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Stability of unbraced frames under non-proportional loading

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Abstract. This paper discusses the elastic stability of unbraced frames under non-proportional loading based on the concept of storey-based buckling. Unlike the case of proportional loading, in which the load pattern is predefined, load patterns for non-proportional loading are unknown, and there may be various load patterns that will correspond to different critical buckling loads of the frame. The problem of determining elastic critical loads of unbraced frames under non-proportional loading is expressed as the minimization and maximization problem with subject to stability constraints and is solved by a linear programming method. The minimum and maximum loads represent the lower and upper bounds of critical loads for unbraced frames and provide realistic estimation of stability capacities of the frame under extreme load cases. The proposed approach of evaluating the stability of unbraced frames under non-proportional loading has taken into account the variability of magnitudes and patterns of loads, therefore, it is recommended for the design practice.

Key words: non-proportional loading; frame stability; storey-based buckling; linear programming; critical load; unbraced frame; lean-on column.

1. Introduction

The determination of the buckling load of a column in an unbraced frame and that of the frame is of primary importance to the design of unbraced frames. It is well known that the maximum strength of the frame and the maximum strength of an axially loaded column are interrelated. Furthermore, the relationship between the two is often complicated, and the theoretical approach of evaluating elastic buckling of frames under proportional loading, which is referred to as the system buckling approach, is generally considered not practical (Galambos 1988). The reason for this is that it involves solving for the critical load multiplier λ_{cr} from either a highly nonlinear equation or a transcendental equation.

In current design practice, the stability analysis and design of framed structures are commonly carried out by evaluating the effective length factor of columns in conjunction with the classical alignment charts in the current Load and Resistance Factor Design (LRFD) Specifications (AISC 1994). The concept of the effective length factor is considered to be an essential part of many analysis procedures and is valid for ideal structures. However, several assumptions on the buckling modelling of the frame were made in developing the alignment charts. When the assumptions are

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violated, the use of alignment charts results in erroneous effective length factors.

Pointing out that lateral sway buckling of unbraced frames is a total storey phenomenon, and a single individual column cannot fail by lateral sway buckling without all of the columns in the same storey also buckling in the same sway mode, the concept of storey-based buckling was introduced (Yura 1971). Various procedures of evaluating the stability of the frame based on this concept were proposed thereafter (LeMessurier 1977, Lui 1992, Aritizabal-Ochoa 1997, Chong-Siat-Moy 1999).

However, all of the studies on frame stability that have been carried out so far, no matter whether it is based on the classical alignment charts approach, system buckling approach, or storey-based buckling approach, are all under the assumption that the frame is subjected to proportional loading. The non-proportional loading case, which has taken into consideration the volatility of loads and is more closely related to actual situations in practice, is left unsolved due to the complexity of the problem.

In the investigation of frame buckling under non-proportional loading, each individual applied load on the frame is allowed to vary independently. Therefore, multiple critical load multipliers need to be determined. Consequently, the traditional procedure, which converts a frame stability problem into an eigenvalue problem by solving a single critical load multiplier λ_{cr}, as the minimum positive eigenvalue of the system is no longer applied.

The main objective of this paper is to investigate the elastic in-plane buckling characteristics of unbraced frames under non-proportional loading based on the concept of storey-based buckling. A simplified procedure of evaluating column elastic buckling load based on storey-based buckling under proportional loading is first presented. Unlike the case in proportional loading, in which the load pattern is predefined and the solution of the critical load multiplier is unique, load patterns for non-proportional loading are unknown, and there may be different load patterns that will correspond to the critical buckling loads of the frame. To overcome the difficulty associated with non-proportional loading, the problem of the lateral buckling of unbraced frames is expressed as a minimization problem subject to stability constraints. Consequently, the linear method is adopted to determine the most critical loads of the frame.

By applying the linear programming method to non-proportional loading cases, the most critical buckling load, or so-called the lower-bound of buckling loads corresponding to the worst load pattern and the minimum load magnitude of the frame, can be determined through solving a minimization problem. On the other hand, the upper-bound of the buckling loads corresponding to the most favourable load pattern and the maximum load magnitude of the frame, can be evaluated as a corresponding maximization problem. These lower and upper bounds of the critical buckling loads provide clear buckling characteristics of the unbraced frame, in which the volatility of the magnitudes and patterns of applied loads have been taken into account. The numerical examples show that the proposed non-proportional loading approach provides more appropriate results compared to those of proportional loading, therefore, it is recommended to engineering practice.

