and special publications present solutions for many cases of the practice (CEB 1982, IABSE 1991), the problem in creating a S&T model of an arbitrary given structural element remains open. For these reasons, the formulation of methods able to find proper S&T models in a systematic and automatic way, is of topical and wide interest.

This work presents a general method for the automatic search for S&T models representative of possible resistant mechanisms in R.C. elements having arbitrary geometry and arbitrary loads and restraints (Biondini 1996, Biondini *et al.* 1996, 1999). The representativeness criterion here adopted is inspired by the principle of minimum strain energy and requires the consistency of the model with a reference stress field. In this way, an optimisation problem is formulated: its solution identifies the searched model. In particular, the theoretical principles and the mathematical formulation of the method are explained and the search for a S&T model suitable for the design of a deep beam shows its capability in handling the reference stress path. Finally, since the analysis may consider the structure as linear-elastic or cracked and non-linear in both the component materials, it is shown how the proposed procedure allows us to verify the possibilities of activation of the design model, oriented to the serviceability condition and deduced in the linear elastic field, by following the evolution of the resistant mechanisms in the cracked non-linear field up to the structural failure.

2. Formulation of the optimisation problem

For the sake of synthesis, but without any loss of generality, we will develop our considerations by referring to the single span uniformly loaded deep beam, having span/depth ratio $l/L = 0.9$,

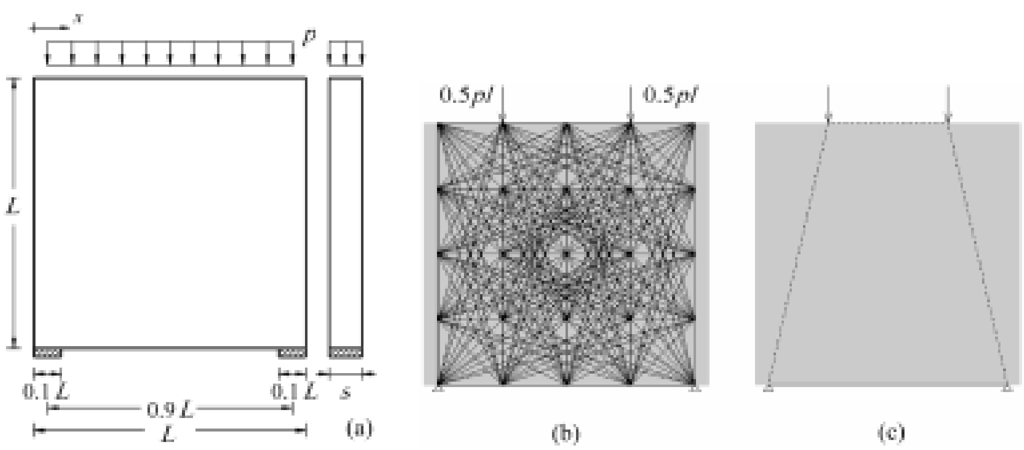

Fig. 1 Structural element (deep beam): (a) geometry and boundary conditions; (b) basic truss; (c) maximum stiffness-minimum volume truss

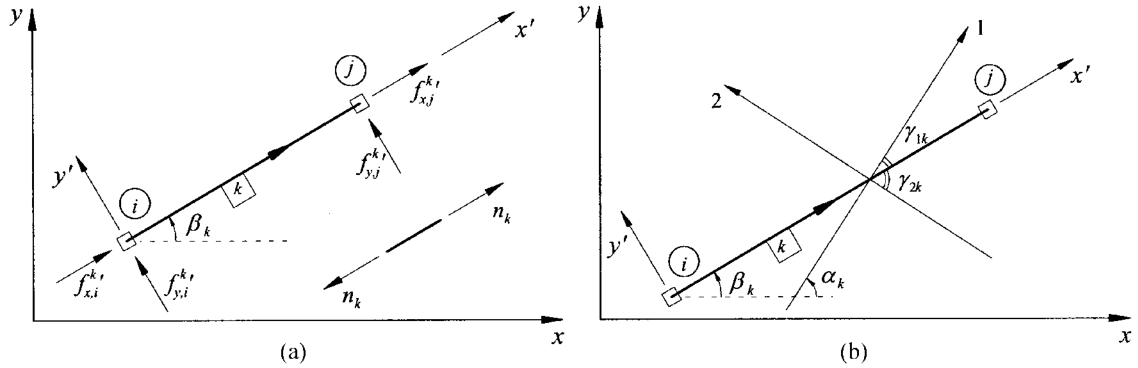

Fig. 2 Generic element of the basic truss: (a) local and global reference systems and sign conventions; (b) orientation in the principal stress field

shown in Fig. 1(a). The region within the element defines the existence domain of the admissible S&T models. A search for absolute optima requires a selection from an infinite set of possible trusses. An approximation of the optimum can be achieved by covering the assigned continuum domain with a closely spaced grid of n nodal points interconnected by $m \le n(n-1)/2$ bar elements and assuming this network as the new existence domain (Hemp 1973). Clearly the net of the nodes of this *basic truss* must include all the load points and all the supports. Therefore distributed loads and continuous supports will be respectively represented by statically equivalent concentrated loads and by suitable nodal restraints (Fig. 1b).

