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Explicit incremental matrices for the postbuckling analysis of thin plates with small initial curvature

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Abstract. The postbuckling behaviour of thin plates is an important phenomenon in the design of thin plated structures. In reality plates possess small imperfections and the behaviour of such imperfect plates is of great interest. To numerically study the postbuckling behaviour of imperfect plates explicit incremental or secant matrices have been presented in this paper. These matrices can be used in combination with any thin plate element. The secant matrices are shown to be very accurate in tracing the postbuckling behaviour of thin plates.

Key words: postbuckling behaviour, imperfect plates, nonlinear finite element analysis, secant or incremental matrices.

1. Introduction

The use of thin walled structures, which are popular in aerospace applications, is increasing in the construction of civil and industrial buildings. Of which, the technology of cold-formed steel structures is finding wide applications. The major advantage of cold-formed steel sections over hot rolled sections is found in the relative thinness of the sections, which can lead to highly efficient and weight effective members and structures. The most important phenomenon in such structures is the local buckling of the constituent plate elements. The evaluation of linear critical stress or the stress at which the local buckling of thin plates is initiated, has been well documented in the contemporary literature.

This critical stress state characterises the neutral equilibrium of an axially loaded plate. At this state, it is observed that the plate would keep the small out-of-plane perturbations and still remain

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stable. In the case of slender columns, increase in the axial load beyond such a critical state would produce a disproportionate lateral deflection resulting in an unstable state. Nevertheless, edge supported plates do not undergo such unstable deformations immediately after attaining the critical buckling stress, i.e., in the vicinity of the neutral equilibrium. The fibres parallel to edge compression shorten because of elastic strain and bow effect. The latter causes fibre lengthening in the direction perpendicular to axial compression. This membrane effect tends to stabilise equilibrium of the plate and results in a possible increase in strength, which is termed as the postbuckling strength. In other words, the plate continues to carry axial load up to certain level, even beyond the critical stress and presents a stable equilibrium. The unavoidable out-of- flatness or imperfections present in the plates cause a qualitative change in the load - deformation characteristics and is also one of the important factors which contributes to the strength of the plate. It is seen from the above that the problem of assessing the strength of thin plates is basically nonlinear and that the linear critical stresses alone are not adequate for the design of plates. There are indeed several analytical solutions available for the postbuckling analysis of plates. Nevertheless they become intractable in the case of practical problems encountering complex boundary conditions. Hence, more often numerical solutions, which exploit the power of computers become handy and the finite element method is the foremost of all. In the finite element method for dealing with nonlinear problems, especially for plates, there are formulations based on virtual work and minimisation of total potential energy. Since total potential energy is a basis, which provides solution to numerous problems in the realm of structural stability, it is the subject of interest in this paper. It is seen from the literature that several publications are available on the formulations for the postbuckling analysis of initially imperfect plates using energy basis (Yang and Han 1983 and Kapania and Yang 1987). The earlier works of Mallet and Marcal (1968) and Rajasekaran and Murray (1973) present explicit coefficients of displacement gradient tensors for initially perfect plates. Nevertheless, explicit displacement gradient tensors for the post buckling analysis of initially imperfect plates, to the authors' knowledge, have not been reported in the literature. Hence the objective of the present study is to derive, present and validate the displacement gradient tensors for the large deflection and postbuckling analysis of thin plates with small initial imperfections. The formulations are intended to be used in the study of post local buckling analysis of thin isotropic plate elements. With minor modifications these formulations can be extended to the study of large deflection behaviour of laminated composite plates.

