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Influence of thickness variation of annular plates on the buckling problem

P.M. Ciancio† and J.A. Reyes‡

Department of Civil Engineering, Universidad Nacional del Centro de la Provincia de Buenos Aires. Av. Del Valle 5737, 7400 Olavarria, Argentina

Abstract. The aim of this work is to establish the coefficient that defines the critical buckling load for isotropic annular plates of variable thickness whose outer boundary is simply supported and subjected to uniform pressure. It is assumed that the plate thickness varies in a continuous way, according to an exponential law. The eigenvalues are determined using an optimized Rayleigh-Ritz method with polynomial coordinate functions which identically satisfy the boundary conditions at the outer edge. Good engineering agreement is shown to exist between the obtained results and buckling parameters presented in the technical literature.

Key words: critical buckling load; annular plates; variable thickness.

1. Introduction

The buckling problem is of great practical significance in civil, mechanical, naval and aerospace engineering applications of plates with either a constant or variable thickness. This topic has received the attention of many researchers who consider it together with the vibration problem.

Stuart and Carney (1974) studied the free vibrations of isotropic annular plates and have presented an exact solution that includes asymmetric modes using the Rayleigh-Ritz method. Ramaiah (1975) considered the asymmetric vibration modes of polar orthotropic annular plates. Loh and Carney (1976) have determined an exact solution for both the circular frequencies and buckling loads for all modes of radially compressed spinning isotropic annular plates reinforced with edge beams. Soni and Amba-Rao (1975) used the Chebyshev collocation method to study the free axisymmetric vibrations of isotropic annular plates with linear thickness profiles. Ramaiah and Vijayakumar (1975) also considered the free vibration of isotropic annular plates with linear thickness profiles and, using the Rayleigh-Ritz method, they have obtained circular frequency parameters for axisymmetric and asymmetric modes. Laura, *et al.* (1975, 1988, 1995, 1996) and Avalos *et al.* (1979) have produced important contributions. These authors have dealt with the problem by taking into account different support conditions and discontinuous variable thicknesses. They have given critical buckling loads.

Dyka and Carney III (1979) have presented an exact solution for an orthotropic annular plate with variable thickness and reinforced with edge beams along the outer and inner boundaries.

† Master of Science
‡ Professor

An exact solution for the fundamental problem treated here is available in the case of plates of uniform thickness, while there is only a very limited number of studies in the case of annular plates of non-uniform thickness. In this study the coefficient that leads to the critical buckling load is determined for isotropic annular plates of both uniform and variable thickness with the outer edge simply supported and the inner contour free. The outer boundary of the plate is subjected to a uniformly applied pressure. The analysis is performed by using the optimized Rayleigh-Ritz method which is based on a variational principle of mechanics.

Numerical data is obtained for plates with both uniform and non-uniform thicknesses for a Poisson's ratio value of $\mu = 1/3$.

2. Approximate solution by means of the optimized Rayleigh-Ritz method

A thin annular plate is considered where a and b are respectively the radii of inner and outer edges. The thickness h, of the plate is assumed to vary in the radial direction in the form $h=H\, r^p$, where H, p and constants and r is the radius, proposed by Dyka and Carney (1979).

Determination of the critical buckling load is defined by minimization of the governing functional

$$U=\frac{1}{2}\iint D\left[\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)^2-\frac{2(1-\mu)}{r}\frac{\partial w}{\partial r}\cdot\frac{\partial^2 w}{\partial r^2}\right]r\,dr\,d\theta-\frac{1}{2}\iint N_r\left(\frac{\partial w}{\partial r}\right)^2 r\,dr\,d\theta \tag{1}$$

where: D: rigidity of the plate to flexure
N_r: compressive radial force
w : amplitude of normal displacement
r, θ : polars coordinates
μ : Poisson's ratio

Under an in-plane force which is uniform in the radial direction, there is radial symmetry

$$w = w(r) \tag{2}$$

Thus

$$U=\pi\int D\left[\left(\frac{d^2 w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)^2-2(1-\mu)\left(\frac{d^2 w}{dr^2}\frac{1}{r}\frac{dw}{dr}\right)\right]r\,dr-\pi\int N_r\left(\frac{dw}{dr}\right)^2 r\,dr \tag{3}$$

Introducing the dimensionless variable x

$$x = r/b,\; r = b\,x;\quad \text{and } dr = b\,dx \tag{4}$$

one obtains

$$U=\pi\left\{\int_k^1 \frac{D}{b^2}\left[\left(\frac{d^2 w}{dx^2}\right)^2 x+\frac{1}{x}\left(\frac{dw}{dx}\right)^2+2\mu\frac{d^2 w}{dx^2}\frac{dw}{dx}\right]dx-\int_k^1 N\left(\frac{dw}{dx}\right)^2 x\,dx\right\} \tag{5}$$

where $k = ri/re$.

