Structural Engineering and Mechanics, Vol. 12, No. 1 (2001) 85-100
DOI: http://dx.doi.org/10.12989/sem.2001.12.1.085

Classes of exact solutions for several static and dynamic problems of non-uniform beams

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Abstract. In this paper, an analytical procedure for solving several static and dynamic problems of non-uniform beams is proposed. It is shown that the governing differential equations for several stability, free vibration and static problems of non-uniform beams can be written in the from of a unified self-conjugate differential equation of the second-order. There are two functions in the unified equation, unlike most previous researches dealing with this problem, one of the functions is selected as an arbitrary expression in this paper, while the other one is expressed as a functional relation with the arbitrary function. Using appropriate functional transformation, the self-conjugate equation is reduced to Bessel's equation or to other solvable ordinary differential equations for several cases that are important in engineering practice. Thus, classes of exact solutions of the self-conjugate equation for several static and dynamic problems are derived. Numerical examples demonstrate that the results calculated by the proposed method and solutions are in good agreement with the corresponding experimental data, and the proposed procedure is a simple, efficient and exact method.

Key words: non-uniform beam; stability; vibration; dynamic.

1. Introduction

Non-uniform beams are widely used in an effort to achieve a better distribution of strength and weight of structures, or structural members, machine parts and sometime to satisfy architectural and functional requirements. The problems of stability, free vibrations and static analysis of non-uniform beams have been the subject of numerous investigations because of its relevance to structural, mechanical and aeronautical engineering. However, in general, it is difficult to find the exact closed-form solutions for the buckling and free vibration of a non-uniform beam.

The experimental results obtained in dynamic testing of structures (e.g., Wang 1978, Li *et al*. 1994b) have shown that it is possible to regard a frame building as a shear beam with varying cross-section for free vibration analysis, the governing differential equation for mode shape function, X(*x*), of the beam can be written as (Li *et al*. 1994b)

$$\frac{d}{dx}\left(K_x\frac{dX(x)}{dx}\right)+\bar{m}_x\omega^2X(x) = 0 \tag{1}$$

in which K_x, $\bar{m}_x$ are shear stiffness and mass per unit length at section x, respectively, and ω is the circular natural frequency. It was discussed by Li *et al*.(1996) that the ordinary differential equations

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of the second-order for several buckling, free vibration and static problems, including the governing differential equation for buckling of a non-uniform beam and the governing differential equation of mode shape function of a shear beam with varying cross-section, can be written as a unified self-conjugate equation of the second-order as follows

$$\frac{d}{dx}\left[f(x)\frac{dy(x)}{dx}\right]+\psi(x)y(x)=0 \tag{2}$$

It should be pointed out that the governing differential equation for mode shape function of a flexural beam with varying cross-section can be divided into two self-conjugate equations of the second order (Li *et al*. 1994b, Li 1999), and the governing differential equation of free longitudinal vibration of a non-uniform beam (Li *et al.* 1998, Li 2000) or a straight rod (Clough *et al*. 1975, Wang 1978) can be also expressed as Eq. (1) which is a special case of Eq. (2).

It can be seen from the above discussion that the exact solutions of Eq. (2) with various expressions of $f(x)$ and $\psi(x)$ are very useful in engineering practices. Exact solutions of Eq. (2) for several special cases were found by Li *et al.* (1994a, 1994b, 1995) and Kumar and Sujth (1997). Panayotounakos (1995) obtained classes of analytical solutions of the linear ordinary differential equations, which were established by Li *et al*. (1995), governing the stability problem of non-uniform bars subjected to axial distributed loading. These classes of buckling solutions extended the closed-form solutions obtained by Li *et al.* (1995).

A review of technical literature dealing with the problems of free vibration and buckling analysis of non-uniform beams indicates that generally the authors of the previous studies have directed their investigations to special functions for describing the distributions of mass and stiffness as well as axial distributed forces in order to derive closed-form solutions for such problems. This is equivalent to select several special functions for the expressions of $f(x)$ and $\psi(x)$ to obtain the analytical solutions for Eq. (2). It should be mentioned that exact solutions for Eq. (2) with arbitrary expression of $f(x)$ or $\psi(x)$ for structural mechanics analysis have not been obtained in the literature. In this paper a successful attempt is made to present classes of analytical solutions for the governing differential Eq. (2) for several static and dynamic problems of non-uniform beams. By means of functional transformation, classes of analytical solutions are found for the case when the function, $f(x)$, in Eq. (2) is arbitrary, while the function, $\psi(x)$, is expressed as a functional relation with $f(x)$. It is noted that some analytical solutions presented previously mentioned above result as special cases of the exact solutions obtained in this paper.

