Structural Engineering and Mechanics, Vol. 11, No. 1 (2001) 89-104
DOI: http://dx.doi.org/10.12989/sem.2001.11.1.089

Deducing thick plate solutions from classical thin plate solutions

C.M. Wang†

Department of Civil Engineering, The National University of Singapore, Kent Ridge, Singapore 119260

Abstract. This paper reviews the author's work on the development of relationships between solutions of the Kirchhoff (classical thin) plate theory and the Mindlin (first order shear deformation) thick plate theory. The relationships for deflections, stress-resultants, buckling loads and natural frequencies enable one to obtain the Mindlin plate solutions from the well-known Kirchhoff plate solutions for the same problem without much tedious mathematics. Sample thick plate solutions, deduced from the relationships, are presented as benchmark solutions for researchers to use in checking their numerical thick plate solutions.

Key words: plates; Kirchhoff; Mindlin; bending; buckling; vibration; relationships; shear deformation.

1. Introduction

In designing plates against bending, buckling and vibration, designers have readily available solutions from handbooks (e.g., Column Research Council 1971, Leissa 1969, Roark and Young 1975) and standard texts (e.g., Timoshenko and Woinowsky-Krieger 1959, Bulson 1970, Reddy 1999). The bulk of these solutions have been obtained using the classical thin (Kirchhoff) plate theory. When the thickness to length ratio increases to 1/20 or higher, designers are faced with the question of the importance of the effect of transverse shear deformation on the solutions. To date, this question cannot be answered easily without recourse to elaborate analytical and numerical methods.

Recently, the author and his colleagues have devised a way to determine the thick plate solutions from their existing thin plate counterparts, without tedious mathematics. Based on an analogy technique, they successfully established exact relationships between the solutions of classical thin plate theory and those of higher order plate theories proposed by Mindlin (1951) and by Reddy (1984). These more refined plate theories allow for the effect of transverse shear deformation. While the Mindlin plate theory requires a shear correction factor, the Reddy plate theory does not require correction factors but its governing equations are more complicated to solve.

The exact relationships for deflections, stress-resultants, buckling loads and natural frequencies of vibration, so far developed, cover a relatively large variety of plate shapes, loading and boundary conditions and are reported in diverse mechanics journals. This paper collates these relationships between the solutions of the classical thin (Kirchhoff) plate theory and the Mindlin plate theory for easy reference source. These newly-discovered relationships are important contributions because

† Professor

they enable engineers to readily deduce the Mindlin (thick) plate solutions from the abundant classical thin plate solutions for the same problem. Most of the thick plate solutions obtained from the developed relationships are hitherto not available and thus are extremely valuable to both design engineers and researchers. In this paper, some sample thick plate solutions generated from the relationships are presented to serve as benchmark checks for researchers who are involved in developing numerical methods and computer codes for thick plate analysis. Often one finds that analysts have to check their numerical solutions against another set of numerical results obtained from other methods due to the lack of exact analytical solutions. Such comparison studies involving numerically obtained results cannot conclusively establish the validity, convergence and accuracy of the numerical methods. In addition to giving benchmark results, the relationships (being in a closed form expression) give an insight into the effect of transverse shear deformation on the classical solutions. More importantly, they may be adopted in design codes for quick estimation of thick plate solutions, or adopted as basic forms for developing formulas with the aid of modification factors for other plate shapes, loading and boundary conditions in which the exact relationships are not valid.

2. Bending relationships

Consider the elastic bending problem of an isotropic plate of uniform thickness h, Poisson's ratio ν, modulus of elasticity E, shear modulus G and subjected to a transverse load $q(x, y)$. The equations of static equilibrium are given by (Timoshenko and Woinowksy-Krieger 1959)

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=0 \tag{1a}$$

$$\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=0 \tag{1b}$$

$$\frac{\partial M_{yy}}{\partial y}+\frac{\partial M_{xy}}{\partial x}-Q_y=0 \tag{1c}$$

where M_{xx}, M_{yy} are the bending moments per unit length, M_{xy} the twisting moment per unit length, and Q_x, Q_y are the transverse shear forces per unit length.

2.1. Kirchhoff plate theory

According to the Kirchhoff (or classical thin) plate theory, the stress-resultants in terms of the displacement are given by

$$M_{xx}^{K}=-D\left(\frac{\partial^2 w^{K}}{\partial x^2}+\nu\frac{\partial^2 w^{K}}{\partial y^2}\right) \tag{2a}$$

$$M_{yy}^{K}=-D\left(\nu\frac{\partial^2 w^{K}}{\partial x^2}+\frac{\partial^2 w^{K}}{\partial y^2}\right) \tag{2b}$$

$$M_{xy}^{K}=-D(1-\nu)\frac{\partial^2 w^{K}}{\partial x\partial y} \tag{2c}$$

where the superscript $\boldsymbol{K}$ denotes Kirchhoff plate quantities, w the transverse deflection, $D=Eh^3/[12(1-\nu^2)]$ and the flexural rigidity of the plate.

Since the Kirchhoff plate theory neglects the presence of transverse shear strain, the transverse shear forces can only be determined from the equilibrium equations. In view of Eqs. (1b), (1c), (2a), (2b) and (2c) and introducing the *Marcus* moment or moment sum $\mathfrak{M}$,

$$\mathfrak{M}^{\boldsymbol{K}}=\frac{M_{xx}^{\boldsymbol{K}}+M_{yy}^{\boldsymbol{K}}}{1+\nu}=-D\nabla^2 w^{\boldsymbol{K}} \tag{3}$$

the transverse shear forces can be expressed as

$$Q_x^{\boldsymbol{K}}=-D\frac{\partial}{\partial x}(\nabla^2 w^{\boldsymbol{K}})=\frac{\partial \mathfrak{M}^{\boldsymbol{K}}}{\partial x} \tag{4a}$$

$$Q_y^{\boldsymbol{K}}=-D\frac{\partial}{\partial y}(\nabla^2 w^{\boldsymbol{K}})=\frac{\partial \mathfrak{M}^{\boldsymbol{K}}}{\partial y} \tag{4b}$$

where $\nabla^2(\cdot)=\partial^2(\cdot)/\partial x^2+\partial^2(.)/\partial y^2$ is the Laplacian operator.