2.1 Equilibrium and conformity equations

We write the equilibrium equation of the generic bar k, in the local (x', y') and in the global (x, y) reference systems, rotated, with respect to each other, by the β_k angle (Fig. 2a):

$$\begin{bmatrix} f_{x,i}^{k\prime} \\ f_{y,i}^{k\prime} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} n_k, \quad \begin{bmatrix} f_{x,j}^{k\prime} \\ f_{y,j}^{k\prime} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} n_k \quad \Rightarrow \quad \begin{matrix} \boldsymbol{f}_i^{k\prime} = \boldsymbol{h}_i^{k\prime} n_k \\ \boldsymbol{f}_j^{k\prime} = \boldsymbol{h}_j^{k\prime} n_k \end{matrix} \tag{1}$$

$$\boldsymbol{T}_k = \begin{bmatrix} \cos\beta_k & -\sin\beta_k \\ \sin\beta_k & \cos\beta_k \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} \boldsymbol{f}_i^{k} = \boldsymbol{T}_k \boldsymbol{f}_i^{k\prime} = \boldsymbol{T}_k \boldsymbol{h}_i^{k\prime} n_k = \boldsymbol{h}_i^{k} n_k \\ \boldsymbol{f}_j^{k} = \boldsymbol{T}_k \boldsymbol{f}_i^{k\prime} = \boldsymbol{T}_k \boldsymbol{h}_j^{k\prime} n_k = \boldsymbol{h}_j^{k} n_k \end{matrix} \tag{2}$$

By assembling the force vectors converging to a generic node s, $\boldsymbol{f}_s = \Sigma_{k \to s} \boldsymbol{f}_s^{k}$, one obtains the overall equilibrium equations for a truss having n nodes and m elements:

$$\begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \cdot \\ \boldsymbol{f}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{h}_1^1 & \boldsymbol{h}_1^2 & \cdot & \cdot & \boldsymbol{h}_1^m \\ \boldsymbol{h}_2^1 & \boldsymbol{h}_2^2 & \cdot & \cdot & \boldsymbol{h}_2^m \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \boldsymbol{h}_n^1 & \boldsymbol{h}_n^2 & \cdot & \cdot & \boldsymbol{h}_n^m \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ \cdot \\ \cdot \\ n_m \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{f} = \boldsymbol{Hr} \tag{3a}$$

where

$$\boldsymbol{h}_s^k = \begin{cases} \boldsymbol{h}_i^k & \text{if } k \to s \text{ with } i \\ \boldsymbol{h}_j^k & \text{if } k \to s \text{ with } j \\ \boldsymbol{0} & \text{if } s \notin k \end{cases} \tag{3b}$$

where $\boldsymbol{f}$ is the vector of the nodal forces, $\boldsymbol{r}$ is the vector of the axial forces and $\boldsymbol{H}$ is the equilibrium matrix (Livesley 1975). Such a system may be modified to take the prescribed displacements into account. In the following, we will implicitly assume that the equilibrium matrix $\boldsymbol{H}$ has rank $2n<m$. This assumption justifies the search for an optimal solution.

Finally, the vector $\boldsymbol{r}$ of the axial forces in the bars must comply with the following conformity conditions:

$$-\boldsymbol{r}^{-} \leq \boldsymbol{r} \leq \boldsymbol{r}^{+} \tag{4}$$

where $\boldsymbol{r}^{+} \geq \mathbf{0}$ and $\boldsymbol{r}^{-} \geq \mathbf{0}$ are respectively the limits due to the tension and compression strength of the m elements.

2.2 Maximum stiffness-minimum volume truss

Since in nature load transmission works in such a way that the associated strain energy results minimum, a rational design philosophy aims to look for the maximum stiffness truss which, for a given load condition, coincides with that of the minimum volume of material (Hemp 1973, Kumar 1978). If one calls respectively $\boldsymbol{a}$ and $\boldsymbol{l}$ the vectors of the areas of the cross sections a_k and of the lengths l_k of the bars $k = 1, ..., n$, the total volume of the system results:

$$V = \sum_{k=1}^{m} a_k l_k = \boldsymbol{l}^T \boldsymbol{a} \tag{5}$$

and the problem in finding the truss having maximum stiffness and satisfying equilibrium and conformity conditions can be reduced to the following linear programming problem:

$$\min \ \{\boldsymbol{l}^T \boldsymbol{a} \,|\, \boldsymbol{H}\boldsymbol{r} = \boldsymbol{f},\ -\sigma^{-}\boldsymbol{a} \leq \boldsymbol{r} \leq \sigma^{+}\boldsymbol{a},\ \boldsymbol{a} \geq 0\} \tag{6}$$

which involves $2n$ constraints and $2m$ variables $\boldsymbol{r}$ and $\boldsymbol{a}$. The stresses $\sigma^{+} \geq 0$ and $\sigma^{-} \geq 0$ are respectively the level of tension and compression strength of the material, assumed here, for simplicity and without loss of generality, the same for all the elements.

In the following $2m$ additional variables $\boldsymbol{a}^{+}$ and $\boldsymbol{a}^{-}$ defined as follows:

$$\boldsymbol{r} - \sigma^{+}\boldsymbol{a} + (\sigma^{+} + \sigma^{-})\boldsymbol{a}^{-} = 0 \tag{7}$$

$$\boldsymbol{r} + \sigma^{-}\boldsymbol{a} - (\sigma^{+} + \sigma^{-})\boldsymbol{a}^{+} = 0 \tag{8}$$

are introduced. They respectively represent the possible areas of the cross section of the ties $\boldsymbol{a}^{+}$ and of the struts $\boldsymbol{a}^{-}$ ($a_k^{+} a_k^{-} = 0$). In particular:

(a) if $a_k^{+} > 0$ and $a_k^{-} = 0$, the element k is a tie, having $a_k = a_k^{+}$, $r_k = \sigma^{+}$;
(b) if $a_k^{+} = 0$ and $a_k^{-} > 0$, the element k is a strut, having $a_k = a_k^{-}$ and $r_k = \sigma^{-}$;
(c) if $a_k^{+} = a_k^{-} = 0$ the element k doesn't belong to the optimal truss.

