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Bending of steel fibers on partly supported elastic foundation

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Abstract. Fiber reinforced cementitious composites are nowadays widely applied in civil engineering. The postcracking performance of this material depends on the interaction between a steel fiber, which is obliquely across a crack, and its surrounding matrix. While the partly debonded steel fiber is subjected to pulling out from the matrix and simultaneously subjected to transverse force, it may be modelled as a Bernoulli-Euler beam partly supported on an elastic foundation with non-linearly varying modulus. The fiber bridging the crack may be cut into two parts to simplify the problem (Leung and Li 1992). To obtain the transverse displacement at the cut end of the fiber (Fig. 1), it is convenient to directly solve the corresponding differential equation. At the first glance, it is a classical beam on foundation problem. However, the differential equation is not analytically solvable due to the non-linear distribution of the foundation stiffness. Moreover, since the second order deformation effect is included, the boundary conditions become complex and hence conventional numerical tools such as the spline or difference methods may not be sufficient. In this study, moment equilibrium is the basis for formulation of the fundamental differential equation for the beam (Timoshenko 1956). For the cantilever part of the beam, direct integration is performed. For the non-linearly supported part, a transformation is carried out to reduce the higher order differential equation into one order simultaneous equations. The Runge-Kutta technique is employed for the solution within the boundary domain. Finally, multi-dimensional optimization approaches are carefully tested and applied to find the boundary values that are of interest. The numerical solution procedure is demonstrated to be stable and convergent.

Key words: beam on elastic foundation; non-linear modulus; boundary conditions; cantilever; higher order differential equation; Runge-Kutta technique; optimization approach; downhill simplex method; genetic algorithms.

1. Introduction

With the extensive application of fiber reinforced cementitious composites in civil engineering, research on their behaviour is developing at the microscopic level. The interaction of fiber and matrix is given great attention by experimentalists and researchers. This study focuses on a partly debonded steel fiber that crosses a tension crack in the matrix material. The fiber bridging the crack may be cut into two parts to simplify the problem (Leung and Li 1992). Each half of the fiber is therefore treated as a cantilever beam partly supported on an elastic foundation with varying modulus (Fig. 1). This models the embedded fiber with its axis inclined to the crack face. If the end

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displacement of the fiber between the crack faces is determined, the relationship between crack opening and the pulling-out force can be developed. The relation between the crack opening and applied force can be predicted by integrating over the contributions of those individual fibers that cross the matrix crack plane (Li *et al*. 1991, Li 1992, Jain and Wetherhold 1992, Elser *et al.* 1996) by incorporating into the integration the probability density functions of the fiber orientation angle and embedded length of the fiber.

For an elastic foundation, Winkler (1867) proposed a well-known linear elastic model. However, this model is considered to be very crude. This situation gives rise to the development of more general plate/beam-foundatiton models, which may be roughly classified into two categories, according to Selvadurai (1979), elastic continuum models and Winkler-type models. The former is regarded as mathematically complex. The later indeed have concise mathematical format. The two-parameter elastic models are the representatives of the second category and their development may be mainly attributed to Filonenko-Borodich (1940, 1945), Pasternak (1954), Kerr (1964), Vlazov and Leontiev (1966), and others. On the other hand, non-linear elastic foundation, in analogy to the two-parameter model has been proposed. For instance, three order nonlinear model (Waas 1990), fourth order nonlinear formulation (Huk 1988), hyperbolic sine-type (Kerr 1969), and hyperbolic type (Soldatos and Selvadurai 1985).

It is noted that a third approach to model the foundation response has also been proposed by some authors. In this category, formal power series expansions are performed in order to capture various deformational components in foundation, or both deformation properties in the foundation and mechanical components in pressure response. The first sub-category proposed by Ratzersdorfer (1929, 1936), Favre (1960, 1961), and Levinson-Bharatha (1978, 1979, 1980) (all of which are quoted by Kerr 1984) may be expressed as $p = Lw$, where L is a linear differential operator containing even order derivatives only. The second sub-category is proposed by Kerr and Rhines (1967) (quoted by Kerr 1984) and is expressed as $Lp = Lw$. Kerr (1984) also provided physical explanations indicating that a general foundation includes spring, shearing, and bending effectiveness as well as their combination.