2. A brief review of literature

It is seen from the literature that a number of studies based on analytical (Dawson and Walker 1973, Rhodes 1981, Bradfield and Stoner 1984, Shanmugam 1987) and numerical approach (Crisfield 1980, Sridharan 1982, Allman 1982) have been reported on the axial compression behaviour of thin plates encountered in aerospace and civil engineering structures. Foremost of them is the work of Murray and Wilson (1969). A complete evaluation of some notable works done in the seventies and early eighties, on the elasto- plastic behaviour of steel plates is also available in Bradfield (1982). Many works have been reported in the late eighties on the axial compression behaviour of both metal (Nara and Fukumoto 1988, Rerkshananda *et al.* 1981) and laminated composite (Ashwini Kumar 1987, Stein 1983) plates. Details of the numerical work on the large deflection behaviour of plates continued to be reported in the nineties also (Chang and Sherbourne 1990, Bilin Chang and Shabana 1990). Surprisingly work is still needed on the axial compression behaviour of thin plated structures especially those with complex boundary conditions and with combination of different loads (Harris and Little 1991, Tsutomu Usami 1993, Jeom Key Pak 1995, Osama Bedair 1994). It is also inferred from a literature survey that there is a spurt of research activity in the nineties on the axial compression behaviour of epoxy based composite plates under thermo-mechanical loads (Singh *et al.* 1994, Sundaresan *et al.* 1996). Regarding the subject of interest of this paper, i.e., the energy based numerical formulations for the post buckling analysis of initially imperfect thin plates, Yang has contributed several papers (Yang 1970, 1971a, 1971b, 1972, Yang and Han 1983, Kapania and Yang 1987). There are also papers on the post buckling analysis of plates using virtual work formulations (Pica and Wood 1980). Yang's initial works (Yang 1970, 1971a, 1971b, 1972) were based on the Mallet and Marcal symbolism and the explicit coefficients for the incremental matrices which were presented, are applicable only for a particular plate element. However the explicit displacement gradient tensors for the postbuckling analysis of thin imperfect plates which could be used for any plate element (similar to the matrices presented by Mallet and Marcal 1968, Rajasekaran and Murray 1973, Narayanan and Krishnamoorthy 1989, Ganapathy and Varadhan 1995) were not reported in the literature. In particular, the earlier work of Yang on postbuckling of thin plates (Yang 1971b, Yang 1972) and the later work (Kapania and Yang 1987) do not contain the explicit displacement gradient tensors for initially imperfect plates. The numerical results reported in Yang (1971b) for the postbuckling analysis of plates were not very accurate and this was attributed to not using higher order approximating functions for membrane displacements. In a later work by Kapania and Yang (1987) the initial imperfection was handled through Love-Kirchoff thin shell theory. Even in that work, the numerical results for initially imperfect plates were found to be not very accurate. It must be noted that, it is possible to solve postbuckling problems of initially imperfect plates with perfect plate displacement gradient tensors (Mallet and Marcal 1968, Rajasekaran and Murray 1973, Narayanan and Krishnamoorthy 1989) by considering the effect of initial imperfection as a pseudo force term and adding it to the force term of the equilibrium equations. However imperfections in reality do not follow any idealised pattern and hence in situations where random imperfections are to be taken into the analysis (Dow and Smith 1984, Ueda and Tao 1985) the initial imperfection needs to be handled as an incremental matrix (Yang 1971b). This paper presents the details of the incremental matrices for the large deflection and postbuckling analysis of plates with small initial curvature.

3. Energy formulations for the large deformation and postbuckling analysis of initially imperfect plates

The strain energy density U_0 of a Hookian material can be written in terms of stress tensor $\boldsymbol{S}_{mn}$ and strain tensor $\boldsymbol{\varepsilon}_{mn}$ as $\delta U_0 = \boldsymbol{S}_{mn}\delta\boldsymbol{\varepsilon}_{mn}$. If we let the virtual strain $\delta\boldsymbol{\varepsilon}_{mn}$ correspond to the actual strain increment $d\boldsymbol{\varepsilon}_{mn}$, the increment in the strain energy can be written as $dU_0 = \boldsymbol{S}_{mn}\delta\boldsymbol{\varepsilon}_{mn}$. As the strains increase from zero to some current value ε_{ij} the strain energy can be written as

$$U_0 = \int \boldsymbol{S}_{mn} d\boldsymbol{\varepsilon}_{mn} \tag{1}$$

Since in the present study the deformation of the initially imperfect plate is described in the total Lagrangian format, a fixed right handed rectangular Cartesian frame of reference is used, with the X_1, X_2 plane coinciding with the middle surface of the plate in its undeformed flat state and the Z (Z