In applying the Rayleigh-Ritz method, it is quite convenient to approximate the displacement amplitude $w(r)$ by means of a summation of simple polynomial coordinate functions. It is assumed in the usual form

$$w_{ap}(r)=\sum_{j=1}^{j} A_{j-1}(\alpha_{j-1}x^{\gamma}+\beta_{j-1}x^{2}+1)x^{j-1} \tag{6}$$

In the purposed displacement function, an unknown parameter γ, appears as an exponent. Taking two terms of the summation, one has

$$w_{ap}(r)=A_0(\alpha_0 x^{\gamma}+\beta_0 x^2+1)+A_1(\alpha_1 x^{\gamma+1}+\beta_1 x^3+x) \tag{7}$$

Values of α_{j-1} and β_{j-1} are determined from the governing outer boundary conditions, applying them to either term of the summation.

For a simply supported outer edge, $r=b$ or $x=1$, the governing boundary conditions are:

$$w(1)=0 \tag{8a}$$

$$w''(1)+\mu/x\ w'(1)=0 \tag{8b}$$

where w' and w'' indicate first and second order derivatives respectively. Substituting Eqs. (8) into each term of Eq. (7) results in

$$\alpha_{0\ \text{sup simple}}=\frac{2(1+\mu)}{\gamma(\gamma-1+\mu)-2(1+\mu)} \qquad \beta_{0\ \text{sup simple}}=-\frac{\gamma(\gamma-1+\mu)}{\gamma(\gamma-1+\mu)-2(1+\mu)} \tag{9}$$

$$\alpha_{1\ \text{sup simple}}=\frac{2(3+\mu)}{\gamma(\gamma+\mu+1)-2(3+\mu)} \qquad \beta_{1\ \text{sup simple}}=-\frac{\gamma(\gamma+\mu+1)}{\gamma(\gamma+\mu+1)-2(3+\mu)} \tag{10}$$

The exponential parameter γ allows for minimization of the calculated eigenvalue.

On the other hand, the radial stress resultant is given by Dyka and Carney III (1979).

$$\sigma_{rr}=\frac{AE_{\theta}(x_1+\mu_{\theta})}{K^2-\mu_{\theta}^2}r^{(x_1-1)}+\frac{BE_{\theta}(x_2+\mu_{\theta})}{K^2-\mu_{\theta}^2}r^{(x_2-1)} \tag{11}$$

$$x_1=\frac{-p+\sqrt{p^2+4(K^2-p\mu_{\theta})}}{2}, \quad x_2=\frac{-p-\sqrt{p^2+4(K^2-p\mu_{\theta})}}{2} \tag{12}$$

p = exponent for the law of the variation of thickness
E_r, E_q = Young's moduli
μ_r, μ_{θ} = Poisson's ratios for the orthotropic polar plate
$K^2 = \mu_{\theta}/\mu_r = E_{\theta}/E_r$

In our case, for isotropic plates, the expressions (11) and (12) will be

$$\sigma_r=\frac{AE(x_1+\mu)}{1-\mu^2}r^{x_1-1}+\frac{BE(x_2+\mu)}{1-\mu^2}r^{x_2-1} \tag{13}$$

$$x_1=\frac{-p+\sqrt{p^2+4(1-p\mu)}}{2}, \quad x_2=\frac{-p-\sqrt{p^2+4(1-p\mu)}}{2} \tag{14}$$

A and B are constants to be determined from the natural boundary conditions

$$\sigma_r\big|_{r=ri}=-p_i \qquad \sigma_r\big|_{r=re}=-p_e \tag{15}$$

p_i = inner pressure
p_e = outer pressure

Thus

$$\sigma_r=\frac{\left(p_i-p_e\, k^{(x_2-1)}\right)}{\left(k^{(x_2-x_1)}-1\right)}\frac{r^{(x_1-1)}}{b^{(x_1-1)}}+\frac{\left(p_e\, k^{(x_1-1)}-p_i\right)}{\left(1-k^{(x_1-x_2)}\right)}\frac{r^{(x_2-1)}}{b^{(x_2-1)}} \tag{16}$$