Another treatment to the elastic foundation is proposed by Vallabhan and Das (1988). They used

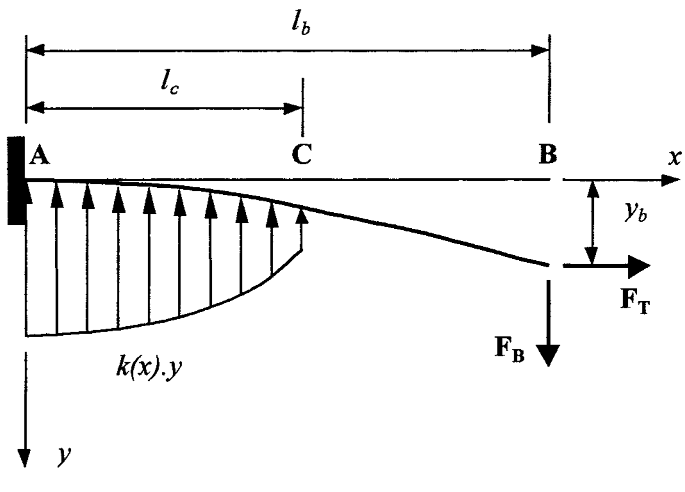

Fig. 1 Analysis of fiber

two displacement functions to obtain matrix foundation response. This method is called the 'matrix foundation model' by the authors.

It is noticed that some models already account for the foundation depth H in the spring constant k (Reissner 1958, Vlazov and Leontiev 1966, Kerr and Rhines 1967). This implies that k may not be a constant with respect to the position if H varies with the coordinate. Hetenyi (1946) has analysed the beam on the foundation with linearly varying modulus, i.e., $k(\xi) = \alpha\xi$. This consideration may result in an essential change in solution method due to the change to the characteristic of the differential equation for the beam deflection. Direct analytic solution seems impossible. Therefore Hetenyi's strategy is to linearly combine four infinite polynomial series:

$$y(\xi) = c_1 y_1(\xi) + c_2 y_2(\xi) + c_3 y_3(\xi) + c_4 y_4(\xi) \tag{1}$$

where y_1 to y_4 are power series, the coefficients c_1 to c_4 are unknowns to be determined by boundary conditions. In the application of this solution by Liu and Lai (1996), the accuracy of the truncation on the 4 infinite series depends on the value of the parameter α in each series, which should be limited to less than 10,000 by truncating the terms after the fifth.

In regard to solving methods, various discretization methods have been tried, for instance, finite difference technique (Lentini 1979, Vallabhan and Das 1991), B-spline technique (Bechtold and Riley 1991), differential quadrature element method (DQEM) (Chen 1998), and finite element method (Ting and Mockry 1984, Mourelatos and Parsons 1987, Leung and Chi 1995, Wasti and Senkaya 1995, Thambiratnam and Zhuge 1996, Al-Nageim *et al.* 1998). However, no recommendation is made on which is better.

If a beam has varying cross section on an elastic foundation with constant modulus, it leads to another sort of differential equation:

$$EI(x) = \frac{d^4 y(x)}{dx^4} + k \cdot y(x) = 0 \tag{2}$$

where E is elastic beam modulus and $I(x)$ is its area moment of inertia of the cross section at x. Again, it is a good choice to use the discretization methods mentioned above.

It is noticed that, for most of the problems in literature, the corresponding boundaries at one end or both form a closed solution system so that no difficulty exists no matter which technique is selected.