2. Theory

It was discussed in the last section that the governing differential equations for several mechanics problems of non-uniform beams can be written in the form of Eq. (2). The procedure for determining the exact solutions of Eq. (2) is as follows:
Eq. (2) can be rewritten as

$$\frac{d^2y(x)}{dx^2}+\frac{1}{f(x)}\frac{df(x)}{dx}\frac{dy(x)}{dx}+\frac{\Psi(x)}{f(x)}y(x)=0 \tag{3}$$

Setting

$$\frac{1}{f(x)} = p(x), \ \psi(x) = p(x)Q(x), \ \zeta = \int p(x)dx \tag{4}$$

In the above equation, the function, $f(x)$ can be selected as an arbitrary function, while the function, $\psi(x)$, is expressed as a functional relation with $f(x)$.

Substituting Eq. (4) into Eq. (3) yields

$$\frac{d^2y(\zeta)}{d\zeta^2} + Q(\zeta)y(\zeta) = 0 \tag{5}$$

Obviously, the exact solutions of Eq. (5) are dependent on the expression of $Q(\zeta)$. Several cases which are important in engineering practice are considered and discussed as follows:

Case 1:

$$Q(\zeta) = (a + b\zeta)^C \tag{6}$$

Using the following functional transformation

$$t = (a + b\zeta)^{\frac{1}{2v}}, \ y = t^v Z, \ v = \frac{1}{C+2}, \ \alpha = \frac{2v}{|b|}$$

Eq. (5) is reduced to a Bessel's equation as

$$\frac{d^2Z}{dt^2} + \frac{1}{t}\frac{dZ}{dt} + \left(\alpha^2 - \frac{v^2}{t^2}\right)Z = 0 \tag{7}$$

The function, $y(x)$, can be expressed as

$$y = \begin{cases} (a+b\zeta)^{\frac{1}{2}}\left\{C_1 J_v\left[\alpha(a+b\zeta)^{\frac{1}{2v}}\right] + C_2 J_{-v}\left[\alpha(a+b\zeta)^{\frac{1}{2v}}\right]\right\} & v = \text{a non-integer} \\ (a+b\zeta)^{\frac{1}{2}}\left\{C_1 J_v\left[\alpha(a+b\zeta)^{\frac{1}{2v}}\right] + C_2 Y_v\left[\alpha(a+b\zeta)^{\frac{1}{2v}}\right]\right\} & v = \text{an integer} \end{cases} \tag{8}$$

where J_v and Y_v are Bessel functions of the first, second kind of order v, respectively, C_1 and C_2 are integral constants that can be determined by the boundary conditions.

If $C=-2$, then substituting Eq. (6) into Eq. (5) yields an Euler equation, the solution of the equation can be easily found.

Case 2:

$$Q(\zeta) = a(1 + b\zeta)^C \tag{9}$$

This case is an alteration of Case 1. The general solution can be written as

$$y = \begin{cases} (1+b\zeta)^{\frac{1}{2}}\left\{C_1 J_v\left[\tilde{\alpha}(1+b\zeta)^{\frac{1}{2v}}\right] + C_2 J_{-v}\left[\tilde{\alpha}(1+b\zeta)^{\frac{1}{2v}}\right]\right\} & v = \text{a non-integer} \\ (1+b\zeta)^{\frac{1}{2}}\left\{C_1 J_v\left[\tilde{\alpha}(1+b\zeta)^{\frac{1}{2v}}\right] + C_2 Y_v\left[\tilde{\alpha}(1+b\zeta)^{\frac{1}{2v}}\right]\right\} & v = \text{an integer} \end{cases} \tag{10}$$

where

$$\tilde{\alpha} = \frac{4a - b^2}{b^2},\ v = \frac{1}{C+2}$$

Case 3:

$$Q(\zeta) = ae^{b\zeta} - c, \qquad c > 0 \tag{11}$$

where a, b and c are parameters that can be determined by the values of $\bar{m}(x)$ at critical sections of the beam.

For this case, setting

$$t = \frac{b\zeta}{2},\ \bar{\alpha} = \frac{2a^{\frac{1}{2}}}{|b|},\ v = \frac{2c^{\frac{1}{2}}}{|b|}$$

Eq. (5) is reduced to a Bessel's equation of v-order and the function, $y(x)$, can be expressed as

$$y = \begin{cases} C_1 J_v\left[\bar{\alpha} e^{\frac{b\zeta}{2}}\right] + C_2 J_{-v}\left[\bar{\alpha} e^{\frac{b\zeta}{2}}\right] & v = \text{a non-integer} \\ C_1 J_v\left[\bar{\alpha} e^{\frac{b\zeta}{2}}\right] + C_2 Y_v\left[\bar{\alpha} e^{\frac{b\zeta}{2}}\right] & v = \text{an integer} \end{cases} \tag{12}$$

If c=0, then v=0.