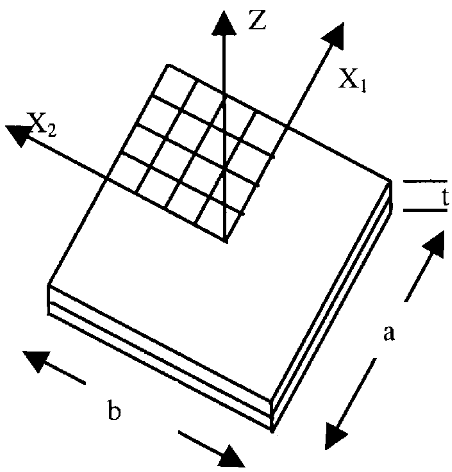

Fig. 1 Co-ordinate axes of the plate and one quarter FEM mesh

is used instead of X_3 for convenience) axis normal to it as shown in Fig. 1. The components of the inplane and out-of plane displacements of a particle of the plate originally at ($x1$, $x2$, z) are denoted by $\boldsymbol{u}_i=\{u_1, u_2\}$ and w respectively. As the plate is assumed to be very thin and the traction on surfaces parallel to the middle surface of the plate are negligible and hence the flexural strains vary linearly along the thickness. Since Marguerre's assumption is valid the strain terms dependent on the square of the membrane displacements are neglected. The corresponding Green's strain tensor of an initially imperfect plate satisfying the above assumptions could be written as

$$\boldsymbol{\varepsilon}_{mn}= \mathbf{1/2}\ (\boldsymbol{u}_{m,n}+\boldsymbol{u}_{n,m})+ \mathbf{1/2}\ \boldsymbol{w}'_{,m}\boldsymbol{w}'_{,n} - \mathbf{1/2}\ \boldsymbol{w}_{0,m}\boldsymbol{w}_{0,n} - z\ \boldsymbol{w}_{,mn} \tag{2}$$

in which the comma denotes the differentiation of the preceding quantity with respect to the indexed co-ordinates and $\boldsymbol{w}'=\boldsymbol{w}+\boldsymbol{w}_o$ where $\boldsymbol{w}_o$ is the initial imperfection in the unloaded state and $\boldsymbol{w}$ is the additional deflection after loading. The strain measures are functions of co-ordinates $x_i=\{x_1, x_2\}$ and hence the range of indices m, n is two. In Eq. (2) the first term represents the first order membrane strains (this is linear in view of the Marguerre's assumption). Since, the out-of-plane deformations are measured from the undeformed plane of the plate, (which coincides with X_1–X_2 Cartesian plane) the axial shortening strain (due to bowing effect of the plate) is represented in the second term as a quadratic function of the total deflection $\boldsymbol{w}'$. The last term represents the flexural strains due to curvature of the plate. Since Green's strain is assumed for the description of motion of the plate, it's work conjugate, the second Piola-Kirchoff stress is used as the stress measure. In the case of isotropic material the second Piola-Kirchoff stress tensor can be related to the Green's strain using a fourth order elastic compliance tensor as

$$\boldsymbol{S}_{mn}= \boldsymbol{C}_{mnrs}\boldsymbol{\varepsilon}_{rs} \tag{3}$$

where $\boldsymbol{C}_{mnrs}= \lambda\delta_{mn}\delta_{rs}+ G(\delta_{mr}\delta_{ns}+\delta_{ms}\delta_{nr})$, λ and G are the usual Lame's constants and δ is the Kronecker delta and the range of indices m, n, r, s is two. Alternatively the second Piola-Kirchoff stress in the case of isotropic material, can be written as a function of Green's strain as

$$\boldsymbol{S}_{mn}= \lambda\varepsilon_{rr}\delta_{mn}+ 2G\varepsilon_{mn} \tag{4}$$

The total potential energy of the plate during deformation, in the absence of body forces can be written as

$$\boldsymbol{\Pi} = \int_{\Omega} \boldsymbol{S}:\boldsymbol{\varepsilon} d\Omega - \int_{\Gamma} \boldsymbol{p}.\boldsymbol{q}\ d\Gamma \tag{5}$$

where the integrand in the first term represents a double dot product or scalar tensor product and hence Eq. (5) can be rewritten as

$$\boldsymbol{\Pi} = \int_{\Omega} S_{mn}\boldsymbol{\varepsilon}_{mn} d\Omega - \int_{\Gamma} \boldsymbol{p}.\boldsymbol{q}\ d\Gamma \tag{6}$$

and the second term represents the potential lost by the tractional loads p on the boundary Γ in undergoing a deformation q. Applying Eq. (4) in Eq. (6), the first term in Eq. (6) namely the elastic strain energy $U_{0,}$ could be obtained as a function of displacement gradients which are the components of the Green's strain tensor in Eq. (2). By examining the quadratic dependence of Green's strain tensor, the scalar tensor product in Eq. (6) could be split as strain energy components which are functions of second, third and fourth order of displacement gradients and accordingly they could be written symbolically as