Moreover, for the equivalence of the search criterion between tensioned and compressed elements, initially the following bounds $\sigma^{+} = \sigma^{-} = \sigma$ are temporarily assumed. Thus, by removing the vector $\boldsymbol{r}$, the previous linear program can be reduced to the following normal form:

$$\min\ \{\boldsymbol{l}^T(\boldsymbol{a}^+ + \boldsymbol{a}^-)\,\big|\,\sigma\boldsymbol{H}(\boldsymbol{a}^+ - \boldsymbol{a}^-) = \boldsymbol{f},\ \ \boldsymbol{a}^+ \geq \boldsymbol{0},\ \boldsymbol{a}^- \geq 0\} \tag{9}$$

again with $2n$ constraints and $2m$ variables $\boldsymbol{a}^+$ and $\boldsymbol{a}^-$. When the optimum truss is found, the actual resistant sections of the bars are deduced on the basis of the actual limit strength of the materials.

The previous linear program can be solved by using the well-known simplex algorithm (Hadley 1962). The optimum truss resulting from the direct optimisation of the basic truss in Fig. 1(b) is shown in Fig. 1(c).

2.3 Consistency of the S&T model with the actual stress field and new optimum criterion

The truss found with the previous criteria ignores the actual two-dimensional behaviour of the assigned continuum system and has to be considered as a qualitative reference. In fact, since the constitutive behaviour of the concrete is quite different from the ideal hypothesis of perfect plasticity, as assumed by the theory of limit analysis, the reliability of a S&T model deduced in this way is assured only from a theoretical point of view, while no guarantees are given about its capability of activation before that the limited strain capacity of the materials is reached. Hence, for a consistent design, some new criteria are needed.

By starting from the same basic truss, a more reliable force path can be deduced if some condition of similarity to the actual stress field in the given continuum is imposed. To this aim, we suppose that the structural element has been studied separately as a 2D continuum. We call it *reference continuum*. The stress field is deduced by means of a F.E.A. which may consider the structure as linear-elastic or cracked and non-linear in both the component materials. Being related to the lower bound theorem, the S&T model is independent with respect to kinematics. However, if the assumed load path differs markedly compared to those which develop in the cracked structure, it may not be allowed among the actual possible redistribution schemes. For these reasons, the design usually refers to the serviceability states and, hence, to the field of the linear elastic principal stresses.

Let $\sigma_1(x, y)$, $\sigma_2(x, y) \leq \sigma_1(x, y)$ and $0 \leq \sigma(x, y) \leq \pi$ respectively be the principal stresses and the angle which the direction 1 forms with the axis x. Hence, along each element k $\sigma_{1k}(x')$, $\sigma_{2k}(x')$ and $\sigma_k(x')$ are known functions as well as the angles $\gamma_{1k}(x') \leq \pi/2$ and $\gamma_{2k}(x') = \pi/2 - \gamma_{1k}(x')$ shown in Fig. 2(b). By assuming:

$$p = \{t \mid \gamma_{tk} = \min\{\gamma_{1k}; \gamma_{2k}\} \leq \pi/4, \qquad t = 1, 2\} \tag{10}$$

$$q = \{t \mid \gamma_{tk} = \max\{\gamma_{1k}; \gamma_{2k}\} \geq \pi/4, \qquad t = 1, 2\} \tag{11}$$

we can define the following parameters:

$$\gamma_k = \frac{1}{l_k}\int_0^{l_k} \gamma_{pk}\,dx' \tag{12}$$

$$\sigma_{//k} = \frac{1}{l_k}\int_0^{l_k} \sigma_{pk}\cos\gamma_{pk}\,dx' \tag{13}$$

$$\sigma_{\perp k} = \frac{1}{l_k}\int_0^{l_k} \sigma_{qk}\sin\gamma_{qk}\,dx' \tag{14}$$

where γ_k represents an average angular deviation of the bar k with respect to trajectories of the principal directions which intersect it, while $\sigma_{//k}$ and $\sigma_{\perp k}$ in average indicate the type of the prevailing stress around such an element (Fig. 2b). At first, the parameter $\sigma_{//k}$ leads us to recognise if a bar is a strut or a tie In this way, the number of variables halves and the linear program assumes the following form:

$$\min\{\boldsymbol{l}^T\boldsymbol{a} \mid \sigma\,\overline{\boldsymbol{H}}\boldsymbol{a} = \boldsymbol{f},\ \ \boldsymbol{a} \geq \boldsymbol{0}\} \tag{15}$$

where the modified equilibrium matrix $\overline{\boldsymbol{H}}$ takes the signs of the axial forces $\boldsymbol{r}$ into account:

$$\overline{\boldsymbol{H}} = \begin{bmatrix} \bar{\boldsymbol{h}}_1^1 & \bar{\boldsymbol{h}}_1^2 & . & . & \bar{\boldsymbol{h}}_1^m \\ \bar{\boldsymbol{h}}_2^1 & \bar{\boldsymbol{h}}_2^2 & . & . & \bar{\boldsymbol{h}}_2^m \\ . & . & . & . & . \\ \bar{\boldsymbol{h}}_n^1 & \bar{\boldsymbol{h}}_n^2 & . & . & \bar{\boldsymbol{h}}_n^m \end{bmatrix}, \qquad \bar{\boldsymbol{h}}_s^k = \frac{\sigma_{//k}}{|\sigma_{//k}|}\boldsymbol{h}_s^k \tag{16}$$

