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A direct modification method for strains due to non-conforming modes

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Abstract. This paper addresses an efficient modification method that eliminates the undesirable effects of strains due to various non-conforming modes so that the non-conforming element can pass the patch test unconditionally. The scheme is incorporated in the element formulation to establish new types of non-conforming hexahedral elements designated as NH*x* and NVH*x* for the regular element and variable node element, respectively. Non-conforming displacement modes are selectively added to the ordinary (conforming) element displacement assumptions to improve the bending behavior of the distorted solid element. To verify the validation of proposed direct modification method and the improvement of element behavior, several numerical tests are carried out. Test results show that the proposed method is effective and its applications to non-conforming solid elements guarantee for the element to pass the patch test.

Key words: direct modification method; non-conforming modes; effective modification method; solid elements.

1. Introduction

The accuracy attainable by the standard and conventional isoparametric eight-node solid element tends to be rather low in case of bending as the element deforms in a shear mode. If a single element is used in the direction of the thickness, the accuracy degrades significantly and may lead to unreliable results. This poor behavior of basic isoparametric element in bending is explained as the parasitic shear deformation. Because the shape functions of basic 8-node solid element are linear, they cannot represent true bending mode shape which is a higher order form (See Fig. 1).

A number of different techniques have been proposed for improvement of the basic behavior of this type of elements. Among these techniques, the use of non-conforming modes may be one of the most successful approaches in bending situations. The basic concept of this approach is to restore true deformation by adding additional deformation modes which is called non-conforming modes (Choi and Lee 1993, Hughes 1987).

The use of non-conforming modes improves the element behavior significantly, but at the same time it may create another problem that the resulting elements do not always pass the patch test. To circumvent this defect several researchers have studied various possible modification schemes for non-conforming modes (Taylor, *et al*. 1976, Wilson and Ibrahimbegovic 1990). Under the constant

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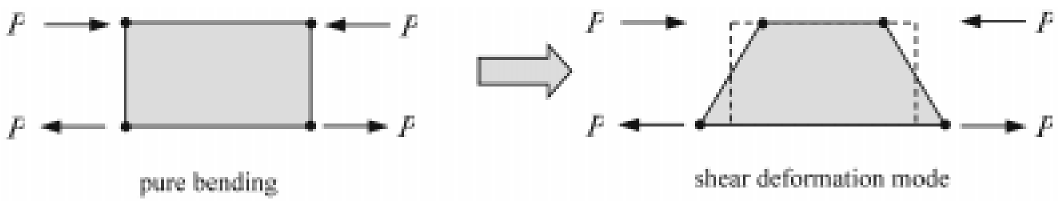

Fig. 1 Parasitic shear in a basic isoparametric element

stress state, the strain energy associated with the non-conforming modes must vanish in an element domain. To satisfy this requirement Taylor, *et al.* (1976) proposed a remedy that in computing strains due to non-conforming modes the Jacobian matrix is replaced with the constant Jacobian matrix computed at the center of an element. The non-conforming elements modified in this manner pass the patch test and show improved accuracy. However, it has been found that this method is not always applicable to general non-conforming modes. When the non-conforming mode is an odd function, the scheme can not be used.

Wilson and Ibrahimbegovic (1990) proposed a different scheme in which the requirement for passing the patch test can be satisfied by adding a constant correction matrix to the strain displacement matrix. This remedy guarantees that the element always pass the patch test for various types of non-conforming modes, but computing time for the correction matrix was relatively large.

In this paper a new scheme is proposed in which the merits of aforementioned two schemes are inherited. The new proposed method corrects the strains due to non-conforming modes in a direct way and is applicable to the cases of more general non-conforming modes. Thus, the present scheme not only satisfies the requirement for an element with general non-conforming modes to pass the patch test, but also requires less computational time. To show the validity and effectiveness of the proposed method, the various solid elements with non-conforming displacement modes are tested and the results are evaluated.

2. Non-conforming displacement modes

The approach to improve the basic behavior of 2-D isoparametric element by eliminating the excessive shear strains through the addition of non-conforming displacement modes was first adopted by Wilson, *et al*. (1971) and expanded to use in the shell elements by Choi and Schnobrich (1975). The non-conforming modes are of the same form as the error distribution or what are missing in the original displacement approximation, and therefore the actual displacement field can be better approximated by the addition of these non-conforming modes.

The interpolation matrices for both conforming and non-conforming displacement are expressed by the shape functions N_i and non-conforming modes $\overline{N}_j$, respectively.

$$\boldsymbol{N}_i = \begin{bmatrix} N_i & & \\ & N_i & \\ & & N_i \end{bmatrix} \tag{1}$$

$$\bar{\boldsymbol{N}}_j = \begin{bmatrix} \bar{N}_j & & \\ & \bar{N}_j & \\ & & \bar{N}_j \end{bmatrix} \tag{2}$$

where

$$N_i = \frac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i) \tag{3}$$

$$\bar{N}_j = 1-\xi^2, \quad 1-\eta^2, \quad 1-\zeta^2 \quad \text{etc.} \tag{4}$$

and ξ, η and ζ are the natural coordinates in the range of (–1, 1).

The total displacement field of the element with additional non-conforming displacement modes can be expressed as

$$\boldsymbol{u}^h = \begin{Bmatrix} u^h \\ v^h \\ w^h \end{Bmatrix} = \sum \boldsymbol{N}_i \boldsymbol{u}_i + \sum \bar{\boldsymbol{N}}_j \bar{\boldsymbol{u}}_j \tag{5}$$

in which $\bar{\boldsymbol{N}}_j$ are the additional non-conforming modes and $\bar{\boldsymbol{u}}_j$ are the additional unknowns corresponding to the additional displacement modes. These additional unknowns are not the physical nodal displacements but can be taken simply as amplitudes of the respective non-conforming modes.