$$U_0 = U_2 + U_3 + U_4 \tag{7}$$

The subscripts of the terms on the right hand side of Eq. (7) denote the order of dependence with respect to displacement gradients. Let g_α be a vector which is the set of all displacement gradients of additional displacements found in the Green's strain tensor in Eq. (2), in which the range of the Greek indices varies as the number of elements in the non-empty set g_α (as given in the Appendix). The gradients of initial imperfection being constants, they are excluded from the gradient vector g_α. However, the total displacements (i.e) $\boldsymbol{w'} = \boldsymbol{w} + \boldsymbol{w_o}$ is a variable and hence needs to be considered in g_α. Now Eq. (7) can be written in terms of displacement gradients and in a quadratic form as

$$U0 = \boldsymbol{g}_\alpha{}^T U \boldsymbol{g}_\alpha \ \forall \boldsymbol{g}_\alpha \tag{8}$$

which could be rewritten using Mallet and Marcal (1968) symbolism as

$$\boldsymbol{U_0} = \boldsymbol{g}_{\boldsymbol{\alpha}}^T \left[\frac{1}{2}\boldsymbol{K}_{\boldsymbol{\alpha\beta}} + \frac{1}{2}\boldsymbol{N0}_{\boldsymbol{\alpha\beta}} + \frac{1}{6}\boldsymbol{N1}_{\boldsymbol{\alpha\beta}} + \frac{1}{12}\boldsymbol{N2}_{\boldsymbol{\alpha\beta}} \right] \boldsymbol{g}_{\boldsymbol{\beta}} \tag{9}$$

where $K_{\alpha\beta}$, $N0_{\alpha\beta}$, $N1_{\alpha\beta}$, $N2_{\alpha\beta}$ are, respectively, the linear incremental, initial imperfection, the first order incremental and the second order incremental matrices which are functions of displacement gradients. It is seen from Eq. (9) that there are indeed two terms which are independent of displacement gradients. The first one contributes the linear stiffness matrix and is purely a function of material compliance relations and the second quantity arises from the third term of the Green's strain tensor in Eq. (2) namely the initial imperfection term. The third and fourth terms are dependent on the first and the second order of displacement gradient g_α. While expressing Eq. (9) in a quadratic form, certain terms in the scalar tensor product in Eq. (6) become zero because of the double differentiation. For example the products containing initial imperfection terms becomes zero in Eq. (9) and would not find a place in Eq. (8) or Eq. (9) because g_α is devoid of pure initial imperfection gradients. This is the reason for the displacement gradient tensors in Eq. (9) being applicable only to small amounts of initial imperfection say 0.2 times the thickness of the plate or less. When larger initial imperfections are present, those product terms may not become small enough to be neglected.

It could be seen from literature that Mallet and Marcal (1968) first presented explicit coefficients of $K_{\alpha\beta}$, $N1_{\alpha\beta}$, $N2_{\alpha\beta}$ for a perfect flat plate element and these were later pointed out to be inadequate by Rajasekaran and Murray (1973) who also presented adequate matrices. However to the author's knowledge, the explicit coefficients for the initially imperfect plate have not been published in literature. By writing Eq. (9) in a quadratic form (the lengthy algebraic calculations are not shown in this paper), the explicit coefficients of displacement gradient tensors $K_{\alpha\beta}$, $N0_{\alpha\beta}$, $N1_{\alpha\beta}$, $N2_{\alpha\beta}$ have been derived and are presented in the Appendix. These are cross checked for their adequacy as elaborated by Rajasekaran and Murray (1973). The potential of the plate as given in Eq. (6) can be represented using the finite element method as a sum of potentials contributed by individual plate finite elements. Let $\boldsymbol{q}_i \in E^n(\Omega)$ be n-tuples of generalised displacements of a typical plate finite element in the domain Ω of the plate. The element displacements $\boldsymbol{q}_e$ can be interpolated using a linear combination of the generalised nodal displacements $\boldsymbol{q}_i$ and an approximating shape function $\boldsymbol{\varphi}_i$ as

$$\boldsymbol{q}_e = \boldsymbol{\varphi}_i \, \boldsymbol{q}_i \tag{10}$$

The displacement gradient can be related to the generalised nodal displacements of a typical plate element as

$$\boldsymbol{g}_\alpha = \boldsymbol{D}_{\alpha i} \, \boldsymbol{q}_i \tag{11}$$

Using Eq. (11) the strain energy equation can be written symbolically as

$$U_0 = \boldsymbol{q}^T(\boldsymbol{D}^T\boldsymbol{U}\boldsymbol{D})\, \boldsymbol{q} \tag{12}$$