Subsequently, the optimisation criterion is improved by weighting the contribution of each bar to the objective function according to its orientation with respect to the stress field of the reference continuum and to its capability in reproducing the continuum stress path. As regards this, higher penalty weights have to be applied to higher average angular deviations γ_k. Moreover, since for the tensioned elements, to be built through steel bars, a regular and straight tracing is preferable, a lower penalty weight has to be attributed to the longer ties. However, curved trajectories can be better fitted by short elements. Hence by giving a lower penalty weight to the shorter struts, one makes the compressed members adapt better to the local stress path. Let

$$\Gamma_k(\gamma_k) = \tan\gamma_k \in [0;1] \tag{17}$$

$$\lambda_k(l_k) = \begin{cases} \dfrac{l_k}{\max\{l_k|k=1,m\}} \in [0;1], & \text{if} \quad \sigma_{//k} > 0 \\ 1 - \dfrac{l_k}{\max\{l_k|k=1,m\}} \in [0;1], & \text{if} \quad \sigma_{//k} \leq 0 \end{cases} \tag{18}$$

the weighting penalty function is formulated as follows:

$$w_k(\gamma_k,\ l_k) = \Gamma_k^{\mu\lambda_k} \in [0;1] \tag{19}$$

where $\mu \geq 0$ is a numerical coefficient heuristically deduced. With these assumptions, the total volume is the sum of the single weighted volumes:

$$V_{eq} = \sum_{k=1}^{m} w_k a_k l_k = (\boldsymbol{Wl})^T\boldsymbol{a} = \boldsymbol{l}_{eq}^T\boldsymbol{a} \tag{20}$$

where $\boldsymbol{l}_{eq} = \boldsymbol{Wl}$ is a vector of *equivalent lengths* and $\boldsymbol{W}$ is the diagonal matrix of the weights. The final form of the linear programming problem is given by:

$$\min\{\boldsymbol{l}_{eq}^T\boldsymbol{a} \mid \sigma\overline{\boldsymbol{H}}\boldsymbol{a} = \boldsymbol{f},\ \boldsymbol{a} \geq \boldsymbol{0}\} \tag{21}$$

The optimal truss, consistent with the stress field in the reference linear elastic continuum of Figs. 3(a) and (b), is shown in Fig. 3(c) ($\mu = 1$).

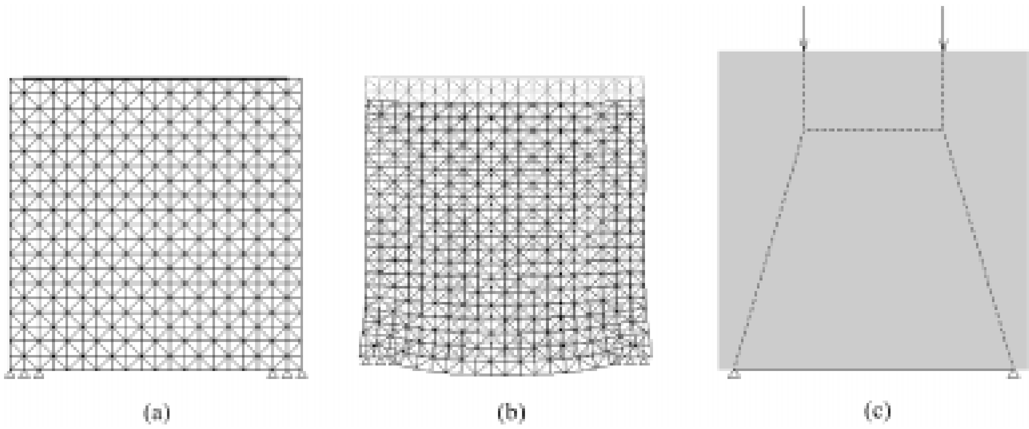

Fig. 3 (a) F.E. discretisation, (b) deformed shape, (c) minimum weighted volume truss

2.4 Improvement of the search procedure

The scheme of Fig. 3(c) seems suitable for actual design, but one can observe that the bottom tie has no cover and the stress diffusion zones in the neighbourhood of the supports are not taken into account by this model. Finer nets can certainly give better results, but their computational demand grows with m and hence with n^2.