Then, the element stiffness matrix can be obtained by direct application of variational principles. The resulting stiffness matrix has been enlarged over the original isoparametric element matrix due to the additional modes and the corresponding unknowns and partitioned as

$$\begin{bmatrix} \boldsymbol{K}_{CC} & \boldsymbol{K}_{CN} \\ \boldsymbol{K}_{CN}^T & \boldsymbol{K}_{NN} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u} \\ \bar{\boldsymbol{u}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{Bmatrix} \tag{6}$$

where

$$\boldsymbol{K}_{CC} = \int_V \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B}\, dV, \qquad \boldsymbol{K}_{CN} = \int_V \boldsymbol{B}^T \boldsymbol{D} \bar{\boldsymbol{B}}\, dV, \qquad \boldsymbol{K}_{NN} = \int_V \bar{\boldsymbol{B}}^T \boldsymbol{D} \bar{\boldsymbol{B}}\, dV \tag{6a}$$

and subscript C denotes conforming whereas N denotes non-conforming part. The null vector in the lower part of the load vector in Eq. (6) indicates that no nodal loads can be applied in association with the non-conforming modes. The enlarged element stiffness matrix in Eq. (6) can be condensed back to the same size as the stiffness matrix of the ordinary conforming elements $\boldsymbol{K}_{CC}$.

$$(\boldsymbol{K}_{CC} - \boldsymbol{K}_{CN} \boldsymbol{K}_{NN}^{-1} \boldsymbol{K}_{CN}^T)\boldsymbol{u} = \boldsymbol{f} \tag{7}$$

The elements formulated in this manner and designated as non-conforming elements show much improved behavior over original conforming element (Choi, *et al.* 1998). However, this type of elements do not always pass the path test.

3. A new direct modification method for strains due to non-conforming modes

3.1 Review of modification schemes

One of the most significant contributions to the finite element method is the introduction of the 'patch test' by Bruce Irons (1972). In order that the finite element method would have an appropriate mathematical basis, the requirement for passing patch test was introduced as a relaxed restriction on displacement compatibility. It is generally acknowledged that an element which passes the patch test is convergent. Unfortunately, most original non-conforming elements fail to pass patch test. This is resulted from the fact that the constant distribution of strains in an element domain is disturbed by the addition of strains due to the non-conforming displacement modes. To obtain the state of constant stress $\boldsymbol{\sigma}_c$ for an element to pass patch test, the strain energy associated with the non-conforming modes should vanish in an element domain V as shown in Eq. (9) (Taylor, *et al.* 1976, 1986, Wilson and Ibrahimbegovic 1990).

$$\delta\bar{\boldsymbol{u}}^T\int_V \bar{\boldsymbol{B}}^T\boldsymbol{\sigma}_c dV=0 \tag{8}$$

or

$$\int_V \bar{\boldsymbol{B}}^T dV=0 \tag{9}$$

where, $\bar{\boldsymbol{B}}$ is the strain-displacement matrix of an element due to non-conforming modes. Most of basic non-conforming elements do not satisfy Eq. (9), and consequently fail to pass the patch test. To solve the above problem for non-conforming modes $1-\xi^2$ and $1-\eta^2$, Taylor, *et al.* (1976) proposed a remedy that replaces Jacobian matrix in computing Eq. (9) with the constant Jacobian matrix computed at a zero point in the natural coordinate system.

In an isoparametric solid element, this requirement on strain-displacement matrix $\bar{\boldsymbol{B}}$ due to non-conforming modes means that the integration should be zero for the derivative of each non-conforming mode $\bar{N}_j$ with respect to global coordinate system x_k (Fig. 2). For eight-node solid element this condition reduces to

$$\int_V \frac{\partial\bar{N}_j}{\partial x_k}dV=\int_V\left(\sum_{\alpha=1}^{3}\frac{\partial\bar{N}_j}{\partial\xi_\alpha}\frac{\partial\xi_\alpha}{\partial x_k}\right)dV=0 \qquad (k=1,\ 2,\ 3) \tag{10}$$

or

$$\iiint\sum_{\alpha=1}^{3}\frac{\partial N_j}{\partial\xi_\alpha}J^{-1}_{k\alpha}(\xi,\eta,\zeta)|J(\xi,\eta,\zeta)|d\xi d\eta d\zeta=0 \tag{10a}$$

where

$$dV=|J(\xi,\eta,\zeta)|d\xi d\eta d\zeta \tag{10b}$$

$$\frac{\partial\xi_1}{\partial x_k}\equiv\frac{\partial\xi}{\partial x_k}=J^{-1}_{k1}(\xi,\eta,\zeta) \tag{10c}$$

$$\frac{\partial\xi_2}{\partial x_k}\equiv\frac{\partial\eta}{\partial x_k}=J^{-1}_{k2}(\xi,\eta,\zeta) \tag{10d}$$