The compressive radial force is

$$N_r=\sigma_r h \tag{17}$$

and expressed as a function of x

$$Nr=\frac{\lambda\, De}{b^2}\left(C_1 x^{p+x_1-1}+C_2 x^{p+x_2-1}\right) \tag{18}$$

where

$$C_1=\frac{k\left(k^{x_2-1}-t\right)}{\left(k^{x_2}-k^{x_1}\right)} \qquad C_2=\frac{k\left(t-k^{x_1-1}\right)}{\left(k^{x_2}-k^{x_1}\right)} \tag{19}$$

Whereas

$$D=\frac{Eh^3}{12(1-\mu^2)} \quad \Rightarrow D=\frac{EH^3 r^{3p}}{12(1-\mu^2)} \quad D=\frac{EH^3 x^{3p} b^{3p}}{12(1-\mu^2)} \tag{20}$$

$$De=\frac{Ehe^3}{12(1-\mu^2)}=\frac{EH^3 b^{3p}}{12(1-\mu^2)} \quad D=De\ x^{3p} \tag{21}$$

Substituting Eq. (18) into Eq. (5) and operating, one has

$$U=\pi\frac{De}{b^2}\left\{\int_k^1\left[\left(\frac{d^2w}{dx^2}\right)^2 xx^{3p}+\frac{1}{x}\left(\frac{dw}{dx}\right)^2+2\mu\frac{d^2w}{dx^2}\frac{dw}{dx}x^{3p}\right]dx\right.$$
$$\left.-\lambda\int_k^1\left(C_1x^{p+x_1-1}+C_2x^{p+x_2-1}\right)\left(\frac{dw}{dx}\right)^2 x\ dx\right\} \tag{22}$$

Requiring that the functional be a minimum with respect to the A_{j-1} one obtains

$$\frac{\partial U}{\partial A_1}\frac{b^2}{\pi De}=0=\left\{\int_k^1\frac{\partial}{\partial A_1}\left[\left(\frac{d^2w}{dx^2}\right)^2 xx^{3p}+\frac{1}{x}x^{3p}\left(\frac{dw}{dx}\right)^2+2\mu\frac{d^2w}{dx^2}\frac{dw}{dx}x^{3p}\right]dx\right.$$
$$\left.-\lambda\int_k^1\frac{\partial}{\partial A_1}\left(C_1x^{p+x_1-1}+C_2x^{p+x_2-1}\right)\left(\frac{dw}{dx}\right)^2 x\ dx\right\} \tag{23}$$

$$\frac{\partial U}{\partial A_2}\frac{b^2}{\pi De}=0=\left\{\int_k^1 \frac{\partial}{\partial A_2}\left[\left(\frac{d^2w}{dx^2}\right)^2 x x^{3p}+\frac{1}{x}x^{3p}\left(\frac{dw}{dx}\right)^2+2\mu\frac{d^2w}{dx^2}\frac{dw}{dx}x^{3p}\right]dx\right.$$
$$\left.-\lambda\int_k^1 \frac{\partial}{\partial A_2}\left(C_1x^{p+x_1-1}+C_2x^{p+x_2-1}\right)\left(\frac{dw}{dx}\right)^2 x\,dx\right\} \qquad (24)$$

The non-triviality condition leads to a transcendental equation. The lowest root is the desired critical buckling parameter λ.

3. Analyzed cases and results obtained

Timoshenko (1961) analyzed the buckling problem in circular and annular plates with uniform thickness. He obtained, for example, $(N_r)_{cr} = (4,\ 2D)/b^2$, for the circular plate with the simply supported edge, where 4,2 represents the critical buckling prrameter λ. The results obtained in this work for annular plates with simply supported outer edge appear in Table 1.

Other studies (Laura *et al.* 1995) give the critical buckling parameter for annular plates with a discontinuous variable thickness. These values are indicative for the case of annular plates with variable thickness according to the exponential law we have considered above.

For this study we supposed the same height at the outer boundary and for a radius $r = (a+c)/2$ as Laura (1995) considered and shows Fig. 2. Taking as a starting point these considerations, we have determined the following equation for the exponent p that appears in the exponential law.