Case 4:

$$Q(\zeta) = (a\zeta^2 + b\zeta + c)^{-2} \tag{13}$$

Substituting Eq. (13) into Eq. (5) and setting

$$\left.\begin{aligned} Q(\zeta) &= (a\zeta^2 + b\zeta + c)^{\frac{1}{2}}\eta(\xi) \\ \xi &= \int \frac{d\zeta}{a\zeta^2 + b\zeta + c} \end{aligned}\right\} \tag{14}$$

one yields

$$\frac{d^2\eta}{d\xi^2} + A\eta = 0 \tag{15}$$

in which

$$A = 1 + ac - \frac{1}{4}b^2 \tag{16}$$

The general solution for this case is given by

$$y = (a\zeta^2 + b\zeta + c)^{\frac{1}{2}}(C_1 \sin\sqrt{A}\xi + C_2 \cos\sqrt{A}\xi) \quad \text{for} \quad A > 0 \tag{17}$$

or

$$y = (a\zeta^2 + b\zeta + c)^{\frac{1}{2}}(C_1 e^{\sqrt{A}\xi} + C_2 e^{-\sqrt{A}\xi}) \qquad \text{for} \qquad A = 0 \tag{18}$$

or

$$y = (a\zeta^2 + b\zeta + c)^{\frac{1}{2}}(C_1 \sin\xi + C_2 \cos\xi) \qquad \text{for} \qquad A = 0 \tag{19}$$

Case 5:

$$Q(\zeta) = a(\zeta^2 + b)^{-2} \qquad a > 0,\ b > 0 \tag{20}$$

The general solution for this case is given by

$$y = (\zeta^2 + b)^{\frac{1}{2}}(C_1 \sin\xi + C_2 \cos\xi) \tag{21}$$

where

$$\xi = \left(\frac{a+b}{b}\right)^{\frac{1}{2}} \arctan\frac{\zeta}{b^{\frac{1}{2}}}$$

Case 6:

$$Q(\zeta) = a(\zeta^2 - b)^{-2} \qquad a > 0,\ b > 0 \tag{22}$$

The general solution for this case is as

$$y = (\zeta^2 - b)^{\frac{1}{2}}(C_1 \sin\xi + C_2 \cos\xi) \tag{23}$$

where

$$\xi = \frac{1}{2}\left(\frac{a-b}{b}\right)^{\frac{1}{2}} \ln\frac{b^{\frac{1}{2}} + \zeta}{b^{\frac{1}{2}} - \zeta}, \qquad |\zeta| < b^{\frac{1}{2}}$$

3. Applications

3.1 Free vibration of a shear beam

As mentioned above, the governing differential equation of a shear beam with varying cross-section is given by Eq. (1). Comparing Eq. (1) with Eq. (2) gives

$$f(x) = K_x, \quad \psi(x) = \overline{m}_x \omega^2 \tag{24}$$

If the distribution of shear stiffness, K_x, is described by

$$K_x = K_0(1 + \beta x)^{-r} \tag{25}$$

in which parameters K_0, β, γ are given, and $Q(\zeta)$ is selected as Eq. (9), where $a=a_1\omega^2$, then the distribution of mass per unit length, $\bar{m}_x$, can be found using Eq. (4) as

$$\bar{m}_x = \frac{a_1}{K_0}(1+\beta x)^{\gamma}(1+b\zeta)^{C}, \quad \zeta = \frac{1}{K_0\beta}(1+\beta x)^{\gamma+1} \tag{26}$$

where a_1, b, C are three independent parameters that can be determined by using the values of $\bar{m}_x$ at three critical sections of the beam.

The general solution for mode shape function, $y(\zeta)$, of the shear beam is given by Eq. (10). The frequency equation can be established by using Eq. (10) and the boundary conditions. It was reported by Li *et al*. (1999) that it is possible to regard a shear plate as two shear beams for free vibration analysis. Thus, the method introduced above can also be used to estimate natural frequencies and mode shapes of such a plate.

3.2 Free longitudinal vibration of a beam with varying cross-section

The governing differential equation for mode shape function of longitudinal vibration of a non-uniform beam is the same as Eq. (1), but K_x represents the longitudinal stiffness of the beam.

3.3 Free torsional vibration of a beam with varying circular section

For this case, K_x, $\bar{m}_x$, $X(x)$ are the torsional stiffness, the polar moment of mass inertia and torsional angle, respectively.

3.4 Free vibration of shear beam with varying cross-section on an elastic foundation (Fig. 1)

For this case, $f(x)$ in Eq. (2) is the shear stiffness, which is given by

$$(x) = \bar{m}_x\omega^2 - \bar{c}_x \tag{27}$$

in which $\bar{c}_x$ is the coefficient of elastic foundation.

As Cao *et al*. (1993) reported that a roof system of an one-story industrial building supported by closely spaced columns can be treated as a shear beam on an elastic foundation as shown in Fig. 1. Thus, the present method and solutions can also be used to analyze free vibration of such a building.