From Eqs. (1a), (4a) and (4b), the well-known fourth-order governing equation of the Kirchhoff plate theory can be established as

$$\nabla^4 w^{\boldsymbol{K}}=\frac{q}{D} \quad \text{or} \quad \nabla^2\mathfrak{M}^{\boldsymbol{K}}=-q \tag{5}$$

2.2. Mindlin plate theory

According to the Mindlin plate theory, the stress-resultants in terms of the displacement and rotations are given by (Mindlin 1951, Reismann 1988)

$$M_{xx}^{\boldsymbol{M}}=D\left(\frac{\partial\phi_x}{\partial x}+\nu\frac{\partial\phi_y}{\partial y}\right) \tag{6a}$$

$$M_{yy}^{\boldsymbol{M}}=D\left(\nu\frac{\partial\phi_x}{\partial x}+\frac{\partial\phi_y}{\partial y}\right) \tag{6b}$$

$$M_{xy}^{\boldsymbol{M}}=\frac{1}{2}D(1-\nu)\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right) \tag{6c}$$

$$Q_x^{\boldsymbol{M}}=\kappa^2 Gh\left(\phi_x+\frac{\partial w^{\boldsymbol{M}}}{\partial x}\right) \tag{6d}$$

$$Q_y^{\boldsymbol{M}}=\kappa^2 Gh\left(\phi_y+\frac{\partial w^{\boldsymbol{M}}}{\partial y}\right) \tag{6e}$$

where superscript $\boldsymbol{M}$ denotes Mindlin plate quantities, ϕ_x and ϕ_y denote the rotations of the normals to the middle-surface about the y- and x- axes, respectively, $w^{\boldsymbol{M}}$ is the transverse deflection of the middle-surface of the plate and κ^2 is the shear correction factor. The shear correction factor depends not only on the material and geometric parameters but also on the loading and boundary conditions.

As the Mindlin plate theory includes the effect of transverse shear deformation, the Mindlin shear forces can be obtained from the constitutive relationships, as shown in Eqs. (6d) and (6e). One may also determine the Mindlin shear forces by substituting the bending and twisting moments in Eqs.

(6a)-(6c) into the equilibrium Eqs. (1b) and (1c), respectively. The Mindlin equilibrium shear forces are thus given by

$$Q_x^{\boldsymbol{M}}=D\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial\phi_y}{\partial y}\right)+\frac{1}{2}(1-\nu)\frac{\partial}{\partial y}\left(\frac{\partial\phi_x}{\partial y}-\frac{\partial\phi_y}{\partial x}\right)\right] \tag{7a}$$

$$Q_y^{\boldsymbol{M}}=D\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial\phi_y}{\partial y}\right)-\frac{1}{2}(1-\nu)\frac{\partial}{\partial x}\left(\frac{\partial\phi_x}{\partial y}-\frac{\partial\phi_y}{\partial x}\right)\right] \tag{7b}$$

Introducing the *Marcus* moment or moment sum $\mathfrak{M}=(M_{xx}+M_{yy})/(1+\nu)$, Eqs. (7a) and (7b) become

$$Q_x^{\boldsymbol{M}}=\frac{\partial\mathfrak{M}^{\boldsymbol{M}}}{\partial x}+\frac{1}{2}D(1-\nu)\frac{\partial}{\partial y}\left(\frac{\partial\phi_x}{\partial y}-\frac{\partial\phi_y}{\partial x}\right) \tag{8a}$$

$$Q_y^{\boldsymbol{M}}=\frac{\partial\mathfrak{M}^{\boldsymbol{M}}}{\partial y}-\frac{1}{2}D(1-\nu)\frac{\partial}{\partial x}\left(\frac{\partial\phi_x}{\partial y}-\frac{\partial\phi_y}{\partial x}\right) \tag{8b}$$

In view of Eqs. (1a), (8a) and (8b), the governing bending equation of a Mindlin plate may be written as

$$\nabla^2\mathfrak{M}^{\boldsymbol{M}}=-q \quad \text{or} \quad \nabla^2\left(\frac{\partial\phi_x}{\partial x}+\frac{\partial\phi_y}{\partial y}\right)=-\frac{q}{D} \tag{9}$$

The Mindlin governing equation may also be obtained by considering the transverse shear forces in Eqs. (6d) and (6e), and the equilibrium Eq. (1a). This gives

$$\kappa^2 Gh\left(\nabla^2 w^{\boldsymbol{M}}+\frac{\mathfrak{M}^{\boldsymbol{M}}}{D}\right)=-q \tag{10}$$

It can also be noted that equating the shear forces in Eqs. (6d) and (6e) to those in Eqs. (7a) and (7b), respectively, and eliminating the Marcus moment in the process, one may deduce the following relation:

$$\nabla^2\Omega=c^2\Omega \tag{11}$$

where

$$\Omega=\left(\frac{\partial\phi_x}{\partial y}-\frac{\partial\phi_y}{\partial x}\right) \tag{12a}$$

and

$$c^2=2\kappa^2 Gh/[D(1-\nu)] \tag{12b}$$

2.3. Kirchhoff-Mindlin bending relationships

Here we derive a general set of relationships for stress-resultants, rotations and deflections of the two plate theories. These bending relationships are general for any plate shape, boundary and loading conditions. Based on load equivalence, Eqs. (5) and (9) lead to the Kirchhoff-Mindlin Marcus moment relationship:

$$\nabla^2\mathfrak{M}^{\boldsymbol{M}}=\nabla^2\mathfrak{M}^{\boldsymbol{K}}\Rightarrow\mathfrak{M}^{\boldsymbol{M}}=\mathfrak{M}^{\boldsymbol{K}}+D\nabla^2\Phi \tag{13}$$

where $\Phi(x, y)$ is a bi-harmonic function that satisfies

$$\nabla^4 \Phi = 0 \text{ in the entire plate domain} \tag{14}$$

In view of Eqs. (5), (10) and (13) and solving the final expression, one can obtain the Kirchhoff-Mindlin deflection relationship as