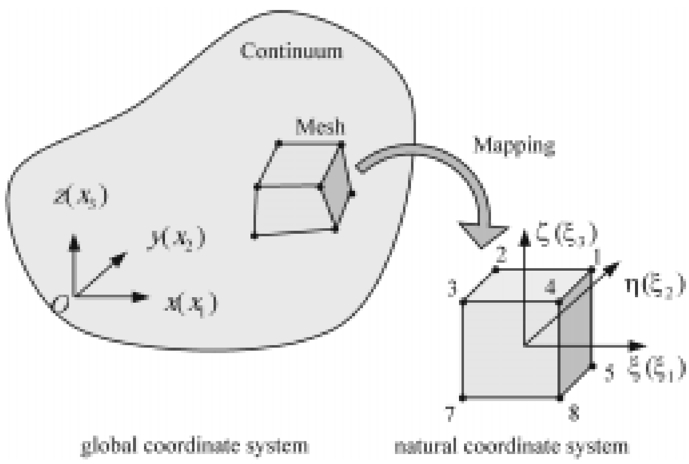

Fig. 2 Configuration of 8-node solid element

$$\frac{\partial \xi_3}{\partial x_k} \equiv \frac{\partial \zeta}{\partial x_k} = J_{k3}^{-1}(\xi, \eta, \zeta) \tag{10d}$$

If the components of inverse Jacobian matrix $\boldsymbol{J}_{k\alpha}^{-1}(\xi,\eta,\zeta)$ in Eq. (10) are replaced by approximated values, e.g., constant values computed at the center of an element $(|\boldsymbol{J}(0,0,0)|/|\boldsymbol{J}(\xi,\eta,\zeta)|)$ $\boldsymbol{J}_{k\alpha}^{-1}(0,0,0)$, the requirement for the patch test to be satisfied can be rewritten as

$$|\boldsymbol{J}(0,0,0)|\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left(\sum_{\alpha=1}^{3}\frac{\partial \overline{N}_j}{\partial \xi_\alpha}\boldsymbol{J}_{k\alpha}^{-1}(0,0,0)\right)d\xi d\eta d\zeta = 0 \tag{11}$$

The above equation will be always satisfied if each integration term becomes zero. For the case of typical non-conforming modes expressed by even functions, such as $(1-\xi^2)$, $(1-\eta^2)$, $(1-\zeta^2)$ and $(1-\xi^2)(1-\eta^2)(1-\zeta^2)$, the modification scheme of Taylor *et al.* given by Eq. (11) is valid. However, for general non-conforming modes it is not always applicable in its original form. The scheme may not work in such cases as non-conforming modes which include the odd function terms $(1-\xi^2)\eta$, the hierarchical shape function to introduce nodal drilling degrees of freedom $1/2(1-\xi^2)(1+\eta)$, and the modification of non-conforming modes due to existence of a variable node $(1-\xi^2)-1/2(1-|\xi|)(1+\eta)$, and so forth.

Wilson and Ibrahimbegovic (1990) proposed a different remedy. The main idea is that the requirement of Eq. (9) can be satisfied by adding a constant correction matrix $\overline{\boldsymbol{B}}_c$ to non-conforming strain-displacement matrix $\overline{\boldsymbol{B}}$.

$$\int_V (\overline{\boldsymbol{B}}+\overline{\boldsymbol{B}}_c)dV = 0 \tag{12}$$

where

$$\overline{\boldsymbol{B}}_c = -\frac{1}{V}\int_V \overline{\boldsymbol{B}}dV \tag{13}$$

The constant correction matrix $\overline{\boldsymbol{B}}_c$ is evaluated numerically before the element stiffness is calculated. In calculating $\overline{\boldsymbol{B}}_c$ the same integration formula used in calculating element stiffness must be used, i.e., the correction matrix is applied at each integration point. This remedy guarantees that the element always pass the patch test for various types of non-conforming modes, but computing costs for the correction matrix is relatively high.

3.2 A new direct modification scheme

As Eq. (11) will be always satisfied if each integration term becomes zero, the requirement for an element to pass the patch test to be satisfied can be simplified as

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial\overline{N}_j}{\partial\xi_\alpha}d\xi d\eta d\zeta=0 \qquad \alpha=1,\,2,\,3 \tag{14}$$

In general, Eq. (14) is not always satisfied for various non-conforming modes. In this paper, the concept of adding a correction constant $c_{j\alpha}$ instead of the correction matrix $\overline{\boldsymbol{B}}_c$ in Eq. (12) to the derivatives of non-conforming mode $\overline{N}_j$ with respect to natural coordinate ξ_α is proposed.

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial\overline{N}_j}{\partial\xi_\alpha}+c_{j\alpha}\right)d\xi d\eta d\zeta=0 \tag{15}$$

From the fact that $c_{j\alpha}$ in Eq. (15) is a constant, the correction constants can be calculated analytically for derivatives of each non-conforming modes.

$$c_{j\alpha}=-\frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial\overline{N}_j}{\partial\xi_\alpha}d\xi d\eta d\zeta \tag{16}$$

Finally, using the correction constants obtained analytically, derivatives of non-conforming modes with respect to the global coordinate system x_k can also be obtained in a simple manner.

$$\frac{\partial\overline{N}_j}{\partial x_k}\cong\frac{|J(0,0,0)|}{|J(\xi,\eta,\zeta)|}\sum_{\alpha=1}^{3}\left\{\boldsymbol{J}_{k\alpha}^{-1}(0,0,0)\left(\frac{\partial\overline{N}_j}{\partial\xi_\alpha}+c_{j\alpha}\right)\right\} \tag{17}$$

When the value $c_{j\alpha}$ in Eq. (17) equals to zero, the present scheme is identical to that of Taylor *et al.* It should be noted that unlike the $\overline{\boldsymbol{B}}_c$ in Eq. (12), the present scheme does not require the numerical volume integration for calculation of correction constants in (Eq. 16). Instead, by the analytical calculation of correction constants, the direct correction for the derivatives of various types of non-conforming modes is possible. Thus, this method can substantially reduce the computation efforts when compared with Wilson's Scheme.