3.5 Free flexural vibration of a beam with varying cross-section

The governing differential equation for mode shape function of free flexural vibration of a beam with varying cross-section can be written as (Li *et al*. 1994b)

$$\frac{d^2}{dx^2}\left(K_x\frac{d^2X(x)}{dx^2}\right) - \bar{m}_x\omega^2 X(x) = 0 \tag{28}$$

in which K_x is the flexural stiffness of the beam.

It is not difficult to prove that if $\bar{m}_x$ and K_x satisfy the following condition

$$\frac{\frac{d}{dx}\sqrt{\bar{m}_x K_x}}{\bar{m}_x} = \text{constant} \tag{29}$$

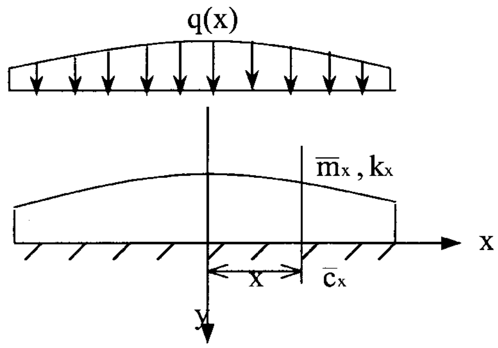

Fig. 1 A shear beam on an elastic foundation

or

$$\overline{m}_x K_x = \text{constant} \tag{30}$$

then, Eq. (28) can be divided into the following two self-conjugate equations of the second-order, which have the same form as Eq. (2)

$$\left.\begin{aligned} \frac{d}{dx}\left(\sqrt{\overline{m}_x K_x}\frac{dX(x)}{dx}\right) + \overline{m}_x \omega_1^2 X(x) = 0 \\ \frac{d}{dx}\left(\sqrt{\overline{m}_x K_x}\frac{dX(x)}{dx}\right) - \overline{m}_x \omega_1^2 X(x) = 0 \end{aligned}\right\} \tag{31}$$

in which

$$\omega_1^2 = \omega \tag{32}$$

3.6 Buckling of a beam with varying cross-section (Fig. 2)

The governing differential equation of a beam with varying cross-section subjected to concentrated and variable axial distributed loading (Fig. 2) is given by (Li *et al*. 1995)

$$\frac{d^2M(x)}{dx^2} - \frac{1}{N(x)}\frac{dN(x)}{dx}\frac{dM(x)}{dx} + \frac{N(x)}{K_x} = \frac{C_0}{N(x)}\frac{dN(x)}{dx} \tag{33}$$

where $M(x)$, $N(x)$ and K_x are the bending moment, axial force and flexural stiffness at section x, respectively. C_0 is given by

$$C_0 = N\frac{dy}{dx} - \frac{dM}{dx} \tag{34}$$

It can be seen from Eq. (34) that $C_0 = -\frac{dM}{dx}$ when $\frac{dy}{dx} = 0$ where $\frac{dy}{dx}$ represents the slope of the deflected beam.

It is evident that $C_0 = 0$ for a cantilever beam. For this case, Eq. (33) becomes

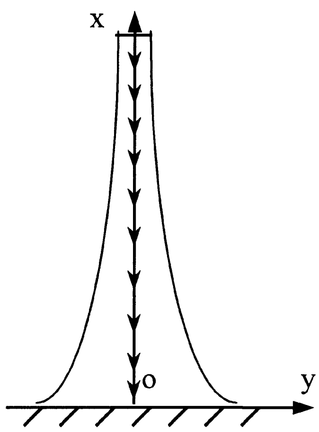

Fig. 2 A beam with variable cross-section

$$\frac{d^2M(x)}{dx^2} - \frac{1}{N(x)}\frac{dN(x)}{dx}\frac{dM(x)}{dx} + \frac{N(x)}{K_x}M(x) = 0 \tag{35}$$

Eq. (35) can be also written as Eq. (2). For this case, we have

$$y(x) = M(x), \quad f(x) = \frac{1}{N(x)}, \quad \psi(x) = \frac{1}{K_x} \tag{36}$$

Timoshenko (1930), Dinik (1955), Timoshenko and Gere (1961) only considered a special case of Eq.(35) that is $N(x)$=constant. Hence, their derived solutions for buckling of non-uniform beams result as special cases of the exact solutions obtained in this paper.