Table 1 Values of buckling parameter λ for the annular plates with simply supported outer edge under in-plane forces and uniform thickness

μ=1/3	$k = a/b$							
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7
λ	4.216	4.014	3.671	3.232	2.778	2.537	2.22	2.20

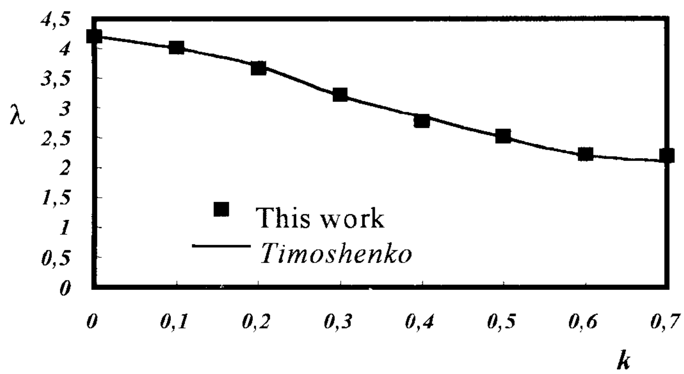

Fig. 1 Values of buckling paramer λ. Plate with uniform thickness and simply supported outer edge

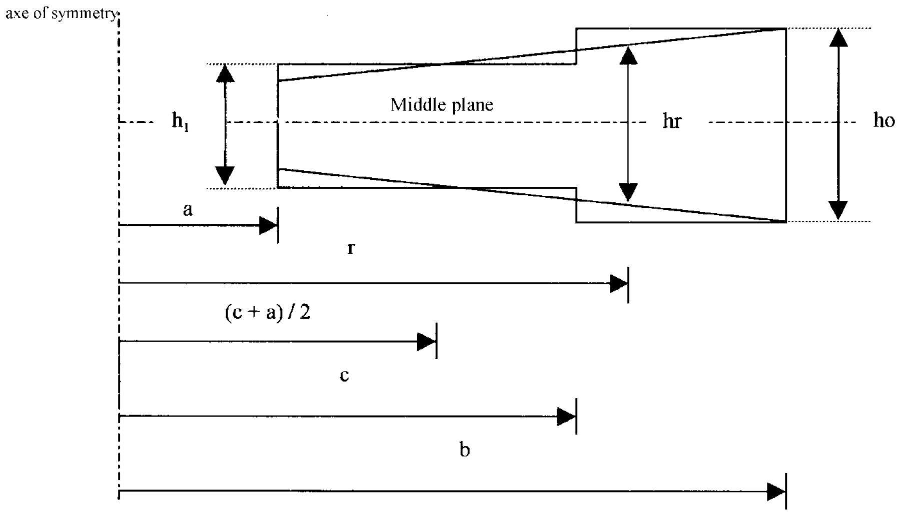

Fig. 2 Variation of thickness as a function of h_1, h_0, a, b, c

Supposing $r=\dfrac{a+c}{2}$ then the thickness is $h_r=h_1$ while for $r=b \Rightarrow h_r=h_0$ one has

$$h_1=H\left(\frac{a+c}{2}\right)^p h_0=Hb^p \Rightarrow \frac{h_1}{h_0}=\left(\frac{a+c}{2b}\right)^p$$

$$\log\frac{h_1}{h_0}=p\log\left(\frac{a+c}{2b}\right) \quad p=\frac{\log\dfrac{h_1}{h_0}}{\log\left(\dfrac{a+c}{2b}\right)} \quad \text{or} \quad p=\frac{\log\dfrac{h_1}{h_0}}{\log\left(\dfrac{1}{2}\left(\dfrac{a}{b}+\dfrac{c}{b}\right)\right)} \tag{25}$$

Table 2 shows the values for p. All calculations have been determined for relations h_1/h_0=0, 8 a/b =0, 1, ..., 0, 7 *y* c/b=0, 2, ..., 0, 8. Table 3 shows results for the buckling parameter λ available in Laura, *et al.* (1995) and those calculated according to the exponential values given in Table 2.

Different values were assigned for the exponent p to be able to consider the influence of the thickness variation on the parameter λ. Table 4 shows the values of λ for different magnitudes of k

Table 2 Values for exponent p as a function of c/b and $k=a/b$

$k=a/b$	Values of c/b						
	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	0.1176	0.1386	0.1609	0.1853	0.2125	0.2435	0.2794
0.2		0.1609	0.1853	0.2125	0.2435	0.2794	0.3219
0.3			0.2125	0.2435	0.2794	0.3219	0.3732
0.4				0.2794	0.3219	0.3732	0.4368
0.5					0.3732	0.4368	0.5179
0.6						0.5179	0.6256
0.7							0.7756