By the combination of the basic non-conforming modes ($\overline{N}_1\sim\overline{N}_4$) and additional ones ($\overline{N}_5\sim\overline{N}_{10}$), it is expected that more general deformation configuration can be described (Fig. 3) for non-conforming elements. The correction constants for the aforementioned general non-conforming modes are summarized in Table 1. The correction constants make the integration of strain modes due to non-conforming displacements in an element domain zero (Fig. 4), *i.e.*, those values are cancelled out in an average sense. Thus, even under the condition that the strains due to non-conforming modes have non-zero values, the requirement for the element to pass the patch test (*i.e.*, Eq. 9) can be satisfied.

3.3 Direct modification scheme for Variable-node non-conforming elements

When several non-conforming modes are used in a variable-node hexahedral element, these non-conforming modes $\bar{N}_j$ need to be modified due to variable nodes. The modified non-conforming modes $\bar{N}_j^*$ for variable-node element will have the following form (Choi *et al.* 1993, 1996, 1998).

$$\bar{N}_j^* = \bar{N}_j - \sum_i b_i N_i \tag{18}$$

where N_i is the conforming shape function of variable node i, b_i is a constant coefficient that is the same value as non-conforming mode $\bar{N}_j$ at node i, and $\bar{N}_j$ is the related original non-conforming modes. This modification enables the modified non-conforming modes to have zero value at each node.

The new direct modification scheme can also be used for the variable-node elements. Without imposing Choi and Lee's (1993) additional constraints, the modification based on the present direct

Table 1 Correction constants for various non-conforming modes

Non-conforming modes $\bar{N}_j$		Correction constants			Remarks
		c_ξ	$c_{j\eta}$	$c_{j\zeta}$	
$\bar{N}_1$	$1-\xi^2$	0	0	0	Basic modes
$\bar{N}_2$	$1-\eta^2$	0	0	0	
$\bar{N}_3$	$1-\zeta^2$	0	0	0	
$\bar{N}_4$	$(1-\xi^2)(1-\eta^2)(1-\zeta^2)$	0	0	0	
$\bar{N}_5$	$(1-\xi^2)\eta$	0	$-\frac{2}{3}$	0	Additional modes
$\bar{N}_6$	$(1-\xi^2)\zeta$	0	0	$-\frac{2}{3}$	
$\bar{N}_7$	$(1-\eta^2)\xi$	$-\frac{2}{3}$	0	0	
$\bar{N}_8$	$(1-\eta^2)\zeta$	0	0	$-\frac{2}{3}$	
$\bar{N}_9$	$(1-\zeta^2)\xi$	$-\frac{2}{3}$	0	0	
$\bar{N}_{10}$	$(1-\zeta^2)\eta$	0	$-\frac{2}{3}$	0	

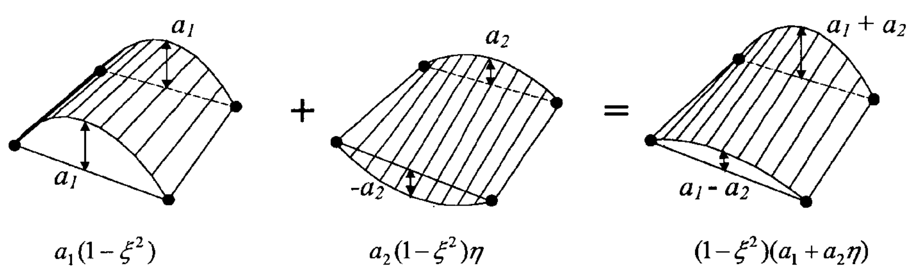

Fig. 3 Combined effect of the basic and additional non-conforming modes

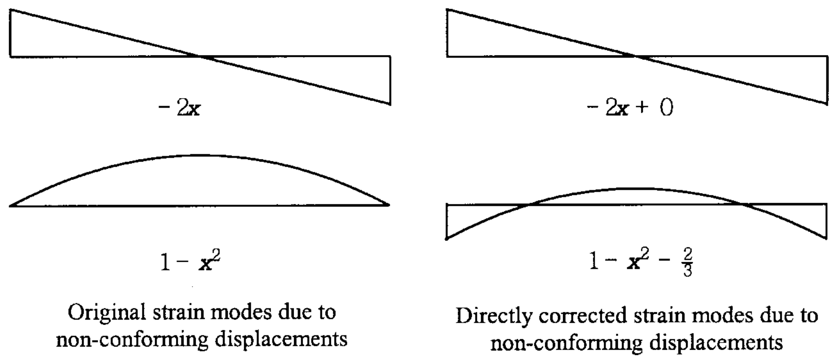

Fig. 4 Basic concept of direct correction scheme

modification scheme can be applied to the elements of different layer patterns, *i.e.*,

$$\frac{\partial \bar{N}_j^*}{\partial x_k} \cong \frac{|J(0,0,0)|}{|J(\xi,\eta,\zeta)|}\sum_{\alpha=1}^{3}\left\{\boldsymbol{J}_{k\alpha}^{-1}(0,0,0)\left(\frac{\partial \bar{N}_j^*}{\partial \xi_\alpha}\right)\right\} \tag{19a}$$

where

$$\frac{\partial \bar{N}_j^*}{\partial \xi_\alpha}=\left(\frac{\partial \bar{N}_j}{\partial \xi_\alpha}-\frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial \bar{N}_j}{\partial \xi_\alpha}d\xi d\eta d\zeta\right)-\sum_i b_i\left(\frac{\partial N_i}{\partial \xi_\alpha}-\frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial N_i}{\partial \xi_\alpha}d\xi d\eta d\zeta\right) \tag{19b}$$

and the integration in Eq. (19b) are obtained analytically and used directly in calculation of the derivatives.