3.7 Static analysis of a beam with variable cross-section on an elastic foundation (Fig. 1)

The governing differential equation for static problem of a non-uniform shear beam on an elastic foundation is given by

$$\frac{d}{dx}\left(K_x\frac{dy(x)}{dx}\right) - \bar{c}_x y(x) = q(x) \tag{37}$$

For this case, $f(x)$ in Eq. (2) is the shear stiffness K_x, $\psi(x)$ is given by

$$\psi(x) = -\bar{c}_x \tag{38}$$

After two linearly independent homogeneous solutions, $y_1(x)$ and $y_2(x)$ of Eq. (37) are found, then the general solution of Eq. (37) can be determined by means of the Lagrange method as follows

$$y(x) = C_1y_1(x) + C_2y_2(x) - y_1(x)\int\frac{y_2(x)}{D(x)}q(x)dx + y_2(x)\int\frac{y_1(x)}{D(x)}q(x)dx \tag{39}$$

in which

$$D(x) = y_1(x)\frac{dy_2(x)}{dx} - y_2(x)\frac{dy_1(x)}{dx} \tag{40}$$

If the beam shown in Fig. 1 is a flexural one, then the governing differential equation of deflection of the beam is given by

$$\frac{d^2}{dx^2}\left(K_x\frac{d^2y(x)}{dx^2}\right) - \bar{c}_x y(x) = q(x) \tag{41}$$

The homogeneous form of Eq. (41) can be divided into two self-conjugate equations.

3.8 Free vibration of a multi-step shear beam

A multi-step shear beam is shown in Fig. 3, each step beam has variably distributed stiffness and mass. The governing differential equation for free vibration of the *i*-th step beam can be written as

$$\frac{d}{dx}\left(K_{xi}\frac{dX_i(x)}{dx}\right) - \bar{m}_{xi}\omega^2 X_i(x) = 0 \tag{42}$$

Eq. (42) has the same form as Eq. (1). The general solution can be obtained by using the method mentioned above and written as follows

$$X_i(x) = C_{i1}S_{i1}(x) + C_{i2}S_{i2}(x) \tag{43}$$

where $S_{i1}(x)$ and $S_{i2}(x)$ are two linearly independent solutions of Eq. (42).

The relation between the parameters X_{i1} (shear displacement) and Q_{i1} (shear force) at the end x_{i1}

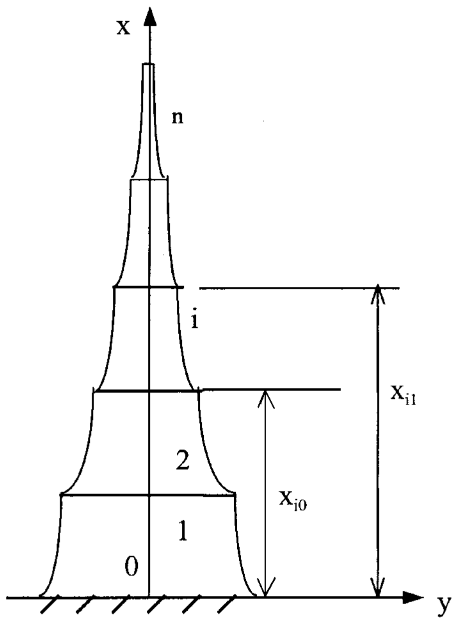

Fig. 3 A multi-step shear beam

and the parameters X_{i0} and Q_{i0} at the end x_{i0} of the *i*-th step beam (Fig. 3) can be expressed as

$$\begin{bmatrix} X_{i1} \\ Q_{i1} \end{bmatrix} = [T_i] \begin{bmatrix} X_{i0} \\ Q_{i0} \end{bmatrix} \tag{44}$$

in which

$$[T_i] = \begin{bmatrix} S_{i1}(x_{i1}) & S_{i2}(x_{i1}) \\ K_{i1}S'_{i1}(x_{i1}) & K_{i2}S'_{i2}(x_{i1}) \end{bmatrix} \begin{bmatrix} S_{i1}(x_{i0}) & S_{i2}(x_{i0}) \\ K_{i0}S'_{i1}(x_{i0}) & K_{i0}S'_{i2}(x_{i0}) \end{bmatrix}^{-1} \tag{45}$$

$$X_{i1} = X_i(x_{i1}), \quad X_{i0} = X_i(x_{i0}), \quad Q_{i1} = Q_i(x_{i1}), \quad Q_{i0} = Q_i(x_{i0}), \quad K_{i1} = K_{ix}(x_{i1})$$

$$K_{i0} = K_{ix}(x_{i0}), \quad S'_{i1} = \left. \frac{dS_{i1}(x)}{dx} \right|_{x = x_{i1}}$$

$[T_i]$ is called the transfer matrix because it transfers the parameters at the end x_{i0} to those at the end x_{i1} of the *i*-th step beam.