$$w^{M} = w^{K} + \frac{\mathfrak{M}^{K}}{\kappa^2 Gh} - \Phi + \Psi \tag{15}$$

where $\Psi(x, y)$ is a harmonic function that satisfies the Laplace equation

$$\nabla^2 \Psi = 0 \text{ in the entire plate domain} \tag{15a}$$

By substituting Eqs. (6d), (13) and (15) into Eq. (8a), it can be found that

$$\phi_x = -\frac{\partial w^{K}}{\partial x} + \frac{\partial}{\partial x}\left[\frac{D}{\kappa^2 Gh}(\nabla^2 \Phi) + \Phi - \Psi\right] + \frac{1}{c^2}\frac{\partial \Omega}{\partial y} \tag{16a}$$

and similarly from Eqs. (6e), (8b), (13) and (15),

$$\phi_y = -\frac{\partial w^{K}}{\partial y} + \frac{\partial}{\partial y}\left[\frac{D}{\kappa^2 Gh}(\nabla^2 \Phi) + \Phi - \Psi\right] - \frac{1}{c^2}\frac{\partial \Omega}{\partial x} \tag{16b}$$

Now for the Kirchhoff-Mindlin stress-resultant relationships, one can simply substitute Eqs. (15), (16a) and (16b) into Eqs. (6a)-(6e) and obtain

$$M_{xx}^{M} = M_{xx}^{K} - D(1-\nu)\frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}\left(\frac{D}{\kappa^2 Gh}\nabla^2 \Phi + \Phi - \Psi\right) - \frac{1}{c^2}\frac{\partial \Omega}{\partial x}\right] + D\nabla^2 \Phi \tag{17a}$$

$$M_{yy}^{M} = M_{yy}^{K} - D(1-\nu)\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{D}{\kappa^2 Gh}\nabla^2 \Phi + \Phi - \Psi\right) + \frac{1}{c^2}\frac{\partial \Omega}{\partial y}\right] + D\nabla^2 \Phi \tag{17b}$$

$$M_{xy}^{M} = M_{xy}^{K} + D(1-\nu)\left[\frac{\partial^2}{\partial x \partial y}\left(\frac{D}{\kappa^2 Gh}\nabla^2 \Phi + \Phi - \Psi\right) + \frac{1}{2c^2}\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x}\right)\Omega\right] \tag{17c}$$

$$Q_x^{M} = Q_x^{K} + \frac{\partial}{\partial x}(D\nabla^2 \Phi) + \frac{D(1-\nu)}{2}\frac{\partial \Omega}{\partial y} \tag{17d}$$

$$Q_y^{M} = Q_y^{K} + \frac{\partial}{\partial y}(D\nabla^2 \Phi) - \frac{D(1-\nu)}{2}\frac{\partial \Omega}{\partial x} \tag{17e}$$

The relationships given in Eqs. (16a), (16b), (17a)-(17e) may also be expressed in the polar coordinates systems as

$$\phi_r = -\frac{\partial w^{K}}{\partial r} + \frac{\partial}{\partial r}\left[\frac{D}{\kappa^2 Gh}(\nabla^2 \Phi) + \Phi - \Psi\right] + \frac{1}{c^2 r}\frac{\partial \Omega}{\partial \theta} \tag{18a}$$

$$\phi_\theta = -\frac{1}{r}\frac{\partial w^{K}}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{D}{\kappa^2 Gh}(\nabla^2 \Phi) + \Phi - \Psi\right] - \frac{1}{c^2}\frac{\partial \Omega}{\partial r} \tag{18b}$$

$$M_{rr}^{M} = M_{rr}^{K} + D(1-\nu)\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\left(\frac{D}{\kappa^2 Gh}\nabla^2 \Phi + \Phi - \Psi\right) + \frac{1}{c^2}\frac{\partial \Omega}{\partial r}\right] + \nu D\nabla^2 \Phi \tag{19a}$$

$$M_{\theta\theta}^{M}=M_{\theta\theta}^{K}-D(1-\nu)\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\left(\frac{D}{\kappa^2 Gh}\nabla^2\Phi+\Phi-\Psi\right)+\frac{1}{c^2 r}\frac{\partial\Omega}{\partial\theta}\right]+D\nabla^2\Phi \tag{19b}$$

$$M_{r\theta}^{M}=M_{r\theta}^{K}+D(1-\nu)\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{D}{\kappa^2 Gh}\nabla^2\Phi+\Phi-\Psi\right)\right]+\frac{1}{2c^2}\left(\frac{1}{r}\frac{\partial\Omega}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Omega}{\partial\theta^2}-\frac{\partial^2\Omega}{\partial r^2}\right)\right\} \tag{19c}$$

$$Q_r^{M}=Q_r^{K}+D\frac{\partial}{\partial r}(\nabla^2\Phi)+\frac{D(1-\nu)}{2}\frac{1}{r}\frac{\partial\Omega}{\partial\theta} \tag{19d}$$

$$Q_\theta^{M}=Q_\theta^{K}+D\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2\Phi)-\frac{D(1-\nu)}{2}\frac{\partial\Omega}{\partial r} \tag{19e}$$

The foregoing Kirchhoff-Mindlin bending relationships contain three intrinsic plate functions, i.e., Φ, Ψ and Ω, that must satisfy Eqs. (14), (15a) and (11), respectively and are dependent on the plate problem. Below, we present specialised Kirchhoff-Mindlin bending relationships for various plate problems that we have so far investigated.

2.4. Simply supported rectangular plates

For rectangular plates with simply supported edges, it can be shown that the intrinsic plate functions

$$\Phi = \Psi = \Omega = 0 \tag{20}$$

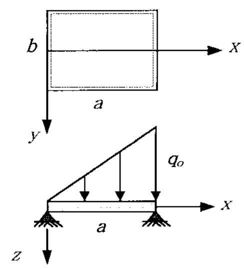
Thus from the foregoing general set of relationships, we find that the stress-resultants between the two plate theories are the same and the deflections are related as

$$w^{M}=w^{K}+\frac{\mathfrak{M}^{K}}{\kappa^2 Gh}=w^{K}-\frac{h^2}{6\kappa^2(1-\nu)}\nabla^2 w^{K} \tag{21}$$

Eq. (20) can be readily used to generate Navier deflection values based on the Mindlin plate theory upon supplying the corresponding Kirchhoff plate deflections. For example, we obtain the following sample deflection values for simply supported, rectangular Mindlin plates under hydrostatic load as shown in Table 1.