Combining Eq. (9) and Eq. (12) the total potential in Eq. (6) can be written in terms of generalised nodal displacements as

$$\boldsymbol{\Pi}^e = \frac{1}{2}\boldsymbol{K}_{ij}\boldsymbol{q}_i\boldsymbol{q}_j + \frac{1}{2}{}^0\boldsymbol{n}_{ij}\boldsymbol{q}_i'\boldsymbol{q}_i' + \frac{1}{6}{}^1\boldsymbol{n}_{ij}\boldsymbol{q}_i'\boldsymbol{q}_i' + \frac{1}{12}{}^2\boldsymbol{n}_{ij}\boldsymbol{q}_i'\boldsymbol{q}_i' - \boldsymbol{p}_i\boldsymbol{q}_i \tag{13}$$

It should be noted that, as defined in Eq. (2) the deformation of the initially imperfect plate has been measured with respect to the original undeflected plate and hence all the incremental matrices in Eq. (13) should be functions of the total displacement vector $\boldsymbol{q}'$ which is the algebraic sum of the initial deflection vector $\boldsymbol{q0}_i$ and the additional deflection vector $\boldsymbol{q}_i$. However by simple algebraic manipulation, it could be shown that the linear stiffness term, i.e., the first term in Eq. (13) and the loss of potential term, i.e., the last term in Eq. (13), are functions of additional deflection vector. It must be noted that the components of the matrices in Eq. (13) can also be obtained through the Castigliano's theorem by double differentiation of component energy terms with respect to the generalised nodal displacements (Yang 1970).

4. Equilibrium equations and method of analysis

The equilibrium equation can be obtained by applying condition of equilibrium to Eq. (13) as Π^e, i (comma represents differentiation with respect to succeeding indices) it becomes,

$$\boldsymbol{p}_i = \boldsymbol{K}_{ij}\boldsymbol{q}_j + {}^0\boldsymbol{n}_{ij}\boldsymbol{q}_j' + {}^1\boldsymbol{n}_{ij}\boldsymbol{q}_j' + {}^2\boldsymbol{n}_{ij}\boldsymbol{q}_j' \tag{14}$$

It is seen that the Eq. (14) is solvable by pure iterative procedures. However upon further differentiation of Eq. (14) with respect to q_i the incremental equilibrium (Rajasekaran and Murray 1973) is obtained as

$$\Delta \boldsymbol{p}_i = \boldsymbol{K}_{ij}\Delta \boldsymbol{q}_j + {}^0\boldsymbol{n}_{ij}\Delta \boldsymbol{q}_j' + {}^1\boldsymbol{n}_{ij}\Delta \boldsymbol{q}_j' + {}^2\boldsymbol{n}_{ij}\Delta \boldsymbol{q}_j' \tag{15}$$

where Δ represents the incremental operator. For finite element representation of the equilibrium and incremental equilibrium equations, if the displacement functions u_1, u_2 and w for a typical plate finite element are assumed, the components of the second order tensor in Eq. (13) and Eq. (14) can be obtained by applying the gradient operator twice to the corresponding component of strain energy in Eq. (13) with respect to the generalised displacements. The above procedure is nothing but the use of Castigliano's theorem. In this study, cubic Hermitian polynomials are assumed for in-plane ($u_i \in C^2$) and out-of-plane displacements ($w \in C^2$) as proposed by Bogner, Fox and Schmidt (1966). In spite of numerous developments that have taken place in the evolution of new plate bending elements the Bogner, Fox and Schmidt (1966) element has been used because that element yields very good results in the case of postbuckling problems (Harris and Little 1991). The equilibrium equation is solved incrementally by first computing the Euler predictor Δq_i corresponding to a load increment Δp_i and then correcting with Newton corrector iterations by computing the unbalanced forces from the equilibrium equations. The operations for a particular load slice are repeated until the Euclidean norm of two consecutive unbalanced force vectors is within a prescribed limit. Since the secondary postbuckling paths of axially loaded thin plates are stable in the case of linear elastic material, the conventional Newton methods are found to be adequate to trace the load deflection behaviour in the postbuckling range. In the case of postbuckling analysis of thin plates under axial compression, non-classical kinematic boundary conditions such as unloaded edges moving in straight and loaded edges remaining straight during axial deformation are encountered. These boundary conditions need to be properly handled in the computer program. The details of the numerical formulations presented in this paper have been incorporated in a computer software called PLOT-COLD (Post LOcal buckling analysis of Thin COLD formed elements)

5. Numerical studies

Studies were carried out using the software PLOT-COLD to calibrate and assess the performance of the numerical formulations derived in this study. First, the large deflection behaviour of simply supported plates subjected to uniform pressure load was studied. The analytical results of Rushton (1970), which are supposed to be accurate to within 0.5% of error, were taken as the basis for comparison. For a perfect simply supported square plate the large deflection behaviour was solved for displacements upto six times the thickness of the plate and the results are presented in Table 1. It is seen from Table 1 that the results of the present study including the computation of stresses, are quite accurate being within an acceptable error of 1%.