4. Numerical test

Some numerical tests are performed to check the validity and applicability of the proposed direct correction scheme. Developed elements are classified into two types by the non-conforming modes used as listed in Table 1 and tentatively designated as Type I and II. Type I has all the nonconforming modes listed in Table 1, while Type II has basic non-conforming modes only ($\bar{N}_1$~$\bar{N}_4$). In the calculation of present element stiffness matrix, the 14-point integration rule (Iron and Ahmad 1980) is adopted.

Since the shape functions of the present variable node solid element are based on those for the connection of different layer patterns (NC-V2, Choi and Lee 1993), it is impossible to directly calculate the Jacobian values at the center point of the element due to slope discontinuities. To circumvent this problem the Jacobian values at a center point $J(0,0,0)$ are calculated based on eight corner nodes excluding mid-edge node for the variable node hexahedral elements.

4.1 Eigenvalue test

To identify the possible spurious zero energy mechanisms, the eigenvalue analysis of element matrix was carried out for the 8-node solid element and the typical variable node solid element

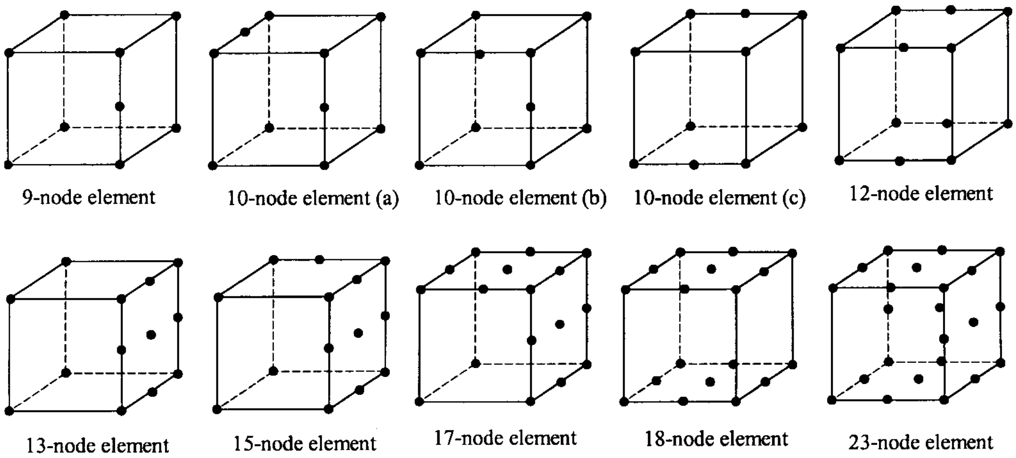

Fig. 5 Examples for typical variable node hexahedral elements

(Fig. 5). For a single element without any boundary conditions, there must be only six zero eigenvalues associated with rigid-body modes. Test results show that no spurious mechanisms were expected to develop in any of the elements presented in this numerical test.

4.2 Patch Test

In order to check whether the proposed 8-node solid elements are capable of representing constant strain states a series of patch tests were carried out. The typical test model (MacNeal and Harder 1985) is shown in Fig. 6, which contains seven distorted elements. Also additional patch tests were carried out to check the validation of present direct correction method for variable node hexahedral elements. The non-conforming modes $\bar{N}_1 \sim \bar{N}_4$ in Table 1 do not need correction since the correction constants calculated by Eq. (16) equal to zero and therefore the correction produces no consequences.