Since

$$X_{i0} = X_{i-1,1}, \qquad Q_{10} = Q_{i-1,1}$$

X_{i1} and Q_{i1} can be expressed as

$$\begin{bmatrix} X_{i1} \\ Q_{i1} \end{bmatrix} = [T_i][T_{i-1}]\ldots[T_1] \begin{bmatrix} X_{10} \\ Q_{10} \end{bmatrix} \tag{46}$$

Setting $i=n$ one obtains the relation between the parameters at the top (Fig. 3) and those at the base of the beam as follows

$$\begin{bmatrix} X_{n1} \\ Q_{n1} \end{bmatrix} = [T] \begin{bmatrix} X_{10} \\ Q_{10} \end{bmatrix} \tag{47}$$

in which

$$[T] = [T_n][T_{n-1}]\ldots[T_1] \tag{48}$$

$[T]$ is a matrix of the second-order as follows

$$[T] = \begin{bmatrix} T_{11} & T_{21} \\ T_{21} & T_{22} \end{bmatrix} \tag{49}$$

The elements T_{ij} (i, j=1, 2) in Eq. (49) can be found by using Eqs. (48) and (45).

The frequency equation can be established by use of Eq. (47) and the boundary conditions as follows

$T_{22} = 0$	for a fixed-free beam (Fig. 3)	(50)
$T_{12} = 0$	for a fixed-fixed beam	(51)
$T_{21} = 0$	for a free-free beam	(52)
$T_{21} + K_u T_{22} = 0$	for a free-spring supported beam (Fig. 4)	(53)

Setting n=1 one obtains the solutions for free vibration of one-step shear beams

4. Numerical Example 1

A 15-story building with 46.0 m height is located in Wuhan, P.R. China. Based on the field measurement of dynamic behavior of this building (Li *et al.* 1994b), it can be treated as a cantilever shear beam in analysis of free vibration of this structure. The values of mass per unit length and shear stiffness are calculated and shown in Fig. 5.

The procedure for determining the natural frequencies and mode shapes of this building is as follows

4.1 Selection of expressions for describing the distributions of mass per unit length and shear stiffness

Because the variation of the mass per unit length is relatively small, it is reasonable to assume $\bar{m}(x)$ as a constant, $\bar{m}$,

$$\bar{m} = \frac{[2.76 \times 10 + (2.84 + 2.79 + 2.75 + 2.80) \times 9] \times 10^5}{46} = 2.79 \times 10^5 (\text{kg/m})$$

The expression for describing the distribution of shear stiffness is selected as an exponential function as follows

$$K(x) = K_0 e^{-\beta \frac{x}{H}} \tag{54}$$

K_0 and β are found as

$$K_0 = 9.86 \times 10^9 N$$

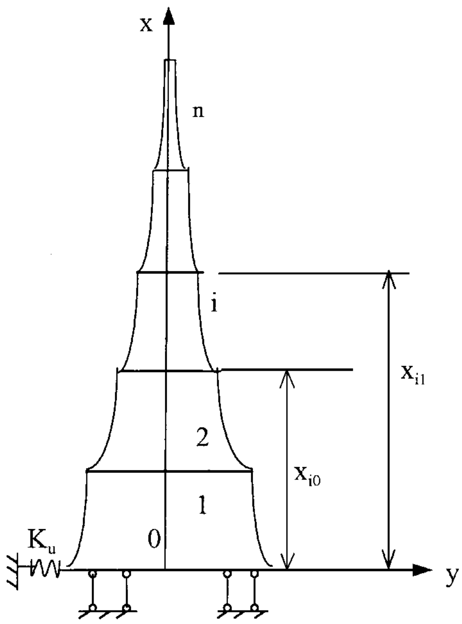

Fig. 4 A multi-step free-spring supported shear beam

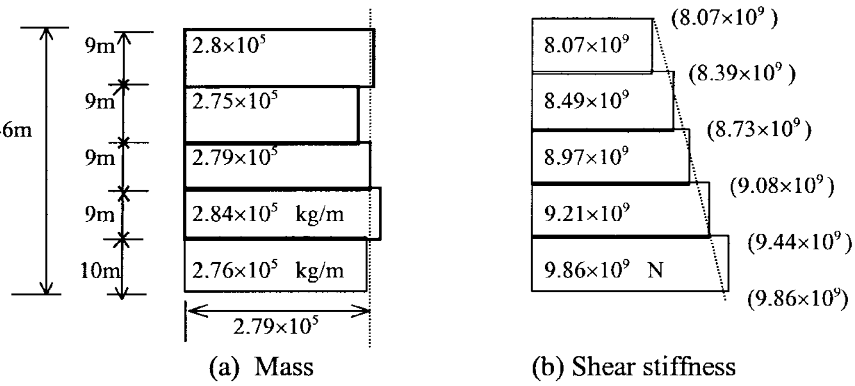

Fig. 5 The distributions of mass and stiffness of the building

$$\beta = \ln\frac{K(0)}{K(H)} = \ln\frac{9.86\times10^9}{8.07\times10^9} = 0.20$$

The distribution of stiffness given by Eq. (54) is also shown in Fig. 5 (dotted line and the values in parentheses). It can be seen from Fig. 5 that the selected expression is suitable for describing the distribution of stiffness of this typical tall building.