2. Lateral stiffness of an axially loaded column

Shown in Fig. 1a is an axial loaded column in an unbraced frame, in which EI/L is the flexural stiffness of the column, P is the column axial load, and R_l and R_u are the rotational restraining stiffness provided by its immediately connected beams at the lower and upper joints, respectively. The deformation and the forces of the column that are associated with a unity lateral deflection at

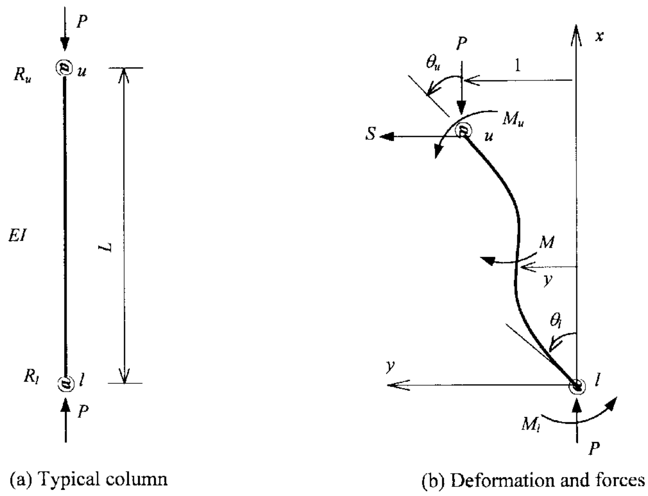

Fig. 1 Analytical model for lateral stiffness of PR column

the upper end of the column are shown in Fig. 1b. Let S be the shear force (lateral stiffness) associated with the unity lateral deflection and M_u and M_l be the corresponding end moments of the column. Thus, the moment at any location x along the column can be expressed as follows:

$$M=M_u-P(1-y)-S(L-x) \tag{1}$$

Considering that $M=EId^2y/d^2x$, Eq. (1) becomes

$$EId^2y/d^2x=M_u-P(1-y)-S(L-x) \tag{2}$$

Let

$$\phi=\sqrt{\frac{PL^2}{EI}}=\pi\sqrt{P/P_e} \tag{3}$$

where P_e is the Euler buckling load of the column. The solution of Eq. (2) can be expressed as

$$y=C_1\cos(\phi x/L)+C_2\sin(\phi x/L)+1-M_u/P+S(L-x)/P \tag{4}$$

where C_1 and C_2 are coefficients to be determined by column end conditions. Considering the equilibrium condition at the lower end of the column, Eq. (1) yields

$$M_l=M_u-P-SL \tag{5}$$

The relation between end moments and rotations are

$$M_u=R_u\theta_u \qquad M_l=-R_l\theta_l \tag{6}$$

The end boundary conditions are

$$y|_{x=0}=0,\quad y|_{x=L}=1,\quad \left.\frac{dy}{dx}\right|_{x=0}=\theta_l,\quad \left.\frac{dy}{dx}\right|_{x=L}=\theta_u \tag{7}$$

Substituting Eq. (6) into Eq. (5) yields

$$\theta_l=(-R_u\theta_u+P+SL)/R_l \tag{8}$$

From Eqs. (4), (6) and (7), we can obtain

$$C_1+1-M_u/P+SL/P=0 \tag{9a}$$

$$C_1\cos\phi+C_2\sin\phi-M_u/P=0 \tag{9b}$$

$$C_2\phi/L-S/P=\theta_l \tag{9c}$$

$$-C_1\frac{\phi}{L}\sin\phi+C_2\frac{\phi}{L}\cos\phi-S/P=\theta_u \tag{9d}$$

Let

$$C=\frac{SL}{12P}=\frac{SL^3}{12EI\phi^2} \tag{10}$$

$$C_l=\frac{1-r_l}{3r_l}\phi^2 \tag{11}$$

$$C_u=\frac{1-r_u}{3r_u}\phi^2 \tag{12}$$

Then it can be solved from Eqs. (9) that

$$C=\frac{(C_l+C_u)\phi\cos\phi+(\phi^2-C_lC_u)\sin\phi}{2\phi-\phi(2+C_l+C_u)\cos\phi+(C_l+C_u-\phi^2+C_lC_u)\sin\phi} \tag{13}$$

Therefore, based on Eq. (10), the lateral stiffness of the column can be expressed as

$$S=12\frac{\phi^2 CEI}{L^3}=\beta\frac{12EI}{L^3} \tag{14}$$

where $\beta=C\phi^2$ is the modification factor of the lateral stiffness that accounts for the effects of axial force and column end rotational restraints.