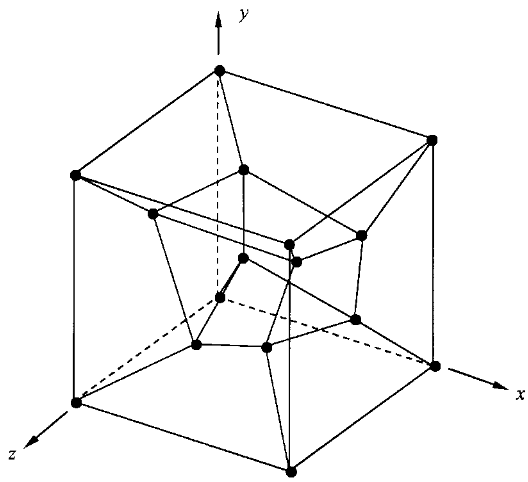

Fig. 6 Patch test model for 3D solid

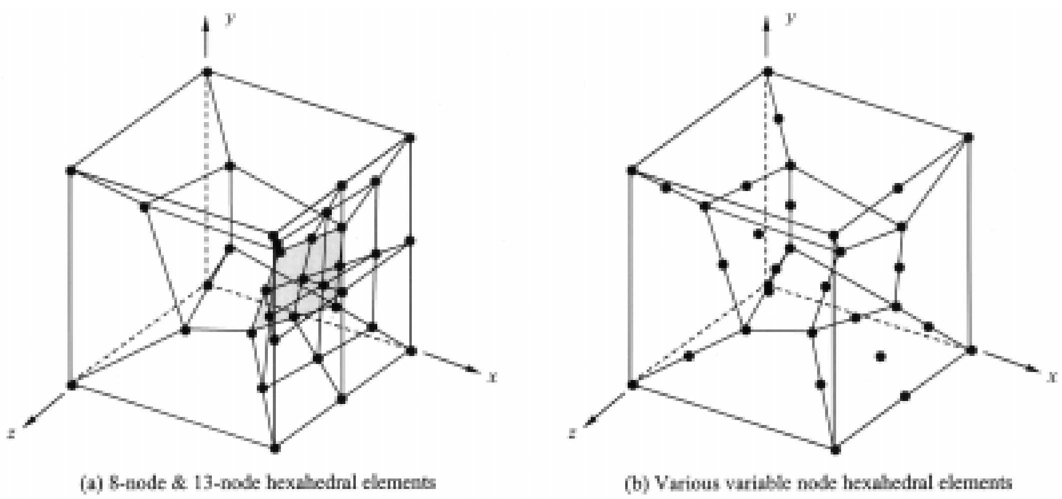

Fig. 7 Patch test models for variable node hexahedral element

Table 2 Boundary conditions and theoretical results

Boundary conditions	Theoretical solution
$u=10^{-3}(2x+y+z)/2$	$\varepsilon_x=\varepsilon_y=\varepsilon_z=\gamma_{xy}=\gamma_{yz}=\gamma_{zx}=10^{-3}$
$v=10^{-3}(x+2y+z)/2$	$\sigma_x=\sigma_y=\sigma_z=2000$
$w=10^{-3}(x+y+2z)/2$	$\tau_{xy}=\tau_{yz}=\tau_{zx}=400$

The remaining non-conforming modes ($\bar{N}_5 \sim \bar{N}_{10}$) in Table 1 are also tested. Patch test models for variable node hexahedral elements are shown in Fig. 7. The problems were solved with the prescribed displacement boundary and the obtained results are identical to the theoretical solutions (See Table 2). In addition, a single element patch (Ibrahimbegovic and Wilson 1991) as shown in Fig. 8 were tested, and all the types of developed elements passed the test. Thus the addition of

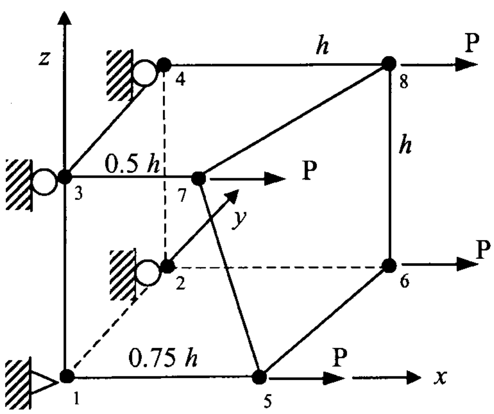

Fig. 8 Single element patch test

non-conforming modes does no longer prevent the non-conforming elements from passing the patch test. The Taylor's scheme is not applicable to these additional non-conforming modes, as the correction constants do not become zero. From these test results, one can say that the new modification scheme for strains due to non-conforming modes are valid and applicable to various types of elements with non-conforming modes.

4.3 Cantilever Beam under Pure Bending

To overview the characteristics of bending behavior, the cantilever beam under pure bending is considered (Fig. 9). The test meshes which are composed of 8-node and 13-node solid elements as

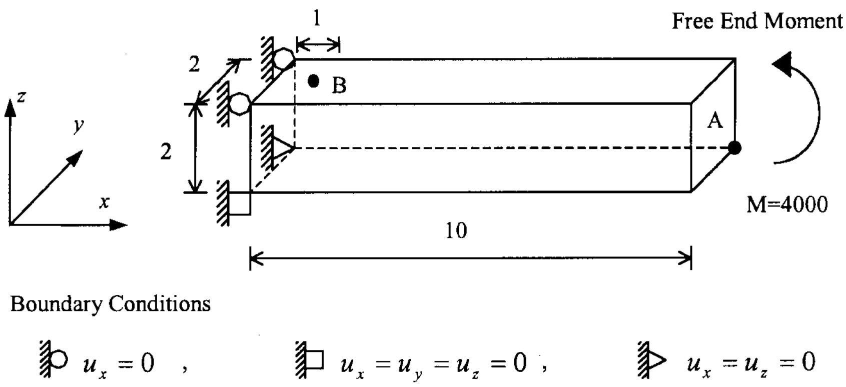

Fig. 9 Cantilever Beam under Pure Bending

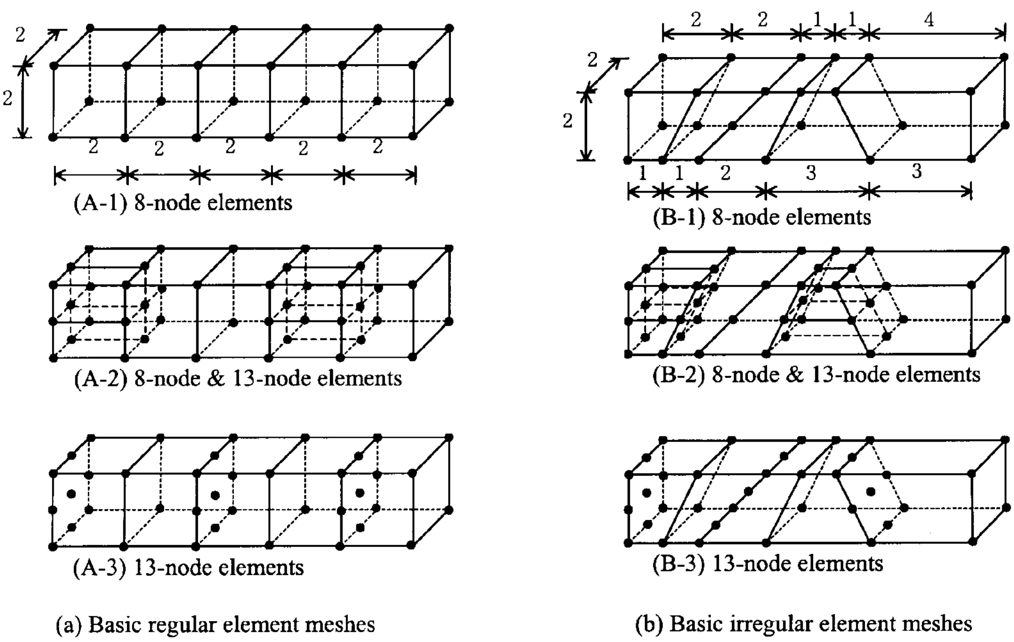

Fig. 10 Several test meshes for pure bending problem

shown in Fig. 10 and the basic material properties are given as E=1500, ν=0.25. For the cantilever beam, the boundary conditions of the fastening end are imposed as shown in Fig. 9.