Table 1 Maximum deflection parameters $wD/(q_o a^4)$of simply supported rectangular plates under hydrostatic load, $q=q_o(x/a)$ (ν=0.3)

b/a	h/a = 0.02	h/a = 0.1	h/a = 0.2
1.0	0.00206	0.00216	0.00249
2.0	0.00507	0.00523	0.00573
3.0	0.00611	0.00628	0.00682
4.0	0.00640	0.00658	0.00712
5.0	0.00648	0.00665	0.00720
∞	0.00650	0.00668	0.00723



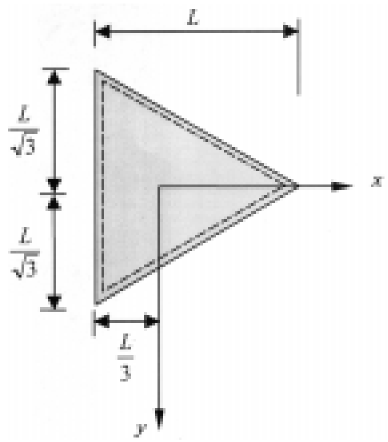


Fig. 1 Simply supported equilateral triangular plate

2.5. Simply supported, polygonal plates

Wang and Alwis (1995) have shown that the deflection relationship (21) applies to simply supported, polygonal plates under any transverse load distribution. For example, consider a simply supported equilateral triangular plate of side length $2L/\sqrt{3}$ as shown in Fig. 1.

The plate is subjected to a uniformly distributed load q. The deflection of the plate according to the Kirchhoff plate theory is given by (Woinowsky-Kreiger 1933)

$$w^{K}=\frac{q}{64LD}\left[x^3-3y^2x-L(x^2+y^2)+\frac{4}{27}L^3\right]\left[\frac{4}{9}L^2-x^2-y^2\right] \tag{22a}$$

The moment sum is

$$\mathfrak{M}^{K}=-D\nabla^2 w^{K}=\frac{q}{4L}\left[x^3-3y^2x-L(x^2+y^2)+\frac{4}{27}L^3\right] \tag{22b}$$

Hence, the deflection of a uniformly loaded equilateral triangular Mindlin plate is

$$w^{M}=\frac{q}{4L}\left[x^3-3y^2x-L(x^2+y^2)+\frac{4}{27}L^3\right]\left[\frac{\frac{4}{9}L^2-x^2-y^2}{16D}+\frac{1}{\kappa^2 Gh}\right] \tag{23}$$

2.6. Rectangular plates with two opposite simply supported edges

Recently Wang, Lim and Lee (1999) developed the bending relationships for rectangular plates with two opposite parallel edges $x=0$ and $x=a$, simply supported while the other two edges, $y=-b/2$ and $y=b/2$, have arbitrary boundary conditions as shown in Fig. 2 under a transverse loading defined by

$$q(x,y)=\sum_{m=1}^{\infty}\bar{q}_m(y)\sin\frac{m\pi x}{a} \tag{24}$$

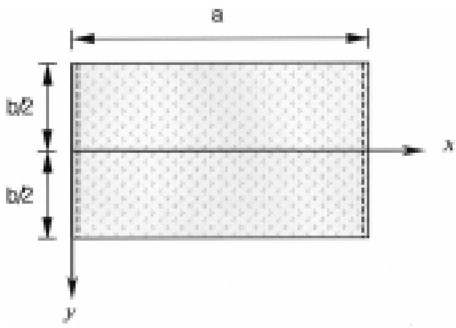


Fig. 2 Rectangular plate with two opposite edges simply supported

It can be shown that the intrinsic plate functions for Levy solutions are given by

$$\Phi(x, y)=\sum_{m=1}^{\infty}\frac{ay}{2m\pi}\left(C_{1m}\cosh\frac{m\pi y}{a}+C_{2m}\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a} \tag{25a}$$

$$\Psi(x, y)=\sum_{m=1}^{\infty}\frac{ay}{2m\pi}\left(C_{3m}\cosh\frac{m\pi y}{a}+C_{4m}\sinh\frac{m\pi y}{a}\right)\sin\frac{m\pi x}{a} \tag{25b}$$

$$\Omega(x, y)=\sum_{m=1}^{\infty}\left(C_{5m}\sinh\lambda_m y+C_{6m}\cosh\lambda_m y\right)\cos\frac{m\pi x}{a} \tag{25c}$$

where

$$\lambda_m^2=\left(\frac{m\pi}{a}\right)^2+\frac{2\kappa^2 Gh}{D(1-\nu)} \tag{25d}$$

Therefore, the general deflection relationship for such rectangular plates is given by

$$w^{\boldsymbol{M}}=w^{\boldsymbol{K}}+\frac{\mathfrak{M}^K}{\kappa^2 Gh}+\sum_{m=1}^{\infty}\left[\left(C_{3m}-C_{1m}\frac{ay}{2m\pi}\right)\cosh\frac{m\pi y}{a}+\left(C_{4m}-C_{2m}\frac{ay}{2m\pi}\right)\sinh\frac{m\pi y}{a}\right]\sin\frac{m\pi x}{a} \tag{26}$$

where C_{im}, i=1, ..., 4 are constants that are determined by the boundary conditions on edges $y=-b/2$ and $y=b/2$. For example, in the case of plates with those two opposite free (i.e., *SFSF* plates), these constants are given by

$$C_{1m}=C_{4m}=0,\quad C_{2m}=\xi_{1m}\frac{D(1-\nu)}{\kappa^2 Gh}\left(\frac{m\pi}{a}\right)^3\operatorname{sech}\frac{m\pi b}{2a}\frac{d\bar{w}_m^{\boldsymbol{K}}}{dy}\bigg|_{y=b/2}$$