The results of a plate with initial imperfections of $w_0/t=0.1$ subjected to uniform pressure load are compared with the results of Rushton (1970) in Table 2. The excellent agreement is in evidence of the correctness of the initial imperfection matrix which is derived in Eq. (13) and reported in the Appendix. For cold formed thin elements the axial compression behaviour is an important aspect in their design. Hence to validate and calibrate the software PLOT-COLD for postbuckling problems

Table 1 Comparison of large deflection results of a perfect plate

Simply supported square plate–Initial imperfection $w0/t$=0.0 Membrane displacements fixed				
Pressure Load qb^4/Et^4	Out -of -plane centre deflections w/t		Stresses at unloaded face at centre $\sigma_x b^2/Et^2$	
	Rushton (1970)	Present	Rushton (1970)	Present
9.16	0.335	0.346	2.46	2.58
36.60	0.818	0.816	6.90	6.88
146.50	1.47	1.464	14.5	14.42
586.00	2.40	2.399	30.0	29.74
2344.00	3.83	3.810	65.2	64.82
9377.00	6.07	6.030	148.3	147.80

Table 2 Comparison of large deflection results of an initially imperfect plate

Simply supported square plate–Initial imperfection w_0/t = 0.1 Membrane displacements fixed				
Pressure Load qb^4/Et^4	Out-of–plane centre deflections w/t		Stresses at unloaded face at centre $\sigma_x b^2/Et^2$	
	Rushton (1970)	Present	Rushton (1970)	Present
9.16	0.316	0.313	2.46	2.45
36.60	0.747	0.744	6.51	6.47
146.50	1.380	1.376	14.0	13.86
586.00	2.30	2.295	29.4	29.11
2344.00	3.73	3.708	64.6	64.15
9377.00	5.98	5.935	147.3	147.10

of thin plates, various studies have been carried out and some typical results are reported in this paper. The analytical results of Yamaki (1959) were taken as the basis for the comparison. It is seen from the literature that several researchers, Harris and Little (1991), have used the results of Yamaki (1959) for assessing the performance of their numerical studies.

In the present study, for postbuckling problems of thin plates, the simply supported condition is assumed for out-of-plane boundary condition. Regarding membrane boundary conditions two cases are considered. In the first case, the unloaded edges are allowed to move in but to remain straight (represented as (*st*) in discussing the results of the present study) and hence the resultant of the distribution of normal stresses across the edge is zero. The other case is that the unloaded edges are free to move in and are free from any normal stresses. This condition is referred to as (*fr*) in the present study. The non-dimensionalised axial load parameter is represented as

$$\lambda = \frac{p_x a^2}{\pi^2 E t^2} \tag{16}$$

where p_x is the uniform compressive stress, a is the width of the loaded edge, E is the Young's modulus and t is the thickness of the plate. In the postbuckling analysis the loaded edges were

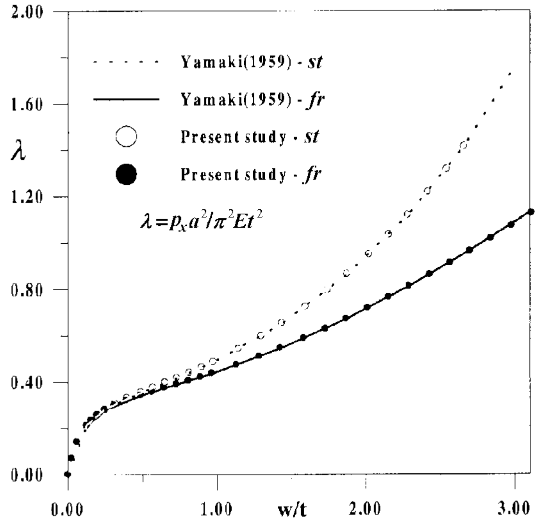

Fig. 2 Comparison of postbuckling paths of initially imperfect plates

maintained straight as if the loads were applied through rigid blocks on either side of the plate. Fig. 2 shows the results of the out-of-plane deformation of the simply supported square plate under uniaxial compression, obtained from PLOT-COLD with those of Yamaki (1959). Since initial imperfection was assumed as w_0/t=0.1 it is seen from Fig. 2 that the plate starts deflecting at the onset of loading. The results displayed in Fig. 2 for both membrane boundary conditions are in excellent agreement with the results of Yamaki (1959).