The vertical displacement at point A and bending stress σ_x at point B under pure bending are shown in Tables 3, 4, and 5 along with the theoretical solutions and the previous studies (Choi, *et al*. 1993, 1996, 2001) for comparison. The element denoted as 'C-V1' does not have any non-conforming modes, and 'NC-V1' and 'NC-V2' have conventional three non-conforming modes ($\bar{N}_1 \sim \bar{N}_3$) and the correction method suggested by Wilson and Ibrahimbegovic (1990) is applied (Choi and Lee 1993). The elements denoted as 'NCH-3', 'NCH-4' (Choi, *et al*. 1996) and MR-H*x* (Choi, *et al.* 2001) which possess rotational degrees of freedom formulated by Allman type functions are also included for comparison.

Test results indicate that all the elements presented in this paper show good performance in bending even with distorted meshes and that the additional non-conforming modes are effective in improving the overall accuracy of the elements.

Table 3 Results of cantilever beam under pure bending for 8-node solid elements

Designation	Regular mesh (mesh A-1)			Distorted mesh (mesh B-1)		
	point A	point A	point B	point A	point A	point B
	Vertical displacement	Rotation	Bending stress	Vertical displacement	Rotation	Bending stress
Type I	100.00	N/A	−3000	94.53	N/A	−2920
Type II	100.00	N/A	−3000	92.61	N/A	−2722
C-V1 (Choi and Lee 1993)	66.67	N/A	−2200	44.38	N/A	−1736
NC-V1 (Choi and Lee 1993)	100.00	N/A	−3000	87.45	N/A	−2262
NC-V2 (Choi and Lee 1993)	100.00	N/A	−3000	87.45	N/A	−2262
NCH-3 (Choi, *et al*. 1996)	100.00	−20.00	−3000	97.33	−18.62	−2270
NCH-4 (Choi, *et al*. 1996)	100.00	−20.00	−3000	91.75	−18.16	−2815
MR-H*x* (Choi, *et al*. 2001)	93.75	−18.75	−3000	81.09	−16.77	−2405
Theory	100.00	−20.00	−3000	100.00	−20.00	−3000

Table 4 Results of cantilever beam under pure bending for 8- and 13-node solid elements

Designation	Regular mesh (mesh A-2)			Distorted mesh (mesh B-2)		
	point A	point A	point B	point A	point A	point B
	Vertical displacement	Rotation	Bending stress	Vertical displacement	Rotation	Bending stress
Type I	99.88	N/A	−3000	91.01	N/A	−3079
Type II	99.88	N/A	−3000	90.52	N/A	−3081
C-V1 (Choi and Lee 1993)	69.62	N/A	−2208	47.37	N/A	−2049
NC-V1 (Choi and Lee 1993)	101.04	N/A	−3175	80.13	N/A	−2953
NC-V2 (Choi and Lee 1993)	99.96	N/A	−2992	79.85	N/A	−2802
MR-H*x* (Choi, *et al*. 2001)	100.46	−19.95	−3070	86.32	−17.24	−3369
Theory	100.00	−20.00	−3000	100.00	−20.00	−3000

Table 5 Results of cantilever beam under pure bending for 13-node solid elements

Designation	Regular mesh (mesh A-3)			Distorted mesh (mesh B-3)		
	point A	point A	point B	point A	point A	point B
	Vertical displacement	Rotation	Bending stress	Vertical displacement	Rotation	Bending stress
Type I	99.81	N/A	−3000	93.67	N/A	−2883
Type II	99.81	N/A	−3000	93.42	N/A	−2876
C-V1 (Choi and Lee 1993)	69.18	N/A	−2208	49.27	N/A	−1738
NC-V1 (Choi and Lee 1993)	99.91	N/A	−3000	91.13	N/A	−2402
NC-V2 (Choi and Lee 1993)	99.92	N/A	−3000	90.82	N/A	−2396
MR-H*x* (Choi, *et al.* 2001)	96.96	−19.50	−3024	90.79	−17.89	−2484
Theory	100.00	−20.00	−3000	100.00	−20.00	−3000

4.4 Cook's problem

This problem was originally proposed by Cook as a test for the accuracy of quadrilateral elements (see Fig. 11). Although it is not common in solid element applications, this test is adopted for the sake of convergence check in shear-dominated problem. Thickness t=1.0, Youngs' modulus E=1.0 and Poissons' ratio μ=1/3 were used, and applied load P=1.0 is distributed along edge side. The result for the tip deflection at point A is compared with the reference value 23.91 obtained by the numerical analysis with a refined model (MacNeal and Harder 1985). Numerical tests with the sequentially refined meshes were carried out, and these test results are shown in Table 6.

Test results show that the behavior of present elements is satisfactory and solutions obtained by present elements converge to the reference value.