$$C_{3m}=\xi_{2m}\frac{D(1-\nu)}{\kappa^2 Gh}\left(\frac{m\pi}{a}\right)\operatorname{csch}\frac{m\pi b}{2a}\frac{d\bar{w}_m^{\boldsymbol{K}}}{dy}\bigg|_{y=b/2} \tag{27a-d}$$

where

$$\xi_{1m}=\frac{\coth\frac{m\pi b}{2a}-\left(\frac{a\lambda_m}{m\pi}\right)\coth\frac{\lambda_m b}{2}}{\left[\frac{3+\nu}{2(1-\nu)}+\left(\frac{m\pi}{a}\right)^2\frac{D}{\kappa^2 Gh}-\lambda_m\left(\frac{m\pi}{a}\right)\frac{D}{\kappa^2 Gh}\tanh\frac{m\pi b}{2a}\coth\frac{\lambda_m b}{2}-\frac{m\pi b}{2a}\operatorname{csch}\frac{m\pi b}{a}\right]}$$

$$\xi_{2m}=1+\xi_{1m}\tanh\frac{m\pi b}{2a}\left[\frac{1}{\nu-1}+\frac{m\pi b}{4a}\coth\frac{m\pi b}{2a}+\frac{1}{2}\right] \tag{27e,f}$$

Table 2 Bending results of SCSC plates and SFSF plates under uniformly distributed load, q_o (ν =0.3, a/b=1 and k^2=5/6)

		SCSC Plates			SFSF Plates		
Point (x/a, y/b)	Plate Results	h/a = 0.02	h/a = 0.1	h/a = 0.2	h/a = 0.02	h/a = 0.1	h/a = 0.2
(0.5,0)	$wD/(q_oa^4)$	0.00193	0.00221	0.00302	0.01311	0.01346	0.01454
(0.5,0.5)	$wD/(q_oa^4)$	0	0	0	0.01507	0.01560	0.01690
(0.5,0)	$M_{xx}/(q_oa^2)$	0.0244	0.0258	0.0292	0.1225	0.1225	0.1229
(0.5,0)	$M_{yy}/(q_oa^2)$	0.0332	0.0333	0.0331	0.0268	0.0256	0.0237
(1.0,0)	$Q_x/(q_oa)$	0.243	0.246	0.254	0.466	0.463	0.460
(0.5,0.5)	$Q_y/(q_oa)$	0.514	0.501	0.475	0	0	0

For the stress-resultants relationships for Levy solutions, the reader may refer to the paper by Wang *et al.* (1999) and Lee *et al.*

Some sample bending results of square plates are given in Table 2.

2.7. Circular and annular plates

For axisymmetric bending of annular plates with an outer radius a, an inner radius b, the intrinsic plate functions can be shown to be given by

$$\Phi(r)=C_1\left[\frac{r^2}{4}(\ln r-1)-\frac{D}{\kappa^2 Gh}\ln r\right]+C_2\frac{r^2}{4} \tag{28a}$$

$$\Psi(r)=-C_3\ln r+C_4 \tag{28b}$$

$$\Omega(r)=0 \tag{28c}$$

In the case of axisymmetric bending of circular plates with simply supported or clamped edges, the above intrinsic plate functions are simplified to

$$\Phi(r)=0 \tag{29a}$$

$$\Psi(r)=C_4 \tag{29b}$$

$$\Omega(r)=0 \tag{29c}$$

Based on the above intrinsic functions, the general deflection relationship is given by (Wang and Lee 1996)

$$w^M=w^K+\frac{\mathfrak{M}^K}{\kappa^2 Gh}+C_1\left[\frac{r^2}{4}(1-\ln r)+\frac{D}{\kappa^2 Gh}\ln r\right]-C_2\frac{r^2}{4}-C_3\ln r+C_4 \tag{30}$$

where C_i, i=1, ..., 4 are constants that are determined by the boundary conditions. For example, in the case of circular plates with simply supported edges or clamped edges,

$$C_1=C_2=C_3=0,\quad C_4=-\left.\frac{\mathfrak{M}^K}{\kappa^2 Gh}\right|_{r=a} \tag{31a,b}$$

where a is the radius of the plate. In particular, for a circular plate of radius a under an axisymmetric linearly varying load defined by $q=q_0(1-r/a)$, the Kirchhoff deflection is given by (Szilard 1974)

$$w^K=\frac{q_0a^4}{14400D}\left[\frac{3(183+43\nu)}{1+\nu}-\frac{10(71+29\nu)}{1+\nu}\left(\frac{r}{a}\right)^2+225\left(\frac{r}{a}\right)^4-64\left(\frac{r}{a}\right)^5\right]$$

for simply supported plate (32a)

$$w^K=\frac{q_0a^4}{14400D}\left[129-290\left(\frac{r}{a}\right)^2+225\left(\frac{r}{a}\right)^4-64\left(\frac{r}{a}\right)^5\right]$$

for clamped plate (32b)

In view of Eqs. (31) and (32), the deflection of the corresponding Mindlin plate is given by

$$w^M=w^K+\frac{q_0a^4}{36\kappa^2Gh}\left[5-9\left(\frac{r}{a}\right)^2+4\left(\frac{r}{a}\right)^3\right] \tag{33}$$

for both cases of simply supported and clamped plates. Note that the deflection components due to the transverse shear deformation for simply supported circular plates and for clamped circular plates are the same. It can be readily proved that this fact holds for any general axisymmetric loading.

The deflection relationships for tapered axisymmetric plates have also been derived by Wang (1997). Based on the relationships, an axisymmetric annular Mindlin plate element was derived by Karunasena *et al.* (1997) and Reddy *et al.* (1997) for bending analysis of axisymmetric thick continuous plates. The element stiffness is unconventional in that it is based on the form of analytical solution, which contains logarithmic terms and yields exact nodal values, whereas the conventional polynomial-based element yields only approximate nodal values.