A more general approach may consider as design variables the nodal coordinates of the basic truss too. In this case, by denoting with $\boldsymbol{x}$ the vector of such coordinates, the correlation $\boldsymbol{l} = \boldsymbol{l}(\boldsymbol{x})$ and $\overline{\boldsymbol{H}} = \overline{\boldsymbol{H}}(\boldsymbol{x})$ yields to a non-linear optimisation problem. However, the linearity with respect to the vector of the areas $\boldsymbol{a}$ can be preserved if the solution procedure is subdivided in two nested levels (Biondini *et al.* 2000). At the first level, or *macro-level*, the non-linear problem is solved as a function of the vector $\boldsymbol{x}$ only. Formally, by denoting with $\boldsymbol{x}^-$ and $\boldsymbol{x}^+$ its lower and the upper limits, respectively, one has:

$$\min \{ V_{eq}(\boldsymbol{x}) \mid \boldsymbol{x}^- \le \boldsymbol{x} \le \boldsymbol{x}^+ \} \tag{22}$$

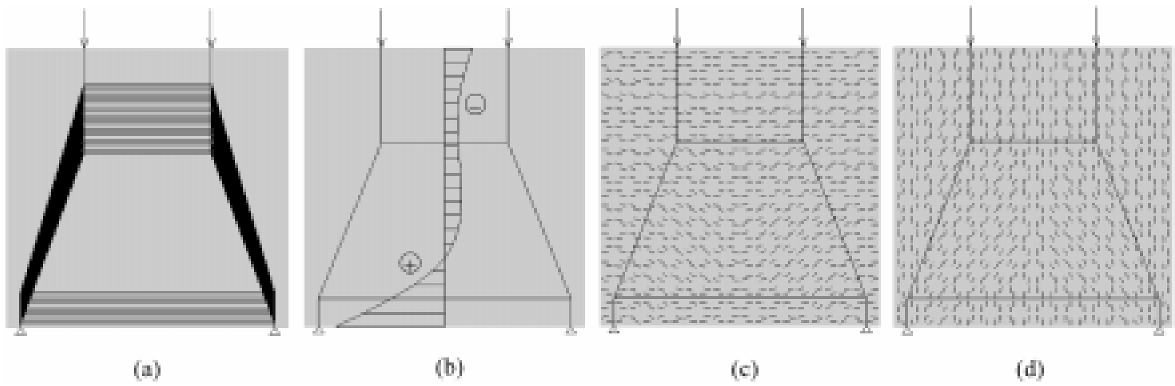

Fig. 4 (a) New basic truss; comparison of the optimal S&T model with (b) the stress diagram along the middle section and the direction of (c) maximum and (d) minimum principal stresses

At the second level, or *micro-level*, the optimal vector $\boldsymbol{a}$ for the basic truss at current iteration is instead found by solving the linear programming problem previously formulated. As regards the solution of the non-linear problem, a derivative-free optimisation method should be used, since the objective function $V_{eq} = V_{eq}(\boldsymbol{x})$ is available only in an implicit way. The results of recent works show that the genetic algorithms can be considered as ideal candidates to handle such kind of problems (Biondini *et al.* 2000).

Alternatively, in order to maintain the numerical advantages of a fully linear formulation, one can consider a different way to solve the same problem: the search criteria already introduced will be now applied to the new basic truss given by the parametric transformation of the nodal coordinates which define the shape of the previously optimised truss (Fig. 4a). The optimal S&T model resulting from this further optimisation process is shown in Fig. 4(b). The reliability of the internal forces path, which the optimal S&T model implicitly defines, is finally evaluated by making a comparison with the normal stress diagram along the middle section (Fig. 4b) and with the direction of the principal stresses, respectively maximum and minimum (Figs. 4c and d).

2.5 Dimensioning the bars

Let f_c the uniaxial compression strength of the concrete. In the struts the limit strength is assumed as $f_c^* = \nu_c(0.85 f_c) = \nu_c f_{c1}$, where $\nu_c \leq 1$ is an *efficiency factor* (Nielsen 1984), which is assumed according to the actual multiaxial stress state of the reference continuum as follows:

$$\begin{cases} \nu_{ck} = 0.2(4 - \Gamma_k) & , \text{ if } \sigma_{\perp k} > 0 \\ \nu_{ck} = 1.0 & , \text{ if } \sigma_{\perp k} \leq 0 \end{cases} \tag{23}$$

where Γ_k is defined by Eq. (17). In a similar way, we can write $f_y = \nu_s f_{c1}$ for the steel. By putting $\sigma = f_{c1}$, it is then possible to evaluate the effective areas of the resistant section of the optimal S&T model (Fig. 5, Table 1):

$$\begin{cases} a_{sk} = a_k / \nu_{sk} & , \text{ if } \sigma_{//k} > 0 \\ a_{ck} = a_k / \nu_{ck} & , \text{ if } \sigma_{//k} \leq 0 \end{cases} \tag{24}$$

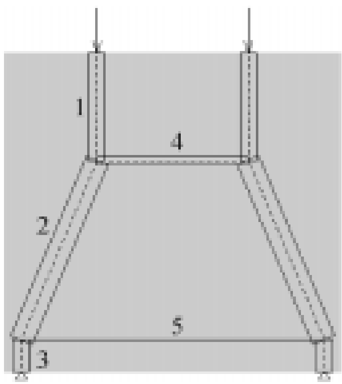

Fig. 5 Dimensioning of the bars

Table 1 Dimensioning of the bars

Bar	n_k/pl	ν_k	$a_k f_{c1}/pl$
1	−0.500	1.000	−0.500
2	−0.539	0.769	−0.701
3	−0.500	1.000	−0.500
4	−0.200	1.000	−0.200
5	0.200	ν_s	$0.200/\nu_s$

3. Evolution of the optimal S&T model in the cracked state

The structural response of the R.C. deep beam designed according to the found S&T model is investigated through N.L.F.E.A. The non-linear analysis is based on the Modified Compression Field Theory (MCFT), which views the cracked R.C. as a composite, orthotropic non-linear hyperelastic material, having orthogonal smeared reinforcement and presenting smeared rotating cracks (Vecchio and Collins 1986).