4.5 Twisted cantilever beam

To evaluate the performance and applicability of the proposed elements in twisted mesh, a twisted

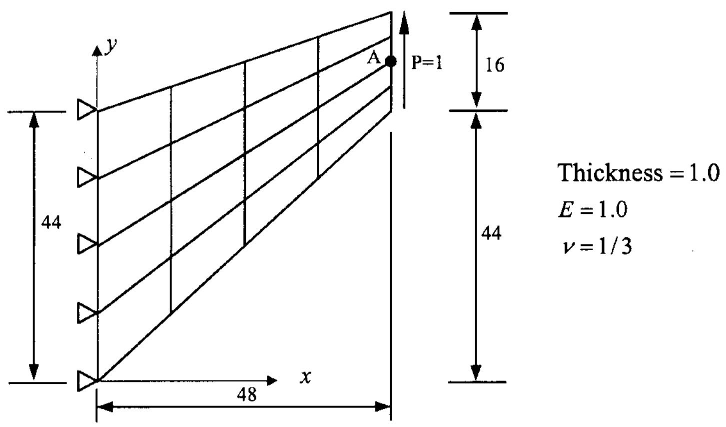

Fig. 11 Cook's problem (4×4×1 Mesh)

Table 6 Results of Cook's problem (end deflection)

Designation	Mesh				Patch test	Element type
	1×1×1	2×2×1	4×4×1	8×8×1		
Type I	16.35	21.04	22.98	23.67	pass	8-node solid element
Type II	15.82	20.96	22.97	23.67	pass	8-node solid element
Iura & Atluri 1992 (M1)	17.93	21.92	23.36	23.78	fail	4-node plane stress element
Iura & Atluri 1992 (M2)	12.80	20.09	22.90	23.66	pass	4-node plane stress element
Ibrahimbegovic *et al.* 1992	14.07	20.68	22.99	23.67	pass	4-node plane stress element w/drilling d.o.f.
Reference value	23.91					

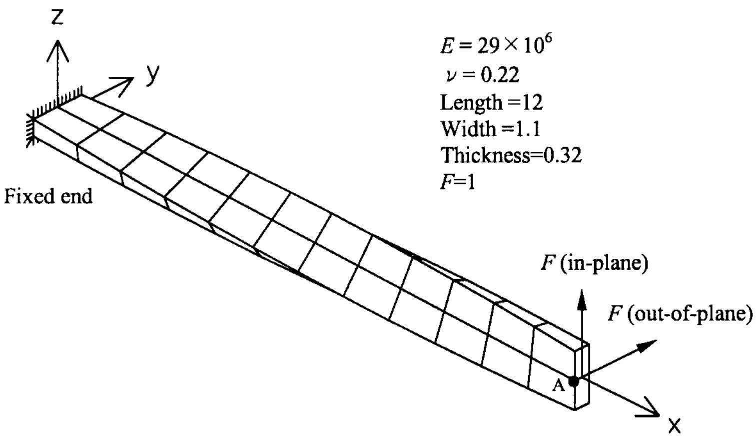

Fig. 12 Twisted cantilever beam

cantilever beam of rectangular cross section is tested. This cantilever beam is twisted 90° over its length, and subjected to a concentrated unit load at its free end. The geometry and loading conditions of this example are depicted in Fig. 12. The reference solutions in the case of in-plane load and of out-of-plane load are 0.5424×10^{-2} and 0.1754×10^{-2}, respectively (MacNeal and Harder 1985). The properties and dimensions used are given as Young's modulus E=29.0×10^6, Poisson's ratio ν=0.22, thickness t=0.32, side length L=12, and concentrated load F=1.0.

Normalized end displacements with reference values are listed at Table 7. The presented elements showed a good performance.

5. Conclusions

In this paper, a new efficient modification method for strains due to various non-conforming modes which may cause the failure in patch tests is presented. This method is successfully applied to the formulation of various non-conforming solid elements. Results obtained from a series of patch tests show that all the non-conforming 8-node elements and variable hexahedral elements formulated by incorporating the present modification scheme pass the patch tests. This implies that

Table 7 Results of twisted cantilever beam (normalized end displacement)

Designation	Loads		Element type
	in-plane	out-of-plane	
Type I	0.997	0.990	8-node solid element
Type II	0.988	0.983	
HEXA(8) (MacNeal and Harder 1985)	0.983	0.977	8-node solid element
HEXA20 (MacNeal and Harder 1985)	0.991	0.995	20-node solid element
HEX20R (MacNeal and Harder 1985)	0.993	0.999	20-node solid element
HEX8R (Yunus 1991)	0.951	0.960	8-node solid element
HEX8RX (Yunus 1991)	1.001	0.957	8-node solid element
reference value	1.000	1.000	

the undesirable effects of additional strains due to non-conforming modes of the elements can be successfully eliminated and the elements can represent constant stress state properly. Several other numerical tests for bending and shear also show that the new modification method for strains due to non-conforming modes is valid and the performance of developed non-conforming eight-node elements is satisfactory. Thus, the new modification scheme set the non-conforming modes free from patch test failures. The best performance element in this study was the element temporarily designated as Type I which has all the possible non-conforming modes and showed better performance than Type II elements. Thus, it is suggested that the element Type I be re-designated as NHx (Non-conforming Hexahedral with 8 node., e.g., NH8) for rigular element and NVHx (Non-conforming Variable node Hexahedral element with x-nodes e.g., NVHI3) for variable node element and used as a typical element for 3-D analysis.

It is expected that the proposed modification scheme can be further applied to various types of non-conforming elements such as the plate element, membrane element, and so forth.

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