2.8. Sectorial plates

Consider a sectorial plate of radius a, thickness h and with a subtended angle α as shown in Fig. 3. The sectorial plate is simply supported along its two radial edges (θ=0, θ=α) while its circular edge can be either simply supported, or clamped or free. The plate is subjected to a transverse load defined by

$$q(r,\,\theta)=\sum_{m=1}^{\infty}\bar{q}_m(r)\sin\frac{m\pi\theta}{\alpha} \tag{34}$$

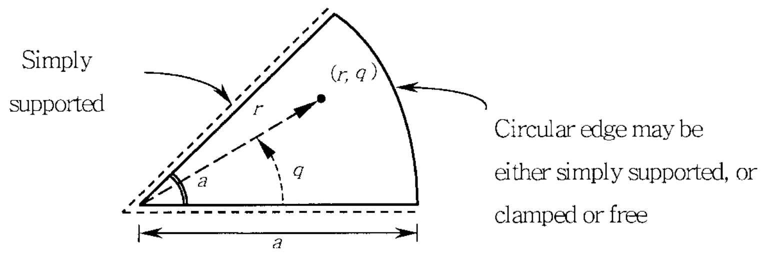


Fig. 3 Sectorial plate with simply supported radial edges

where α is the sectorial angle.
The intrinsic plate functions for this plate problem can be shown to be given by (Wang and Lim 1999)

$$\Phi(r,\ \theta)=\sum_{m=1}^{\infty}\left[\frac{C_{1m}}{4(\mu+1)}\right]r^{2+\mu}\sin\mu\theta \tag{35a}$$

$$\Psi(r,\ \theta)=\sum_{m=1}^{\infty}C_{2m}r^{\mu}\sin\mu\theta \tag{35b}$$

$$\Omega(r,\ \theta)=\sum_{m=1}^{\infty}C_{3m}I_{\mu}(cr)\cos\mu\theta \tag{35c}$$

where $\mu=m\pi/\alpha$.

The deflection relationship takes the form of

$$w^{M}=w^{K}+\frac{\mathfrak{M}^{K}}{\kappa^2 Gh}+\sum_{m=1}^{\infty}\left[\left(C_{2m}-C_{1m}\frac{r^2}{4(1+\mu)}\right)r^{\mu}\sin\mu\theta\right] \tag{36}$$

where C_{1m}, C_{2m} are constants which can be determined using the boundary conditions along the circular edge. The stress-resultant relationships are given in Wang and Lim (1999).

Some sample bending results of sectorial plates with simply supported radial edges and clamped circular edge are given in Table 3.

2.9. Annular sectorial plates

Consider an annular sectorial plate with an outer radius a, an inner radius b, thickness h and with a subtended angle α as shown in Fig. 4. The plate is simply supported along its two radial edges (defined by θ=0 and $\theta=\alpha$) while its circular edges may be either simply supported, or clamped or free. Polar co-ordinates system has been adopted with the origin being located at the intersection of the projected lines along the radial edges as shown in Figure 3.

The intrinsic plate functions for this plate problem are given by

$$\Phi(r,\ \theta)=\sum_{m=1}^{\infty}\left[\frac{C_{1m}r^{\mu+2}}{4(1+\mu)}+\frac{C_{2m}r^{2-\mu}}{4(1-\mu)}\right]\sin\mu\theta \tag{37a}$$

$$\Psi(r,\ \theta)=\sum_{m=1}^{\infty}\left[C_{3m}r^{\mu}+4r^{-\mu}\right]\sin\mu\theta \tag{37b}$$

Table 3 Bending results of SSC sectorial plates under uniformly distributed load, q_o (ν = 0.3 and κ^2=5/6)

		$\alpha=\pi/3$			$\alpha=\pi/2$		
Point (r/a, θ/α)	Plate Results	h/a = 0.02	h/a = 0.1	h/a = 0.2	h/a = 0.02	h/a = 0.1	h/a = 0.2
(0.5,0.5)	$wD/(q_oa^4)$	0.000568	0.000664	0.000953	0.001323	0.001492	0.002004
(0.5,0.5)	$M_{rr}/(q_oa^2)$	0.0143	0.0141	0.0137	0.0272	0.0273	0.0273
(0.5,0.5)	$M_{\theta\theta}/(q_oa^2)$	0.0197	0.0202	0.0214	0.0240	0.0244	0.0257
(1.0,0.5)	$Q_r/(q_oa)$	−0.315	−0.308	−0.293	−0.363	−0.359	−0.352
(0.5,0)	$Q_\theta/(q_oa)$	0.213	0.215	0.218	0.242	0.243	0.248

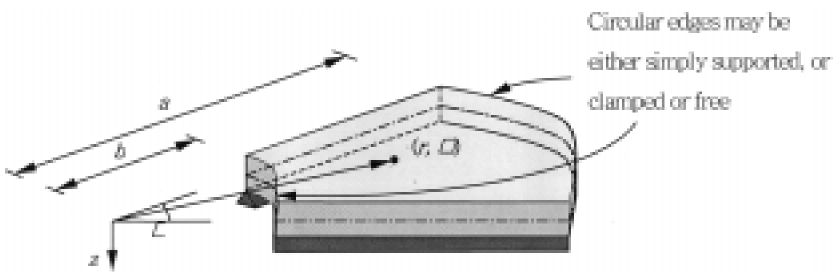


Fig. 4 Annular sectorial plate with simply supported radial edges

Table 4 Bending results for SSSS and SCSC annular sectorial plates (α=π/3, b/a=0.5, m=18)

	$\bar{w}(\beta, \pi/6)$		$\bar{M}_{rr}(\beta, \pi/6)$		$\bar{M}_{\theta\theta}(\beta, \pi/6)$	
h/a	SSSS	SCSC	SSSS	SCSC	SSSS	SCSC
0*	8.0021	2.5223	8.3728	4.0869	4.8716	1.7492
0.025	8.0765	2.6167	8.3724	4.0933	4.8731	1.7684
0.050	8.2995	2.8955	8.3712	4.1088	4.8774	1.8263
0.075	8.6709	3.3508	8.3690	4.1275	4.8844	1.9206
0.100	9.1906	3.9744	8.3660	4.1444	4.8938	2.0462

*based on thin plate analysis $\bar{w}=1000wD/\lfloor q_o(a-b)^4 \rfloor$; $\bar{M}_{rr}=100M_{rr}/\lfloor q_o(a-b)^2 \rfloor$; $\bar{M}_{\theta\theta}=100M_{\theta\theta}/\lfloor q_o(a-b)^2 \rfloor$; $\beta=(b+a)/2a$.

$$\Omega(r,\ \theta)=\sum_{m=1}^{\infty}[C_{5m}I_\mu(cr)+C_{6m}K_\mu(cr)]\cos\mu\theta \tag{37c}$$

where $\mu=m\pi/\alpha$, $I_\mu(\cdot)$ and $K_\mu(\cdot)$ are the modified Bessel functions of the first and second kind of order μ, respectively. C_{im}, (i=1, ..., 6) are the constants of integration which can be determined uniquely by using the boundary conditions along the circular edges when $r=a$ and $r=b$.