4.2 Determination of $Q(\zeta)$

It is assumed that $Q(\zeta)$ can be expressed as special case of Case 1, i.e.,

$$Q(\zeta) = \frac{\bar{m}H}{\beta}\cdot\frac{1}{\zeta} \tag{55}$$

Using Eq. (4) leads to

$$\alpha = \frac{2\omega\bar{m}H}{\beta},\ \zeta = \frac{H}{K_0\beta}e^{\beta\frac{x}{H}},\ \zeta_0 = \frac{H}{K_0\beta} = 2.3327\times10^{-8},\ \zeta_H = \frac{H}{K_0\beta}e^{\beta} = 2.8492\times10^{-8} \tag{56}$$

The frequency equation is Eq. (50), i.e.,

$$J_1\left[\alpha\left(\frac{\bar{m}H}{\beta}\zeta_0\right)^{\frac{1}{2}}\right]Y_0\left[\alpha\left(\frac{\bar{m}H}{\beta}\zeta_H\right)^{\frac{1}{2}}\right] = Y_1\left[\alpha\left(\frac{\bar{m}H}{\beta}\zeta_0\right)^{\frac{1}{2}}\right]J_0\left[\alpha\left(\frac{\bar{m}H}{\beta}\zeta_H\right)^{\frac{1}{2}}\right] \tag{57}$$

A set of roots, α_j (j=1, 2, ...) can be obtained by solving Eq. (57). The minimum root, α_1, is found as

$$\alpha_1 = 7.9092\times10^8$$

Substituting α_1 into Eq. (56) one obtains the fundamental circular natural frequency and the fundamental period as

$$\omega_1 = 6.1627 \text{ rad sec}^{-1} \qquad T_1 = 1.0196 \text{ sec.}$$

The measured value of fundamental period is 1.05 sec (Li *et al.* 1994b).

It is evident that the computed value of the fundamental period shows good agreement with the measured one.

4.3 Calculation of the vibration mode shapes

The vibration mode function can be determined by using Eq. (45) and setting X_{10}=0 as follows

$$y = \left(\frac{\overline{m}H}{\beta}\zeta\right)^{\frac{1}{2}} \left\{ J_1\left[\alpha\left(\frac{\overline{m}H}{\beta}\zeta\right)^{\frac{1}{2}}\right] - \frac{J_1\left[\alpha\left(\frac{\overline{m}H}{\beta}\zeta_0\right)^{\frac{1}{2}}\right]}{Y_1\left[\alpha\left(\frac{\overline{m}H}{\beta}\zeta_0\right)^{\frac{1}{2}}\right]} Y_1\left[\alpha\left(\frac{\overline{m}H}{\beta}\zeta\right)^{\frac{1}{2}}\right] \right\} \tag{58}$$

The values of the fundamental mode shape are obtained by substituting α_1 into Eq. (58) and are listed in Table 1.

The field measured values of the fundamental mode shape are also listed in Table 1 for comparison purposes. It can be seen from Table 1 that the computed values of the fundamental mode shape are very close to the corresponding field measured ones.

It should be mentioned that using the aforementioned procedure the higher natural frequencies and corresponding mode shapes can also be determined.

5. Numerical Example 2

This numerical example will illustrate how to determine the fundamental natural frequency and mode shape of the building considered in the Numerical Example 1 by using the model of five-step uniform beam shown in Fig. 6. The distributions of mass and shear stiffness of this building are shown in Fig. 5.

The procedure for determining the natural frequencies and mode shapes by using the model of five-step uniform beam shown in Fig. 6 is as follows:

5.1 Determination of special solutions for free vibration of each step beam

Because the distributions of mass per unit length and shear stiffness of each step are uniform for this case, the special solutions can be found from Eq. (5) and setting $Q(\zeta)$=constant as follows

$$S_{i1}(x) = \sin\left(\sqrt{\frac{\overline{m}_i}{K_i}}\,\omega x\right) \qquad S_{i2}(x) = \cos\left(\sqrt{\frac{\overline{m}_i}{K_i}}\,\omega x\right) \tag{59}$$

Table 1 The fundamental mode shape

Story level	1	3	5	7	9	11	13	15
x/H	0	0.1522	0.2826	0.4130	0.5435	0.6739	0.8043	1.0
Computed values	0 (0)	0.2431 (0.244)	0.4299 (0.431)	0.6026 (0.604)	0.7529 (0.754)	0.8708 (0.872)	0.9507 (0.953)	1.0 (1.0)
Measured values	0	0.242	0.431	0.603	0.754	0.871	0.951	1.0

Note: the values in parentheses are calculated ones using the model of five-step beam to be described in the Numerical Example 2.