In this study, a cantilever beam partly supported on an elastic foundation with varying modulus is investigated (Fig. 1). To simplify the model, one-parameter non-linear modulus foundation is selected. Since the loading mode and the boundary conditions cannot be explicitly expressed for the solution. In order to simplify the solution procedure, transformation of the higher order differential equation into one order simultaneous differential equations is performed and the Runge-Kutta method is utilized for the solution. For the cantilever part (Fig. 1), an analytical formula is used. To deal with the implicit boundaries, a two dimensional optimization technique is employed. The final numerical computation process displays stable convergence.

2. Governing equation of Bernoulli-Euler beam on elastic foundation

According to the finite element analysis result provided by Leung and Li (1992) the foundation stiffness, in the case of a steel fiber lying on an elastic foundation with varying thickness, should be expressed using a non-linear function. Thus the elastic foundation stiffness (Fig. 1) is proposed to be:

$$k(x) = C_a \cdot \left[C_b(l_b - \Delta - x) \cdot \frac{2}{D_f \tan(\varphi)} \right]^r \tag{3}$$

where C_a, C_b, r, and Δ are constants, D_f and φ are diameter and the inclined angle of fiber, respectively.

In anticipation of mathematical difficulty, the general fourth order beam equation is not used here. Instead, the moment at x is equated to the curvature of the fibre. The advantage of this manipulation is that the higher order of foundation reaction will not be lost. Hence the deflection curve of the fiber loaded as in Fig. 1 can be described by the following differential equation (Timoshenko 1956):

$$EI\frac{d^2y}{dx^2} = F_B(l_b - x) - F_T(y_b - y) - H(l_c - x) \cdot \int_x^c k(t)y(t)tdt \tag{4}$$

where E and I are the modulus of elasticity and the moment of inertia of the fiber respectively, $k(x)$ is the elastic stiffness of the foundation, and $H(t)$ is Herath's function:

$$H(l_c - x) = \begin{cases} 0, & l_c - x < 0 \\ 1, & l_c - x > 0 \end{cases} \tag{5}$$

While $x > l_c$, Eq. (4) has a general solution:

$$y = c_1 e^{px} + c_2 e^{-px} + y_b - \frac{F_B}{F_T}(l_b - x) \tag{6}$$

where $p^2 = F_T/EI$. Substituting the boundary conditions into Eq. (6) yields:

$$y|_{x=l_c} = c_1 e^{pl_c} + c_2 e^{-pl_c} + y_b - \frac{F_B}{F_T}(l_b - l_c) = y_c \tag{a}$$

$$y'|_{x=l_c} = c_1 p e^{pl_c} - c_2 p e^{-pl_c} + \frac{F_B}{F_T} = y_c', \quad \text{or} \quad c_1 e^{pl_c} - c_2 e^{-pl_c} + \frac{F_B}{pF_T} = \frac{y_c'}{p} \tag{b}$$

$$y''|_{x=l_b} = c_1 p^2 e^{pl_b} + c_2 p^2 e^{-pl_b} = 0, \quad \text{or} \quad c_1 e^{pl_b} + c_2 e^{-pl_b} = 0 \tag{c}$$

adding Eqs. (a) to (b) and let $\lambda_1 = \frac{F_B}{F_T}(l_b - l_c)$, gives

$$c_1 = \frac{e^{-pl_c}}{2}\left[-y_b + y_c + \frac{y_c'}{p} + \lambda_2 \right] \tag{d}$$

where $\lambda_2 = \lambda_1 - \frac{F_B}{pF_T}$. By subtracting (b) from (a), another constant is obtained:

$$c_2 = \frac{e^{pl_c}}{2}\left[-y_b + y_c - \frac{y_c'}{p} + \lambda_3 \right] \tag{e}$$

where $\lambda_3 = \lambda_1 + \dfrac{F_B}{pF_T}$. Substituting Eqs. (d) and (e) into Eq. (c) gives:

$$\frac{e^{p(l_b-l_c)}}{2}\left[-y_b + y_c + \frac{y_c'}{p} + \lambda_2\right] + \frac{e^{-p(l_b-l_c)}}{2}\left[-y_b + y_c - \frac{y_c'}{p} + \lambda_3\right]=0 \tag{f}$$

let $\alpha = p(l_b - l_c)$ and substituting it into Eq. (f), results in:

$$y_b = y_c + \frac{y_c'}{p} tgh\alpha + \frac{1}{2\cosh\alpha}(\lambda_2 e^{\alpha} + \lambda_3 e^{-\alpha}) \tag{7}$$

where $y_c = y(l_c)$ and $y_c' = y'(l_c)$, which satisfy Eq. (4).

By back-substituting (7) into (d) and (e), and finally into (6), the solution of deflection of the fiber is obtained, provided that the value of y_c and y_c' are known.

3. Numerical solution

While $x < l_c$, Eq. (4) gives the following form:

$$EI\frac{d^2y}{dx^2} = F_B \cdot (l_b - x) - F_T \cdot (y_b - y) + \int_c^x k(t)\cdot y(t)\cdot t dt \tag{8}$$

Differentiating Eq. (8) yields:

$$EI\frac{d^3y}{dx^3} = -F_B + F_T\frac{dy}{dx} + k(x)\cdot xy \tag{9}$$

Formula (9) is a high order differential equation with a variable coefficient function $k(x)$. It is difficult to find direct analytic solution for this equation. Though this equation may be solved with series solutions, it is complex to assess the truncation errors and so to manipulate the inside boundary conditions at location c (Fig. 1). Therefore, numerical methods for the solution are explored.

Let $d_1 = -\dfrac{F_B}{EI}$, $d_2 = \dfrac{1}{EI}$, $d_3 = \dfrac{F_T}{EI}$, Eq. (9) becomes:

$$y''' = d_1 + d_2 k(x) xy + d_3 y' \tag{10}$$

Let us perform the following transformation (Kreyszig 1993):

$$Y_1 = y,\ Y_2 = Y_1' = y',\ Y_3 = Y_2' = y'' \text{, and}$$

$$Y_1' = y' = Y_2,\ Y_2' = y'' = Y_3,\ Y_3' = y''' = d_1 + d_2 k(x)\cdot xY_1 + d_3 Y_2 \text{, or}$$

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ d_2 x k(x) & d_3 & 0 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ d_1 \end{pmatrix} \tag{11}$$

or written in vectorial form:

$$\vec{Y}' = A\vec{Y} + \vec{b} \tag{12}$$

This formula is of standard first order and can be solved with standard numerical methods. In the following, the forth order Runge-Kutta method is employed (Boyce and DiPrima 1997):

$$\vec{Y}_{n+1} = \vec{Y}_n + \frac{\Delta}{6}(\vec{K}_{n1} + 2\vec{K}_{n2} + 2\vec{K}_{n3} + \vec{K}_{n4}) \tag{13}$$

where

$$\vec{K}_{n1} = f\,(\vec{x}_n, \vec{Y}_n),\ \ \vec{K}_{n2} = f\left(\vec{x}_n+\frac{\Delta}{2},\ \vec{Y}_n+\frac{\Delta}{2}\vec{K}_{n1}\right),\ \ \vec{K}_{n3}=f\left(\vec{x}_n+\frac{\Delta}{2},\ \vec{Y}_n+\frac{\Delta}{2}\vec{K}_{n2}\right),\ \vec{K}_{n4}=f\,(\vec{x}_n+\Delta,\ \vec{Y}_n+\Delta\vec{K}_{n3}),$$

where Δ is discrete interval of the variable x.