3.1 Design of the structural element and verification of the ultimate capacity

The deep beam is considered to have total length $L = 1.00$ m and thickness $s = 120$ mm. The following material properties are also assumed: $f_c = 30$ MPa, $f_{ct} = 2.9$ MPa, $E_c = 34$ GPa, $\nu = 0.2$, $\varepsilon_{cu} = 2\varepsilon_{c0} = 4‰$, $f_y = 430$ MPa, $E_s = 206$ GPa, $\varepsilon_{su} = 1\%$. The minimum reinforcement section corresponding to a design load level $p_0 = 350$ kN/m results $A_s = 147$ mm^2. We adopt 2∅10 ($A_s = 157$ mm^2). According to the lower bound theorem, with adequate values of the efficiency factors ν_c, the collapse load of the beam so reinforced should result no lower than p_0. In order to verify this expectation, we assume the geometrical reinforcement ratios $\rho_{x1} = 1.31\%$ and $\rho_{y1} = \rho_{x2} = \rho_{y2} = 0$ (Fig. 6a). The non-linear analysis in the cracked state gives a collapse load $p_u \cong 440$ kN/m .

As mentioned, the designers usually refer to the serviceability limit state, ensuring, with this choice, equilibrium conditions which can hold until the ultimate limit state. Anyway, even if the

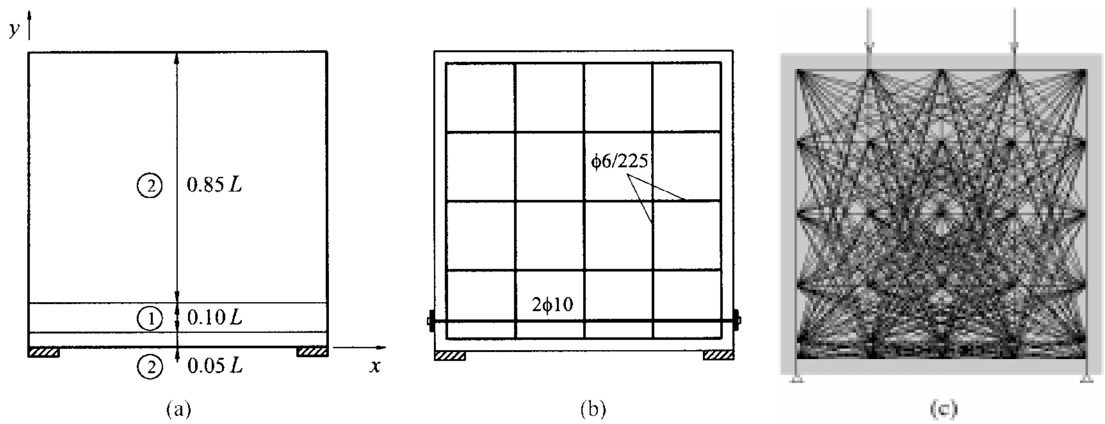

Fig. 6 (a) Zones having different ρ values; (b) reinforcement layout; (c) *basic truss* with fixed ties in correspondence of the actual reinforcement assumed S&T model is optimised with respect to the service stress field, in the zones without reinforcement a crack pattern can start which doesn't allow the stress redistribution required to reach higher values of the load. For this reason, diffused meshes of reinforcement are usually added to compensate for the the limited tensile resistance of the concrete. In our example, a secondary mesh reinforcement ∅6/225 mm ($\omega = \rho f_y / f_c \cong 0.03$) on both the faces has been adopted (Fig. 6b). With the new geometrical reinforcement ratios $\rho_{x1} = 1.55\%$ and $\rho_{y1} = \rho_{x2} = \rho_{y2} = 0.24\%$, the N.L.F.E.A. reaches a higher collapse load at $p_u \cong 870$ kN/m. The relevant growth of p_u from 350 to 870 kN/m is a direct consequence of the additional reinforcement which makes the evolution towards new and stronger resistant mechanisms possible.

3.2 Evolution of the S&T model

With reference to the stress fields associated to successive load steps ($\Delta p = 150$ kN/m), the optimisation process is repeated by assuming basic trusses whose potential ties lie on the actual position of the bars of the reinforcement (Fig. 6c). Hence for a given stress field we consider a basic truss composed by m_1 potential ties aligned in accordance to the actual reinforcement layout and by m_2 potential struts defined by the condition $\sigma_{//k} \le 0$. Other bars for which $\sigma_{//k} > 0$ are discarded during the solving procedure ($m_1 + m_2 = m^* \le m$). The optimisation problem in this case leads us to a linear program expressed in the following generalised normal form:

$$\min \ \{\boldsymbol{l}_{eq}^T \boldsymbol{a}^* | \sigma \overline{\overline{\boldsymbol{H}}} \boldsymbol{a}^* = \boldsymbol{f}, \quad \boldsymbol{0} \le \boldsymbol{a}_1 \le \boldsymbol{a}_{1\max}, \ \boldsymbol{a}_2 \ge \boldsymbol{0}\} \qquad \boldsymbol{a}^* = \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \end{bmatrix} \tag{25}$$