Similarly the non-dimensional loads versus the axial shortening Δ_a, as presented in Fig. 3, are also in excellent agreement with the results of Yamaki (1959). The results presented in Fig. 2 and Fig. 3 further exemplify the correctness of the incremental matrices for the initially imperfect plate presented in this study. Finally the effective width of the plate is computed. The effective width of

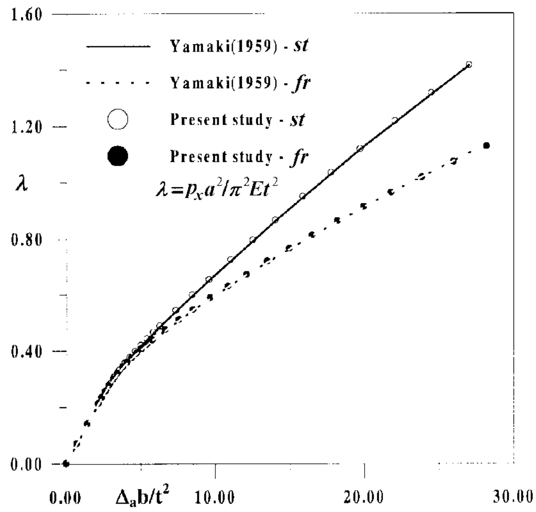

Fig. 3 Comparison of axial shortening of initially imperfect plates

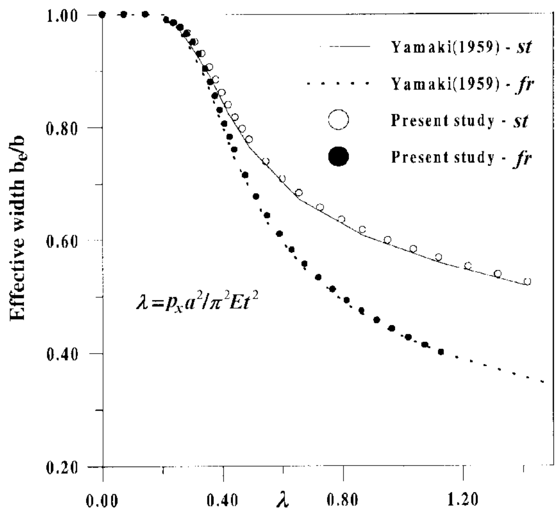

Fig. 4 Comparison of effective widths of initially imperfect plates

the plate is defined as the ratio of the actual load carried by the plate (the stresses are non-uniform in the postbuckled state) to the load the plate would have carried if the stress had been uniform. This can be computed as the Young's modulus times the average edge strain. Denoting the effective width as b_e, the ratio can be written as

$$b_e = \frac{\lambda}{(\Delta_a b/\pi^2 t^2)\beta^2} \tag{17}$$

where δ_x is the axial shortening, b is the width of the loaded edge, t is the thickness of the plate and β is the aspect ratio of the plate (a/b), where a and b are the length and width of the plate. In the present study the effective widths of plates were computed using PLOT-COLD and the results have been compared with those of Yamaki (1959) in Fig. 4. The results are in good agreement except for an acceptable deviation in the initial stages of loading.

6. Conclusions

Incremental matrices for thin initially imperfect plates with a small out-of-flatness have been derived using minimum potential energy principles. The explicit coefficients of the displacement gradient tensors are presented in this study. The formulations have been incorporated in a software PLOT-COLD and the veracity of the incremental matrices has been demonstrated to be excellent.