The boundary conditions for Eqs. (11) and (12) are given as (Fig. 1):

$$\begin{aligned}
&Y_1\big|_{x=0} = y\big|_{x=0} = 0,\ \ Y_2\big|_{x=0} = y'\big|_{x=0} = 0,\\
&Y_3\big|_{x=0} = y''\big|_{x=0} = [F_B l_b - F_T y_b - \textstyle\int_0^c k(x)y(x)xdx]/EI\\
&Y_1\big|_{x=l_c} = y\big|_{x=l_c} = y_c,\ \ Y_2\big|_{x=l_c} = y'\big|_{x=l_c} = y_c{}',\\
&Y_3\big|_{x=l_c} = y''\big|_{x=l_c} = [F_B(l_b - l_c) - F_T(y_b - y_c)]/EI
\end{aligned} \tag{14}$$

The boundary condition at $x = l_c$ indicates that the numerical solution at position c (Fig. 1) gives the initial values to the Eqs. (6) and (7), and hence the whole solution for the beam. However, it is shown that the left boundary (at $x = 0$) contains the undetermined function $y(x)$ and the end displacement y_b. Therefore further solution method should be introduced.

Let the exact solution be $\bar{u} = \int_0^c k(x)y(x)xdx$ and $\bar{v} = y_b$. If trial value u_i and v_i infinitely approach $\bar{u}$ and $\bar{v}$, i.e., $\lim(u_i - \bar{u}) \Rightarrow 0$, and $\lim(u_i - \bar{u}) \Rightarrow 0$, certain values of them within an expected small deviation can be accepted as the solution after finite iterations. For this situation the following function may be constructed:

$$f(u, v) = [(w_1|u - \bar{u}|)^a + (w_2|v - \bar{v}|)^b]^{2/a+b} \tag{15}$$

where w_1 and w_2 are weight, a and b are any constant. If $w_1 = w_2 = 1$ and $a = b = 2$, $f(u, v)$ is distance between (u, v) and $(\bar{u}, \bar{v})$. The above problem is now transformed into a task of root finding, i.e., searching solutions $(u^*, v^*) \Rightarrow (\bar{u}, \bar{v})$ so that $f(u^*, v^*) \Rightarrow 0$. As $(\bar{u}, \bar{v})$ are unknowns also, an iteration method must be employed. However Eq. (15) is a 2-dimensional problem, so the initial guess, which is crucial to the problem, will be difficult to determine and thus convergence or the desired result may not be guaranteed with the available numerical techniques (Press *et al.* 1989). Secondly, since (u, v) and $(\bar{u}, \bar{v})$ are associated by the complex implicit function described above, the function derivatives are not available for the solution procedure. Hence all solution strategies relying on the features of continuous function are not suitable.

The purpose is to solve Eq. (15) to find a very good approximate combination of $(\bar{u}, \bar{v})$, or in other words, to find an effective strategy of generating a sequence of $(\bar{u}_i, \bar{v}_i)$, $i = 1, 2, ..., n$ so that the differences from the output (u, v) approach zero. It must be noted that the function value will not be equal to zero except when $\bar{u} = u$ and $\bar{v} = v$. In fact, this function can be treated as minimization problem with extremum as zero. With optimization techniques, some attractive strategies for generating sequences of $(\bar{u}_i, \bar{v}_i)$, $i = 1, 2, ..., n$ are available. Since the problem is two-dimensional, non-linear and non-differentiable, the Downhill Simplex Method (DSM) in multi-dimensions (Press *et al.* 1989) and Genetic Algorithms (GAs) (Goldberg 1989, Michalewicz 1996) are suitable techniques. One of the reasons for the selection of more than one method is that most optimization techniques converge to a local minimum instead of global minimum. This is verified in the application of DSM to our problem being studied. GAs are regarded as an effective method for multi-modal function and the authors have a self-written and tested computer code (Vardy, Hu and Brown 1999). Hence a GA is chosen for the purpose of comparison so that reliable results can be achieved.

3.1 Downhill Simplex Method (DSM) in multi-dimensions for minimization of functions

This DSM technique (Press *et al.* 1989, Mathew and Fink 1999) is due to Nelder and Mead (1965). It is used only function evaluations and not the derivative. It is slower than other techniques, eg. Powell's method, but tends to be more robust and convenient. It is found that the computing time is not a major concern while using the high performance computer system at the University of Queensland. The principal steps of this method include formation of initial simplex, reflection, expansion, rebounding, and contraction. Readers are referred to Press *et al.* (1989) for details.