Table 3 Values of parameter for annular plate with simply supported outer edge

k=*a*/*b*		Values of *c*/*b*						
		0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.1	(1)	3.02646	2.880401	2.734349	2.58486	2.429609	2.266787	2.095034
	(2)	3.806	3.485	3.224	2.964	2.706	2.505	2.351
0.2	(1)		2.605524	2.479356	2.347538	2.208295	2.060024	1.901283
	(2)		3.220	2.946	2.696	2.465	2.284	2.136
0.3	(1)			2.220959	2.109447	1.989157	1.85838	1.71533
	(2)			2.786	2.493	2.258	2.064	1.896
0.4	(1)				1.879879	1.778762	1.666018	1.539298
	(2)				2.438	2.131	1.927	1.737
0.5	(1)					1.58946	1.492975	1.380846
	(2)					2.171	1.888	1.656
0.6	(1)						1.34345	1.243316
	(2)						1.951	1.651
0.7	(1)							1.11922
	(2)							1.745

(1) Values obtained applying the optimized Rayleigh-Ritz method for the case of variable thickness with continuity
(2) Values obtained by Laura *et al.* (1995), for the case of variable thickness with discontinuity

Table 4 Values of parameter for different laws of variation thickness

P	*k* = *a*/*b*				
	0.1	0.2	0.3	0.4	0.5
0	4.014	3.654	3.158	2.766	2.537
0.5	1.338	1.348	1.418	1.422	1.499
1	0.658	0.703	0.765	0.835	0.901
1.5	0.422	0.447	0.491	0.552	0.625
2	0.314	0.328	0.356	0.400	0.461
2.5	0.253	0.261	0.279	0.311	0.358
3	0.214	0.219	0.230	0.252	0.289

and different values of *p*.

4. Conclusions

In this paper an optimized Rayleigh-Ritz method is used to obtain the buckling parameter λ. This method offers simplicity in its application, and is a powerful tool for determining the buckling load for a number of complex structures.

This work is an attempt to fill an apparent void with respect to the buckling load for variable thickness plates. This is achieved by using the exact expression for the in-plane resultant stress in the governing functional including an exponential law for the thickness profile of annular plates as described above.

When the thickness of the inner edge increases towards the outer boundary, the parameter λ raises. This variation in thickness produces lower values for buckling loads. This difference diminishes when relation k increases.

References

Avalos, D.R., and Laura, P.A.A. (1979), "A note on transverse vibrations of annular plates elastically restrained against rotation along the edges", *J. of Sound and Vib.*, **66**(1), 63-67.

Dyka, C.T., and Carney III, J.F. (1979), "Vibrations of annular plates of variable thickness", *J. of the Mech. Eng.* ASCE, **105**(EM3), 361-370.

Lamé, G. (1852), "Leçons sur la théorie mathématique de l'élasticité des corpes solides", París.

Laura, P.A.A., Paloto, J.C., and Santos, R.D. (1975), "A note on the vibration and stability of a circular plate elastically restrained against rotation", *J. of Sound and Vib.*, **41**, 177-180.

Laura, P.A.A., Ficcadenti, G.M., and Alvarez, S.I. (1988), "Effect of geometric boundary disturbances on the natural frequencies and buckling loads of vibrating clamped circular plates", *J. of Sound and Vib.*, **126**, 67-72.

Laura, P.A.A., Ercoli, L., and Gutierrez, R. (1995), "Optimized Rayleigh-Ritz method", Monograph, Inst. of Appl. Mech., 34-95.

Laura, P.A.A., Gutierrez, R.H., Sonzogni, V., and Idelsohn, S. (1996), "Buckling of circular, annular plates of non-uniform thickness", Inst. of Appl. Mech., **96**(6).

Loh, H.C., and Carney III, J.F. (1976)," Vibration and stability of spinning annular plates reinforced with edge beams", Developments in Theoretical and Applied Mechanics, *Proc. of the 8th Southeastern Conf. on Theoretical and Appl. Mech.*, **8**, 365-377.

Ramaiah, G.K. (1975), "Some investigations on vibrations and buckling of polar orthotropic annular plates", Ph.D. Thesis, Indian Inst. of Science at Bangalore, India, May.

Ramaiah, G.K., and Vijayakumar, K. (1975), "Vibration of annular plates with linear thickness profiles", *J. of Sound and Vib.*, **40**(2), 293-298.

Soni, S.R., and Amba-Rao, C.L. (1975), "Axisymmetric vibrations of annular plates of variable thickness", *J. of Sound and Vib.*, **38**(4), 465-473.

Stuart, R.J., and Carney III, J.F. (1974), "Vibration of edge reinforced annular plates", *J. of Sound and Vib.*, **35**(1), 23-33.

Timoshenko, S. (1961), "Teoria de la establidad elástica", Editorial Ediar.