Introducing non-dimensional end-fixity factors for the lower and upper ends of the column to simulate column end rotational conditions,

$$r_l=\frac{1}{1+3EI/R_lL};\quad r_u=\frac{1}{1+3EI/R_uL} \tag{15a, b}$$

where R_l and R_u are the rotational stiffness at the lower and upper ends, respectively. EI/L is the column flexural stiffness. For a column with rigid ends, R_l and R_u are infinite, and the corresponding values of r_l and r_u are one. For a column with pinned ends, R_l and R_u are zero, and the corresponding values of r_l and r_u are also zero. Upon the introduction of end-fixity factors, the modification factor β in Eq. (14) can be expressed as

$$\beta=\frac{\phi^3}{12}\frac{a_1\phi\cos\phi+a_2\sin\phi}{18r_lr_u-a_3\cos\phi+a_4\phi\sin\phi} \tag{16}$$

where

$$a_1=3[r_l(1-r_u)+r_u(1-r_l)] \tag{17a}$$

$$a_2=9r_lr_u-(1-r_l)(1-r_u)\phi^2 \tag{17b}$$

$$a_3=18r_lr_u+[3r_l(1-r_u)+3r_u(1-r_l)]\phi^2 \tag{17c}$$

$$a_4=-9r_lr_u+3r_l(1-r_u)+3r_u(1-r_l)+(1-r_u)(1-r_l)\phi^2 \tag{17d}$$

It is difficult to evaluate the column critical buckling load due to the transcendental relationship between β and ϕ in Eq. (16), especially in a multicolumn unbraced frame. Thus, a linear approximation of Eq. (16) can be obtained by employing Taylor series expansion of Eq. (16) as

$$\beta=\beta_0-\beta_1\phi^2 \tag{18}$$

where

$$\beta_0=\lim_{\phi\to 0}\beta=\frac{(r_l+r_u+r_lr_u)}{4-r_lr_u} \tag{19}$$

$$\beta_1=\frac{4(20+3r_l+r_l^2)+r_u(12-107r_l+29r_l^2)+r_u^2(4+29r_l-9r_l^2)}{60(4-r_lr_u)^2} \tag{20}$$

Eq. (18) provides a good approximation of the column lateral stiffness modification factor β. Fig. 2 shows the relationship between the modification factor β and the ratio of P/P_e evaluated based on Eq. (16) and Eq. (18), respectively. It can be seen that there is a good agreement between Eqs. (16) and (18) in the entire range of ratio P/P_e between zero and one except when P approaches P_e in the case of a column with rigid ends.

3. Storey-based buckling of unbraced frames under proportional loading

Upon the derivation of the lateral stiffness of a column with consideration of the effect of column compressive load, the storey-based buckling loads of columns in an unbraced frame can be readily evaluated. Unlike that of the alignment chart method, which ignores the fact that columns in a storey of the frame will restrain each other in resisting buckling, the interaction among the columns in a storey of the frame is taken into account in storey-based buckling. The condition for the multicolumn storey-based buckling in a lateral sway mode is that the total lateral stiffness of the storey vanishes. Thus, the stability equation becomes

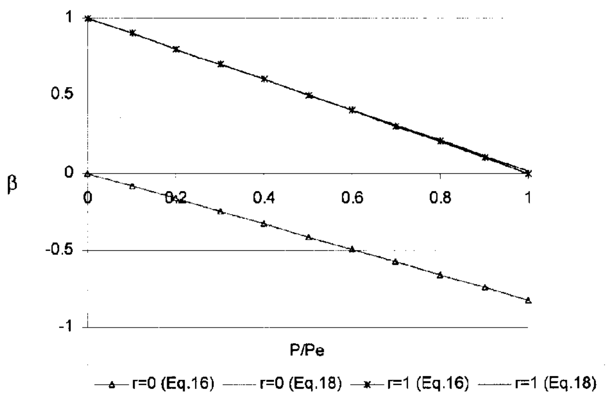

Fig. 2 β-P/P_e curves

$$\sum_{i=1}^{n} S_i = 0 \tag{21}$$

where n is the number of columns in a storey, and S_i is the column lateral stiffness expressed in Eq. (14). Substituting Eq. (14) and Eq. (18) into Eq. (21),