3.1.1 Modification and testing of DSM algorithm

As the problem was thought to be very sensitive to the input number and the objective point may be difficult to find, the efficiency of the algorithm should be carefully checked. Usually, if the objective function is smooth and the gradient of hill slope is not very steep, the convergence of DSM algorithm can be guaranteed. However, whether the process converges to the global minimum may not be ensured, as previously mentioned. Therefore, it is suggested (Press *et al.* 1989) that restarting a multidimensional minimization routine at a point where it claims to have found a minimum. In this study, a random number generator has been used, (Dr. Jim Brown, University of Dundee, Scotland) to produce the initial points for DSM in the search domain. Secondly, for each re-run, the re-starting points have been randomly selected except for the best point of the last run. To verify the effectiveness of the modification, the following simple smooth function was tested:

$$f(x_1, x_2) = -[(\sin x_1 + \sin x_2)/2]^{\alpha} - 1,\ x_i \in (0, \pi),\ i = 1, 2 \tag{16}$$

While α is given a value of 2000, this function produces a very sharp needle in the centre of the search domain (Fig. 2). Obviously, the accurate minimum is –2. The original computer code for DSM is directly quoted from 'Numerical Recipes' (Press *et al.* 1989). The computations were carried out at the high performance computer system at the University of Queensland using 16 significant digit calculation. For randomly selected three start points, the minimum found by DSM was –1. in the first run. Apparently, DSM found nothing because the function value on the whole search space is –1. except at the needle centre. Hence restarting searches were performed. When the best point of each run was kept but other points were randomly chosen again, the exact minimum value, –2. with 16 significance, was obtained after 29 re-runs. When the re-start points were completely re-chosen, i.e., the best point of last run was not kept, the exact minimum value, –2. was never found after 999 re-runs. In fact the results from all re-runs were –1.

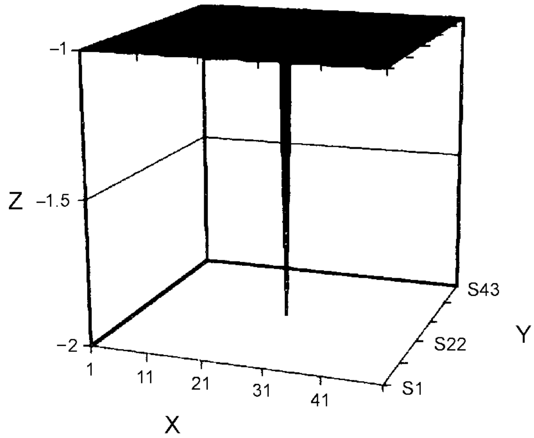

Fig. 2 Optimization on a very sharp function

The test indicates that the modified DSM algorithm is able to fulfil some tough tasks, such as the above instance. It is worth commenting that the modification still has its limitation. The test revealed that, while the parameter α of the above function is greater than 2300, i.e., the slope around the minimum becoming steeper, the algorithm fails to reach the needle pinpoint.

3.2 Genetic Algorithms (GAs)

Genetic Algorithms may be classified into a category of *evolutionary computation* (Karr and Freeman 1999), which has some vigorous members, e.g. evolution strategies, evolution programming (Michalewicz 1996), GAs (Goldberg 1989), and Genetic programming (Soh and Yang 2000), etc. The general background of this field is natural evolution process or genetic mechanisms of biological organisms. During last thirty years, from concepts to algorithms, this field is becoming mature and the corresponding techniques have been applied into a wide range of disciplines in various fields including arts, economics, engineering, medicine, and chemistry, etc. GAs are very popular in the optimization community since their ability to solve a large number of difficult searching problems. They are particularly suitable to multi-modal functions since they use parallel search instead of point-by-point search methods of traditional optimization programs. Another merit of GAs is the derivative-independence. This makes them applicable to the current problem and many other large scale and perplexing problems of non-derivative in engineering practice. Furthermore, GAs do not require the continuity of the objective function. The last two features eliminate the need to prove the analytic properties of their problems. These characteristics of GAs are suitable to the current problem.