where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are, respectively, the vectors of the section areas of the potential ties and struts, $\boldsymbol{a}_{1\max}$ is the vector of the given section areas and $\overline{\overline{\boldsymbol{H}}}$ is an equilibrium matrix once again modified to take the signs of the axial forces $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ into account:

$$\overline{\overline{\boldsymbol{H}}} = \begin{bmatrix} \overline{\overline{\boldsymbol{h}}}_1^1 & \overline{\overline{\boldsymbol{h}}}_1^2 & . & . & \overline{\overline{\boldsymbol{h}}}_1^{m*} \\ \overline{\overline{\boldsymbol{h}}}_2^1 & \overline{\overline{\boldsymbol{h}}}_2^2 & . & . & \overline{\overline{\boldsymbol{h}}}_2^{m*} \\ . & . & . & . & . \\ \overline{\overline{\boldsymbol{h}}}_n^1 & \overline{\overline{\boldsymbol{h}}}_n^2 & . & . & \overline{\overline{\boldsymbol{h}}}_n^{m*} \end{bmatrix}, \qquad \overline{\overline{\boldsymbol{h}}}_s^k = \begin{cases} \boldsymbol{h}_s^k, & k \le m_1 \\ -\boldsymbol{h}_s^k, & m_1 < k \le m^* \end{cases} \tag{26}$$

The evolution of the optimal S&T model shown in Fig. 7 allows us to understand the progressive development of the carrying mechanisms and the role of the secondary reinforcement in increasing the ultimate load. In fact, the added bars contribute in containing the transversal tensions and allow the evolution of the initial carrying scheme, shaped as a *fuse*, to the final *arch shaped form*, characterised by the increased internal lever arm and by a corresponding higher intensity of the ultimate load.

4. Conclusions

A procedure for the automatic search for optimal S&T models in R.C. elements has been proposed. The structural problem is discretised by replacing the assigned continuum domain with a suitable *basic truss* and then translated into a mathematical linear programming problem. In particular, the layout optimisation process is carried out by weighting the contribution of each bar to

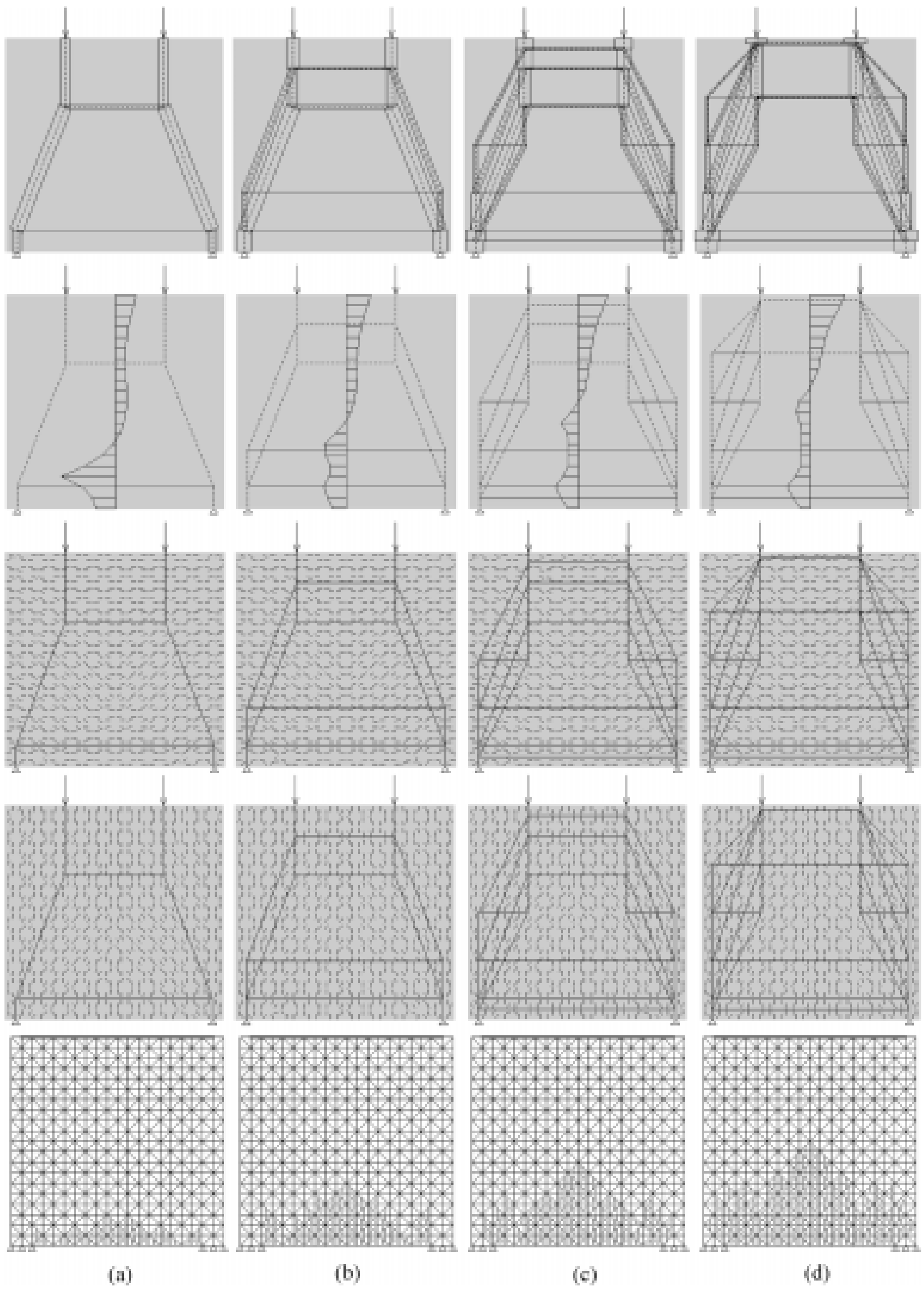

Fig. 7 Evolution of the optimal S&T model and comparison with the normal stress diagram along the middle section, with the direction of maximum and minimum principal stresses and with the cracking state at several loading stages *p*: (a) 350, (b) 500, (c) 650 and (d) 800 kN/m

the objective function (the volume of the truss), according to a given reference stress field.