$$\sum_{i=1}^{n} S_i = 12\sum_{i=1}^{n}\left(\frac{EI_i}{L_i^3}\beta_{0i} - \frac{P_{si}}{L_i}\beta_{1i}\lambda\right) = 0 \tag{22}$$

in which L_i and P_{si} are the length and the axial force, respectively, due to the specified load of column i. λ is the proportional load factor. Let

$$a = \sum_{i=1}^{n}\frac{P_{si}}{L_i}\beta_{1i} \tag{23a}$$

$$b = \sum_{i=1}^{n}\frac{EI_i}{L_i^3}\beta_{0i} \tag{23b}$$

Substituting Eqs. (23) into Eq. (22) results in

$$a\lambda - b = 0 \tag{24}$$

Therefore, the critical load parameter that corresponds to the storey-based lateral sway buckling of the unbraced frame is obtained as

$$\lambda_{cr} = \frac{b}{a} = \frac{\sum_{i=1}^{n}\frac{EI_i}{L_i^3}\beta_{0i}}{\sum_{i=1}^{n}\frac{P_{si}}{L_i}\beta_{1i}} \tag{25}$$

Numerical studies have demonstrated that Eq. (25) yields sufficient accuracy in the evaluation of the critical load multiplier for storey-based buckling of unbraced frames under proportional loading and is recommended for engineering practice due to its simplicity (Xu and Liu 2000).

4. Storey-based buckling of unbraced frames under non-proportional loading

In the case of lateral sway buckling of an unbraced frame under non-proportional loading, each individual applied load on the frame is allowed to vary independently in order to capture the worst load case that causes the frame buckling. Therefore, multiple critical load multipliers λ_i (i=1… n), need to be determined. Because each load is independent from the others in the non-proportional case, it is more convenient to adopt the magnitudes of loads rather than load multipliers as the unknown variables to be determined. Thus, let $P_i = \lambda_i P_{si}$, and Eq. (22) can be expressed as follows for the non-proportional loading case:

$$\sum_{i=1}^{n} S_i = 12\sum_{i=1}^{n}\left(\frac{EI_i}{L_i^3}\beta_{0i} - \frac{P_i}{L_i}\beta_{1i}\right) = 0 \tag{26}$$

Unlike the case of proportional loading, in which the load pattern is predefined, and the solution of the critical load multiplier is unique, load patterns in non-proportional loading are unknown, and there may be different load patterns that will correspond to the critical buckling loads of the frame. Therefore, the two factors that need to be determined in the non-proportional loading case are the total magnitude of loads and corresponding load patterns. Comparing Eq. (26) with Eq. (22), there is no unique solution for Eq. (26) because it involves only one equation that is associated with n variables of P_i. Theoretically, Eq. (26) may have an infinite number of solutions. Among all of the possible solutions, the minimum value of the summation of each individual load together with the associated load pattern would be the critical load and the worst load pattern for the frame in lateral sway buckling. On the other hand, the maximum value of the summation of each individual load together with the associated load pattern would be the most favourable load pattern in which the maximum total load magnitude can be achieved prior to the lateral sway buckling of the frame.

Apparently, the procedure that was introduced in the previous section for solving the critical load multiplier of the frame under proportional loading is no longer applicable. To overcome the difficulty due to non-proportional loading, mathematical programming methods can be adopted to determine the minimum and the maximum magnitude of the total loads and corresponding load patterns. By applying mathematical programming methods to non-proportional loading cases, the most critical buckling loads, or the so-called lower-bound of buckling loads, which correspond to the worst load patterns and the minimum capacity of the frame, can be determined through solving a constrained minimization problem. The upper-bound of the buckling loads, which corresponding to the most favourable load patterns and the maximum capacity of the frame, can be solved as a constrained maximization problem. The lower and upper bounds of the buckling loads and their associated load patterns have clearly characterized lateral buckling of the frame under extreme conditions, which is crucial to evaluate the strength of the frame.