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Appendix

The displacement gradient vector g_α as in Eq. (8) can be written as

$$\boldsymbol{g}_\alpha^T = \{ \boldsymbol{u}_{,x} \ \boldsymbol{u}_{,y} \ \boldsymbol{v}_{,x} \ \boldsymbol{v}_{,y} \ \boldsymbol{w}_{,x} \ \boldsymbol{w}_{,y} \ \boldsymbol{w}_{,xx} \ \boldsymbol{w}_{,yy} \ \boldsymbol{w}_{,xy} \}_{9\times 1}$$

and the comma represents differentiation of the preceding quantity by the succeeding indices. The matrices $\boldsymbol{K}_{\alpha\beta}$, $\boldsymbol{N0}_{\alpha\beta}$, $\boldsymbol{N1}_{\alpha\beta}$, $\boldsymbol{N2}_{\alpha\beta}$ can be written as

The non-zero coefficients of the $[\ K\ ]_{9\times 9}$ matrix are given below

$K(1,1)=\boldsymbol{C_{11}}$ $K(1,4)=\boldsymbol{C_{12}}$ $K(2,2)=\boldsymbol{C_{33}}$
$K(2,3)=\boldsymbol{C_{33}}$ $K(3,3)=\boldsymbol{C_{33}}$ $K(4,4)=\boldsymbol{C_{22}}$
$K(4,4)=\boldsymbol{D_{11}}$ $K(4,5)=\boldsymbol{D_{12}}$ $K(5,5)=\boldsymbol{D_{22}}$
$K(6,6)=\boldsymbol{D_{33}}$

In which $K_{ij}=K_{ji}$ and C_{ij} the membrane rigidity is given by

$$[\boldsymbol{C}]_{3\times 3} = \frac{\boldsymbol{Et}}{(\boldsymbol{1-v^2})}\begin{bmatrix} \boldsymbol{1} & \boldsymbol{v} & \boldsymbol{0} \\ \boldsymbol{v} & \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \frac{(\boldsymbol{1-v})}{\boldsymbol{2}} \end{bmatrix}$$

and D_{ij} the out-of plane or the flexural rigidity matrix is given by

$$[\boldsymbol{D}]_{3\times 3} = \frac{\boldsymbol{Et^3}}{\boldsymbol{12}(\boldsymbol{1-v^2})}\begin{bmatrix} \boldsymbol{1} & \boldsymbol{v} & \boldsymbol{0} \\ \boldsymbol{v} & \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \frac{(\boldsymbol{1-v})}{\boldsymbol{2}} \end{bmatrix}$$

The non-zero components of initial imperfection matrix $\boldsymbol{N0}_{ij} = \boldsymbol{N0}_{ji}$ is given by

$N0(5,5)=\frac{C_{11}}{2}wo_{,x}^2+\frac{C_{12}}{2}wo_{,y}^2$ $\quad$ $N0(5,6)=C_{33}wo_{,x}wo_{,y}$

$N0(6,6)=\frac{C_{21}}{2}wo_{,x}^2+\frac{C_{22}}{2}wo_{,y}^2$

The non-zero coefficients of first order incremental matrix $\boldsymbol{N1}_{ij}=\boldsymbol{N1}_{ji}$ is given by

$N1(1,5)=C_{11}(w_{,x}+wo_{,x})$ $\quad$ $N1(1,6)=C_{12}(w_{,y}+wo_{,y})$

$N1(2,5)=C_{33}(w_{,y}+wo_{,y})$ $\quad$ $N1(2,6)=C_{33}(w_{,x}+wo_{,x})$

$N1(3,5)=C_{33}(w_{,y}+wo_{,y})$ $\quad$ $N1(3,6)=C_{33}(w_{,x}+wo_{,x})$

$N1(4,5)=C_{12}(w_{,x}+wo_{,x})$ $\quad$ $N1(4,6)=C_{22}(w_{,x}+wo_{,x})$

$N1(5,5)=C_{11}u_{,x}+C_{12}v_{,y}$ $\quad$ $N1(5,6)=C_{33}u_{,y}+C_{33}v_{,x}$

$N1(6,6)=C_{12}u_{,x}+C_{22}v_{,y}$

The non-zero coefficients of $\boldsymbol{N2}_{ij} = \boldsymbol{N2}_{ji}$ is given by

$$N2(5,5) = \frac{3C_{11}}{2}(w_{,x}+wo_{,x})^2 + \frac{C_{12}}{2}(w_{,y}+wo_{,y})^2 + C_{33}(w_{,y}+wo_{,y})^2$$

$$N2(5,6) = 2C_{33}(w_{,x}+wo_{,x})(w_{,y}+wo_{,y}) + C_{12}(w_{,x}+wo_{,x})(w_{,y}+wo_{,y})$$

$$N2(6,6) = \frac{3C_{22}}{2}(w_{,y}+wo_{,y})^2 + \frac{C_{12}}{2}(w_{,x}+wo_{,x})^2 + C_{33}(w_{,x}+wo_{,x})^2$$