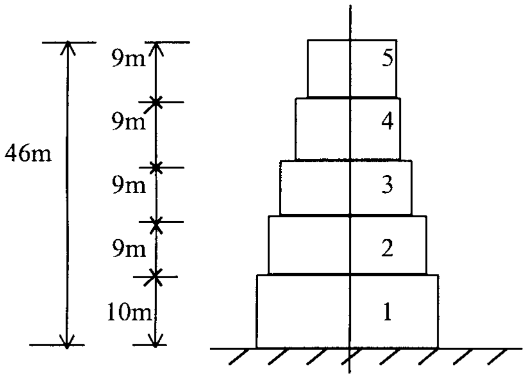

Fig. 6 The five-step beam model of the building

Substituting the values of $\overline{m}_i$ and K_i of the i-th step beam into Eq. (59) one obtains

$$\begin{aligned}
S_{11}(x) &= \sin(0.5291\times10^{-2}\omega x) \qquad & S_{12}(x) &= \cos(0.5291\times10^{-2}\omega x)\\
S_{21}(x) &= \sin(0.5553\times10^{-2}\omega x) & S_{22}(x) &= \cos(0.5553\times10^{-2}\omega x)\\
S_{31}(x) &= \sin(0.5577\times10^{-2}\omega x) & S_{32}(x) &= \cos(0.5577\times10^{-2}\omega x)\\
S_{41}(x) &= \sin(0.5691\times10^{-2}\omega x) & S_{42}(x) &= \cos(0.5691\times10^{-2}\omega x)\\
S_{51}(x) &= \sin(0.5890\times10^{-2}\omega x) & S_{52}(x) &= \cos(0.5890\times10^{-2}\omega x)
\end{aligned} \tag{60}$$

5.2 Evaluation of the natural frequencies and mode shapes

Using Eq. (47) and the boundary conditions for a cantilever shear beam leads to

$$\begin{bmatrix} y_{11} \\ Q_{11} \end{bmatrix} = [T_1]\begin{bmatrix} 0 \\ Q_{10} \end{bmatrix} \tag{61}$$

$$\begin{bmatrix} y_{11} \\ Q_{21} \end{bmatrix} = [T_2][T_1]\begin{bmatrix} 0 \\ Q_{10} \end{bmatrix} \tag{62}$$

$$\begin{bmatrix} y_{31} \\ Q_{31} \end{bmatrix} = [T_3][T_2][T_1]\begin{bmatrix} 0 \\ Q_{10} \end{bmatrix} \tag{63}$$

$$\begin{bmatrix} y_{41} \\ Q_{41} \end{bmatrix} = [T_4][T_3][T_2][T_1]\begin{bmatrix} 0 \\ Q_{10} \end{bmatrix} \tag{64}$$

$$\begin{bmatrix} y_{41} \\ 0 \end{bmatrix} = [T_5][T_4][T_3][T_2][T_1]\begin{bmatrix} 0 \\ Q_{10} \end{bmatrix} \tag{65}$$

$[T_i]$ can be determined by using Eq. (45).

The frequency equation which is the same as Eq. (50) can be determined from Eq. (65), i.e., T_{22} is determined from Eq. (65). Solving the frequency equation one obtains the fundamental natural frequency

$$\omega_1 = 6.1632 \text{ rad sec}^{-1}$$

Substituting ω_1 into Eqs. (61)-(65) one obtains the fundamental mode shape that are tabulated in Table 1 for comparison purposes.

It can be seen from the above results that the fundamental natural frequency and mode shape determined by using the model of continuously varying cross-section (Example 1) are almost the same as those determined by using the model of five-step beam. This fact implies that it is possible to regard a multi-step beam as a one-step beam with continuously varying cross-section and vice versa for free vibration analysis.

6. Conclusions

In this paper, the governing differential equations for several static and dynamic problems of a beam with variable cross-section are written as the unified form of a self-conjugate equation, Eq. (2), of the second-order. There are two functions in Eq. (2), unlike most previous researches dealing with this problem, one of the functions is selected as an arbitrary expression in this paper. Using the method of functional transformation, Eq. (2) is reduced to Eq. (5). Since there is only one function, $Q(\zeta)$, in Eq. (5), it is easier to find the solutions of Eq. (5) as compared with Eq. (2). Because $Q(\zeta)$ is a functional expression, a solution of Eq. (5) actually represents a class of solutions. Some solutions for buckling and free vibration problems of non-uniform beams presented previously by Dinik (1955), Kumar and Sujth (1997), Li (1999), Li *et al*. (1994a, 1994b, 1995, 1998), Timoshenko (1930), Timoshenko and Gere (1961), Wang (1978), etc. result as special cases of the exact solutions obtained in this paper. Numerical examples demonstrate that the results calculated by the proposed method are in good agreement with the corresponding experimental data, and the proposed procedure is a simple, efficient and exact method.

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