A comparison between the models obtained disregarding the actual two-dimensional behaviour ($\mu = 0$) and the optimal one ($\mu = 1$) lead us to appreciate the capacities of the proposed optimum criterion in handling complex stress paths. In particular, the following fundamental aspects are accomplished:

– the optimal S&T models show a good agreement with the solutions presented in literature;
– the alignment of the bars averages the flow of the actual stresses accurately;
– the principal struts and ties are localised near the centroid of the stress diagram of typical sections;
– the resultant tension and compression forces, as well as the relative lever arms, are accurate and in accordance with the theory and the experiments.

As well known, the load carrying capacities evaluated through the S&T models are underestimated in comparison to their real ones, as shown by the experiments or deduced through a full stress non-linear analysis. A lower value of the collapse load intensity is a direct consequence of the lower bound theorem of the limit analysis. However, in many cases the true collapse load can be very much higher than the theoretical one. The higher performances of the structure are due to additional mechanisms which the assumed initial model is no longer able to activate. Such a model, in fact, represents only one of the possible redistribution schemes. In particular, the activation of new schemes is made possible by the presence of other resistant elements which initially don't work but afterwards are activated as the load increases.

The S&T modelling can contribute to highlight such a hierarchical sequence of the static functions. For this reason, the structural response of a R.C. element, designed according to the previous criteria, has been investigated through full stress membrane analyses in the cracked state and up to collapse. Subsequently, by assuming the stress field given by a non-linear analysis in the cracked state as a new reference, the optimisation process is again applied to obtain S&T models also able to reproduce the evolution of the actual load path. The corresponding evolution of the optimal S&T models allows us to understand the progressive development of the carrying mechanisms and the role of the secondary reinforcement details in increasing the ultimate load.

Acknowledgements

This research is supported by the MURST project 40%-1997, Udine Unit, and by the Department of Civil Engineering of the University of Udine (Italy).

References

Biondini, F. (1996), "*Search for optimal Strut-and-Tie models in reinforced concrete elements*", Dissertation, Post-Graduate School for Reinforced Concrete Structures "F.lli Pesenti", Tecnical University of Milan (in Italian).

Biondini, F., Bontempi, F., and Malerba, P.G. (1996), "Search for Strut-and-Tie models by linear programming", *Studi e Ricerche*, Scuola di Specializzazione in Costruzioni in Cemento Armato, Politecnico di Milano, **17**, 121-156 (in Italian).

Biondini, F., Bontempi, F., and Malerba, P.G. (1999), "Optimal Strut-and-Tie models in reinforced concrete structures", *Computer Assisted Mechanics and Engineering Sciences*, **6**(3-4), 279-293.

Biondini, F., Bontempi, F., and Malerba, P.G. (2000), "The search for structural schemes by optimality criteria and soft-computing techniques", *Proc. of Structural Morphology Conference*, 179-187, Delft, The Netherlands, August 17-19.

Blaauwendraad, J., and Hoogenboom, P.C.J. (1996), "Stringer panel model for structural concrete design", *ACI Sruct. J.*, **93**(3), 295-305.

CEB (1982), *Detailing of Concrete Structures*, Bulletin DInformation, **150**.

Hadley, G. (1962), *Linear Programming*, Addison-Wesley, Reading (MA).

Hemp, W.S. (1973), *Optimum Structures*, Clarendon Press, Oxford.

IABSE (1991), *Structural Concrete,* Stuttgart 1991, **62**.

Kumar, P. (1978), "Optimal force transmission in reinforced concrete deep beams", *Comput. & Struct.*, **8**(2), 223-229.

Livesley, R.K. (1975), *Matrix Methods of Structural Analysis*, Pergamon Press.

Malerba, P.G. (1999), "Modelling of the diffusion zones in R.C. elements through discrete schemes", *Proc. of the First Int. Conf. on Adv. in Struct. Eng. & Mech. (ASEM99),* August 23-25, Seoul, Korea.

Marti, P. (1985), "Basic tools of reinforced concrete beam design", *ACI Struct. J.*, **82**(1), 46-56.

Nielsen, M.P. (1984), *Limit Analysis and Concrete Plasticity*, Prentice-Hall, Upper Saddle River (NJ).

Schlaich, J., Schäfer, K., and Jennewin, M. (1987), "Toward a consistent design of structural concrete", *PCI J.*, **32**(3), 72-150.

Schlaich, J., and Schäfer, K. (1991), "Design and detailing of structural concrete using Strut-and-Tie models", *The Struct. Engineer*, **69**(6), 113-125.

Vecchio, F.J., and Collins, M.P. (1986), "The modified compression field theory for reinforced concrete elements subjected to shear", *ACI Struct. J.*, **83**(2), 219-231.