For optimzation problems, the above process is analogous to hill climbing by a group of people. Those who reach higher elevation (corresponding to fitness) have more chance to be selected to generate new population for the next time climbing. Thus, this technique is suitable for multi-modal problems. However, due to the feature of high randomness, this procedure does not guarantee all or most of the population reach the global peak even though incorporating some very sophisticated strategies.

In the current study, GAs have been chosen to attempt to locate any second or more significant peaks in the search domain and avoid the verification of single modal of our function.

3.2.1 Test of GAs

A test on the GA is performed on the same 2-D function (Eq. 16) as used for DSM. The index will be $\alpha \geq 2000$. Using the high performance computer system at the University of Queensland with 16 significant digits calculation, the computation lasts 50–60 minutes. The randomly chosen GA parameters are: population size = 80, evolution generations = 80, total re-runs = 39, crossover probability = 75%, mutation probability = 1%. The exact minimum of the objective function (equal to –2. at $x_1 = x_2 = 1.570796326795$) was found at the 71st generation and the 38th re-run. If the evolution generations were increased to 100 while keeping other parameter values, the exact minimum was found at the 92nd generation and the 16th re-run. Surprisingly, when the index α was set to 3000, the GA still found the accurate minimum –2. at the 92nd generation and the 9th re-run, and found the same result for $\alpha = 4000$ at the 82nd generation and the first re-run, while the GA parameters were given as: population size = 120, evolution generations = 100, total re-runs = 39, crossover probability = 75%, mutation probability = 1%. However, the previous Downhill Simplex Method (DSM) failed to reach the needle pin for $\alpha > 2300$ of the objective function. It seems that GAs may be much more effective than DSM.

3.3 Solution on equations for cantilever on non-linear elastic foundation

Re-writing Eq. (15) as a distance between inputs and outputs, the goal is to minimise it:

$$\min \ f(\bar{u}, \bar{v}) = \{[w_1(u-\bar{u})]^2 + [w_2(v-\bar{v})]^2\}^{1/2} \tag{17}$$

For the example computation, the parameters of the above fiber bending model are initiated as: l_b = 15 mm, l_c = 5 mm, F_t = 2.5 N, F_b = 4.33 N (Fig. 1), elastic modulus of the foundation matrix E_m = 30 GPa, elastic modulus of fiber E_f = 210 GPa, the diameter of fiber D_f = 0.5 mm. Both fiber and matrix are theoretically restrained in elastic domain.

Lengthy test runs have been carried out with DSM code. As the search domain was unknown, trial runs were performed in a very big extent. It was found that $u < 60$ (N-mm) and $v < 2.1$ (mm). Hence the search domain was firstly determined as $u \in (0, 80)$ and $v \in (0, 10)$ (Table 1). The results were found at 2371st re-run in which the code called the DSM algorithm 1,153,440 times. The function value was f = 1.181474E-2, u = 58.266733 N-mm, and v = 2.014760 mm = y_b, where the individual deviation of u was du = 1.075387E-2, and that of v was dv = –1.179019E-2. This result seems not sufficiently accurate. Thus the search domain was reduced to $u \in (0, 70)$ and $v \in (0, 10)$. The result is shown in the third column of Table 1. It is noted that the accuracy was raised by one order of magnitude. Further reduction of the search domain of v to (0, 3), raised the accuracy (dv) by one more order of magnitude (column 4, Table 1). Actually, the outputs of u and v are stable (see row 7 and 8, Table 1). Therefore search domain can be further contracted (column 5, Table 1). To urge both variables to approach the same order of convergence, a weight of 20 was given to v since its domain is about 1/20 of that for u. Now the outputs of both variables have the accuracy higher than $\pm 5.0 \times 10^{-5}$ and are considered acceptable (column 5, Table 1).