Table 4 presents the bending solutions for uniformly loaded annular sectorial plates whose circular edges are either simply supported or clamped. The shear correction factor κ^2 is assumed to be 5/6. For more detailed treatment of this problem, refer to the paper by Lim and Wang (2000).

3. Buckling load relationship

For a simply supported, polygonal plate under isotropic inplane load N, Wang (1995a) showed that the buckling load N^M of Mindlin plate is related to its corresponding Kirchhoff buckling load N^K by

$$N^M=\frac{N^K}{1+\frac{1}{\kappa^2 Gh}N^K} \tag{38}$$

where h is the plate thickness. Wang (1996a) and Wang and Xiang (1999) also showed that the above same buckling relationship applies to (a) radially loaded, circular plates with any

Table 5 Critical buckling loads $N_{cr}b^2/(\pi^2 D)$ of simply supported rhombic plates

h/b	α=0°	α=15°	α=30°	α=45°
0.001	2.0000	2.1147	2.5240	3.5253
0.1	1.8932	1.9957	2.3563	3.2066
0.2	1.6319	1.7074	1.9646	2.5224

Table 6 Critical buckling loads $N_{cr}b^2/D$ of simply supported and clamped circular plates

h/b	Simply supported plate	Clamped plate
0.001	4.1978	14.6819
0.1	4.1480	14.0909
0.2	4.0056	12.5725

homogeneous edge restraint such as simply supported, clamped or simply supported with elastic rotational restraint and (b) simply supported, sectorial plates. The relationship was also extended to include the effect of prebuckling deformation on the buckling load (see Wang 1995a) and modified for buckling of rectangular plates under uniform shear (Wang *et al.* 1994)

Tables 5 and 6 present buckling loads of rhombic plates (of length b) and circular plates (of radius b), respectively. These results were obtained using Eq. (38), with ν = 0.3 and κ^2=5/6. When the skew angle α=0°, the case corresponds to a square plate. Note that the results for h/b = 0.001 represent the Kirchhoff (or thin classical) plate solutions.

When the plate rests on a two-parameter elastic foundation, Wang, Kitipornchai and Xiang (1997) showed that the augmented buckling relationship takes the form of

$$N^M = \frac{N^K}{1 + \frac{1}{\kappa^2 Gh} N^K} + \frac{kD}{N^K} + G_b \tag{39}$$

where k is the modulus of the subgrade reaction for the foundation and G_b the shear modulus of the subgrade. Owing to the second term on the right hand side of the relationship, the buckling load for the plate with foundation may not be the same as that for the classical thin plate without foundation. So it is necessary to substitute the Kirchhoff buckling mode solutions in increasing magnitudes until the lowest load value of Mindlin plate is found. This lowest value corresponds to the buckling load of the Mindlin plate on elastic foundation.

For simply supported, rectangular plate under compressive biaxial loading N_x=N and N_y=βN, the buckling load relationship for $\beta \geq 0.5$ is given by

$$N^M = \frac{N^K}{1 + \frac{N^K}{2\kappa^2 Gh}\left[1 + \sqrt{1 - \frac{4\pi^2(1-\beta)D}{N^K b^2}}\right]} \tag{40}$$

where b is the width of the plate in which N_y is applied.

Table 7 Fundamental frequency parameters $\omega b^2\sqrt{\rho h/D}$ of simply supported, regular polygonal plates

Shape	h/b=0	h/b=0.05	h/b=0.10	h/b=0.15
Triangle	52.638	51.443	48.372	44.432
Square	19.739	19.526	19.080	18.359
Pentagon	10.863	10.810	10.658	10.420
Hexagon	7.129	7.106	7.040	6.933
Octagon	3.624	3.618	3.601	3.572

4. Vibration frequencies relationship

The relationship between vibration frequencies of simply supported, polygonal plates is given by (Wang 1994, Wang 1995d)

$$(\omega^M)^2=\frac{6\kappa^2 G}{\rho h^2}\left\{1+\omega^K\frac{h^2}{12}\sqrt{\frac{\rho h}{D}}\left(1+\frac{2}{\kappa^2(1-\nu)}\right)-\sqrt{\left[1+\omega^K\frac{h^2}{12}\sqrt{\frac{\rho h}{D}}\left(1+\frac{2}{\kappa^2(1-\nu)}\right)\right]^2-(\omega^K)^2\frac{\rho h^2}{3\kappa^2 G}}\right\} \quad (41)$$

where ω is the angular frequency of the plate. Wang (1994) showed that the relationship may also be used to predict quite accurately the vibration of frequencies of Mindlin plates with simply supported curved edges. Note that if one follows Mindlin's suggestion of equating the angular frequency of the first antisymmetric mode of thickness-shear vibration according to the exact three-dimensional theory to the corresponding frequency according to his theory, then the shear correction factor is given by

$$(\kappa^2)^3-8(\kappa^2)^2+\frac{8(2-\nu)}{1-\nu}\kappa^2-\frac{8}{1-\nu}=0 \quad (42)$$

Table 7 presents, for example, the fundamental frequencies of various polygonal plate shapes with side length b and ν=0.3 [i.e., κ^2=0.86 from Eq. (42)].

5. Conclusions

In this paper, exact relationships between solutions of Kirchhoff and Mindlin for isotropic elastic plates of various shapes and boundary conditions are presented. Upon supplying the Kirchhoff solutions, the Mindlin solutions may be readily obtained. These relationships can be readily modified for symmetrically, isotropic laminated plates, sandwich plates and functionally graded plates by modifying the stiffnesses appropriately (see Wang 1995b, 1995c, 1995d, Reddy *et al.* 1999).