What is surprising in the computation with DSM code is that, unlike the test with DSM on Eq. (16), passing the best results from one run to the next made the new outputs worse. Thus for each re-run, the initial points were completely re-input. However, DSM can always approach the small

Table 1 solutions for fiber bending problem with DSM

(1)	(2)	(3)	(4)	(5)
u-domain (N-mm)	0−80	0−70	0−70	0−60
v-domain (mm)	0−10	0−10	0−3	0−2.5
u-weight	1	1	1	1
v-weight	1	1	1	20
Min-f	1.181474e-2	1.421535e-3	6.723531e-4	6.294568e-4
u-output (N-mm)	58.266733	58.233830	58.239035	58.237442
v-output (mm)	2.014760	2.027921	2.025839	2.026476
du (N-mm)	1.075387e-2	−2.659295e-6	2.055564e-6	2.854741e-5
dv (mm)	−1.179019e-2	1.421533e-3	−6.72499e-4	−3.144045e-5
Total runs	2,372	558	2,395	1,792
Cycles of DSM	1,153,441	28,751	176,200	79,153

area near the minimum.

The results of the above computation may also be verified with an approximate analytic calculation. Since in this example, the deflection at end B (Fig. 1) is dominated by the cantilever section, namely, section CB. Let $y_c = 0$ and $y_c' = 0$, which assume section CB as a pure cantilever, Eq. (7) becomes:

$$y_b = \frac{1}{2\cosh\alpha}(\lambda_2 e^{\alpha} + \lambda_3 e^{-\alpha}) \tag{18}$$

where $p = \sqrt{F_T/EI} = 0.0622924$, $\alpha = p(l_b - l_c) = 0.622924$, $\lambda_1 = (l_b - l_c)F_B/F_T = 17.32$, $\lambda_2 = \lambda_1 - F_B/(p^*F_T) = -10.484355$, $\lambda_3 = \lambda_1 + F_B/(p^*F_T) = 45.124355$, hence:

$$y_b = \frac{1}{2\cosh(0.622924)}(-10.484355e^{0.622924} + 45.124355e^{-0.622924}) = 1.939726 \text{ mm}$$

This value is slightly smaller than the corresponding value (*v*-output in Table 1) of the original structure (Fig. 1), which implies that the above computed results are reasonable.

Unfortunately, though the GA has shown better performance than the DSM on the test problem (Eq. 16), it fails to find a function value smaller than 0.1 in spite of great efforts made by the authors. The cause is still unknown to the authors.

4. Conclusions

The bending of steel fiber is one of the micro-mechanisms within fiber reinforced composites. In this study, the authors tried to use conventional analytic models and numerical solvers to simulate the bending behaviour of Bernoulli-Euler beam on elastic foundation problem. Without care, it may be taken as a conventional beam on foundation problem. However, since the existence of nonlinearly distributed foundation stiffness and inclusion of transverse second order deformation, the unknowns are implicitly involved in an integral-differential Eq. (4). The equation is analytically unsolvable. Therefore, a higher order of differential equation is chosen to eliminate the integral operation and an order reduction technique for the differential equation is adopted. The Runge-Kutta method is employed for the solution within the boundaries. Finally, optimization techniques, namely the Downhill Simplex Method (DSM) and Genetic Algorithm (GA), are applied to search for the unknowns concerning the boundary conditions. Before the optimization techniques were used for this problem, they were carefully tested and some modifications were made to increase their efficiency. Computations indicate the good performance of DSM and the poor behavior of GA on the studied problem though both succeeded in the test problem. Fortunately, the results for the unknowns are found with a good precision while the contraction technique of search domain is introduced. The computation process is shown to be stable and convergent.

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