The exact Mindlin solutions obtained via these relationships should serve as useful benchmark values for researchers to check the validity, convergence and accuracy of their numerical results. The exact relationships also show clearly the intrinsic features of the effect of transverse shear deformation on the classical solutions.

Acknowledgement

The author is grateful to Mr Lim Goy Teck for his contribution in deriving some of the relationships and sample thick plate solutions.

References

Bulson, P.S. (1970), *The Stability of Flat Plates*, Chatto and Windus, London, U.K.

Column Research Committee of Japan (1971), *Handbook of Structural Stability*, Corona, Tokyo, Japan.

Karunasena, W., Wang, C.M., Kitipornchai, S. and Xiang, Y. (1997), "Exact solutions for axisymmetric bending of continuous annular plates", *Computers and Structures*, **63**(3), 455-464.

Kirchhoff, G. (1850), "Uber das gleichgwich und die bewegung einer elastischen scheibe", *J. Angew. Math.*, **40**, 51-88.

Lee, K.H., Lim, G.T. and Wang, C.M. "Thick Levy plates revisited", *International Journal of Solids and Structures*, submitted.

Leissa, A.W. (1969), *Vibration of Plates*, U.S. Government Printing Office, NASA SP-160, reprinted by the Acoustical Society of America in 1993.

Lim, G.T. and Wang, C.M. (2000), "Bending of annular sectorial Mindlin plates using Kirchhoff results", *European Journal of Mechanics: A-Solids*, in press.

Mindlin, R.D. (1951), "Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates", *Trans. ASME*, *Journal of Applied Mechanics*, **18**, 31-38.

Reddy, J.N. (1984), "A simple higher-order theory for laminated composite plates", *Trans. ASME*, *Journal of Applied Mechanics*, **51**, 745-752.

Reddy, J.N. (1999), *Theory and Analysis of Elastic Plates*, Taylor and Francis, Philadelphia, U.S.

Reddy, J.N., Wang, C.M. and Lam, K.Y. (1997), "Unified finite elements based on the classical and shear deformation theories of beams and axisymmetric circular plates", *Communications in Numerical Methods in Engineering*, **13**, 495-510.

Reddy, J.N., Wang, C.M. and Kitipornchai, S. (1999), "Axisymmetric bending of functionally graded circular and annular plates", *European Journal of Mechanics: A/Solids*, **18**, 185-199.

Reismann, H. (1988), *Elastic Plates: Theory and Applications*, John Wiley & Sons, Inc., New York.

Roark, R.J. and Young, W.C. (1975), *Formulas for Stress and Strain*, 5th Edition, McGraw-Hill, New York.

Szilard, R. (1974), *Theory and Analysis of Plates*, Prentice-Hall, Englewood Cliffs, New Jersey.

Timoshenko, S. and Woinowsky-Krieger, S. (1959), *Theory and Plates and Shells*, McGraw-Hill, New York.

Wang, C.M. (1994), "Natural frequencies formula for simply supported Mindlin plates", *Trans. ASME*, *Journal of Vibration and Acoustics*, **116**(4), 536-540.

Wang, C.M., Xiang, Y. and Kitipornchai, S. (1994), "Buckling solutions of rectangular Mindlin plates under uniform shear", *Journal of Engineering Mechanics*, *ASCE*, **120**(11), 2462-2470.

Wang, C.M. (1995a), "Allowance for prebuckling deformations in buckling load relationship between Mindlin and Kirchhoff simply supported plates of general polygonal shape", *Engineering Structures*, **17**(6), 413-418.

Wang, C.M. (1995b), "Deflection of sandwich plates in terms of corresponding Kirchhoff plate solutions", ,*Archive of Applied Mechanics*, **65**(6), 408-414.

Wang, C.M. (1995c), "Buckling of polygonal and circular sandwich plates", *AIAA Journal*, **33**(5), 962-964.

Wang, C.M. (1995d), "Vibration frequencies of simply supported polygonal sandwich plates via Kirchhoff solutions", *Journal of Sound and Vibration*, **190**(2), 255-260.

Wang, C.M. (1996a), Discussion on "Postbuckling of moderately thick circular plates with edge elastic restraint", *Journal of Engineering Mechanics*, *ASCE*, **122**(2), 181-182.

Wang, C.M. and Alwis, W.A.M. (1995), "Simply supported Mindlin plate deflections using Kirchhoff plates", *Journal of Engineering Mechanics*, *ASCE*, **121**(12), 1383-1385.

Wang, C.M. and Lee, K.H. (1996), "Deflection and stress resultants of axisymmetric Mindlin plates in terms of corresponding Kirchhoff solutions", *International Journal of Mechanical Sciences*, **38**(11), 1179-1185.

Wang, C.M. (1997), “Relationships between Mindlin and Kirchhoff bending solutions for tapered circular and annular plates”, *Engineering Structures*, **19**(3), 255-258.

Wang, C.M., Kitipornchai, S. and Xiang, Y. (1997), “Relationships between buckling loads of Kirchhoff, Mindlin and Reddy polygonal plates on Pasternak foundation”, *Journal of Engineering Mechanics*, *ASCE*, **123**(11), 1134-1137.

Wang, C.M. and Lim, G.T. (1999), “Bending solutions of sectorial Mindlin plates from Kirchhoff plates”, *Journal of Engineering Mechanics*, *ASCE*, **126**(4), 367-372.

Wang, C.M. and Xiang, Y. (1999), “Deducing buckling loads of sectorial Mindlin plates from Kirchhoff plates”, *Journal of Engineering Mechanics*, *ASCE*, **125**(5), 596-598.

Wang, C.M., Lim, G.T. and Lee, K.H. (1999), “Relationships between Kirchhoff and Mindlin bending solutions for Levy plates”, *Trans. ASME*, *Journal of Applied Mechanics*, **66**(2), 541-545.

Woinowsky-Krieger, W. (1933), “Berechnung der ringsum frei aufliegenden gleihseitigen dreiecksplatte”, *Ing. Archiv.*, **4**, 254-262.