In general, the problem of the minimum and maximum buckling loads and their associated load patterns subject to the storey-based buckling of the frame can be stated as follows:

$$\begin{matrix}\text{Maximize} \\ \text{Minimize}\end{matrix}: \quad Z = \sum_{i=1}^{n} P_i \tag{27a}$$

$$\text{Subject to:} \quad \sum_{i=1}^{n} S_i = 12 \sum_{i=1}^{n} \left(\frac{EI_i}{L_i^3}\beta_{0i} - \frac{P_i}{L_i}\beta_{1i} \right) = 0 \tag{27b}$$

$$P_{li} \le P_i \le P_{ui} = \frac{\pi^2 EI_i}{L_i^2} \qquad (i = 1,\ 2,\ \ldots n) \tag{27c}$$

Where, the objective function Z that corresponds to either the minimum or the maximum elastic buckling loads of the frame is a linear function of variables, applied loads P_i. Eq. (27b) defines the storey-based buckling constraint imposed on the frame and is a linear function of the variables P_i. Eq. (27c) imposes a constraint on the variables such that the load to be applied on an individual column cannot exceed its upper-bound, Euler buckling load (Yura 1996), and should be greater than its lower-bound due to the dead load.

The problem stated in Eqs. (27) is a linear programming problem. Because a single portal frame such as shown in Fig. 3 involves only two variable loads P_1 and P_2, the solutions of Eqs. (27) can be plotted graphically as shown in Fig. 4, in which vertexes a and b correspond to the minimum and maximum storey-based buckling loads of the frame, respectively. In the case of a frame with

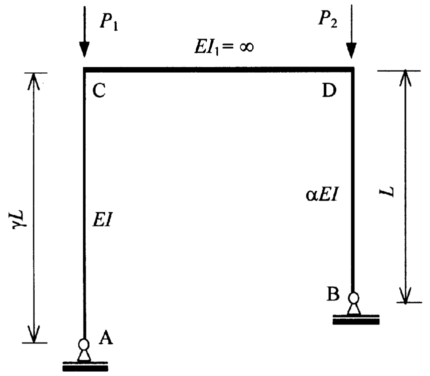

Fig. 3 Single portal frame

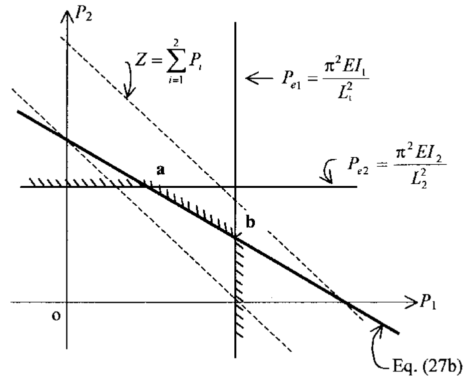

Fig. 4 Graphical solution of single portal frame

multiple columns, the graphical solutions are not available, and standard numerical algorithms such as the simplex method can be employed to solve the problem.

5. Numerical examples

5.1. Example 1

Shown in Fig. 5 is a two-bay single-storey frame. The moment of inertia for columns 1, 2, and 3 are I_1=28.886×10^6 mm^4, I_2=94.485×10^6 mm^4, I_3=76.586×10^6 mm^4, respectively, while column base end-fixity factors are r_1, r_2 and r_3. The moment of inertia for both beams are I_4=I_5=1361.2×10^6 mm^4, and the end rotational conditions of the beams are characterized by the end-fixity factors r_{4L}, $r_{4R,}$ r_{5L} and r_{5R}. Youngs Modulus is E=200,000 Mpa. The storey-based frame buckling analysis is carried out for the following cases.

Case 1: Consider the case that column bases are pinned (r_1=r_2=r_3=0), and the beams are rigidly connected to the columns (r_{4L}=r_{4R}=r_{5L}=r_{5R}=1). This is the case that has been studied by other researchers for proportional loading (LeMessurier 1977; Shanmugam and Chen 1995). The coefficients that relate to the column lateral stiffness β_0 and β_1 are evaluated based on Eqs. (19) and (20) and listed in Table 1. Listed also in Table 1 are coefficients a_i and b_i calculated based on Eqs. (23) to be used

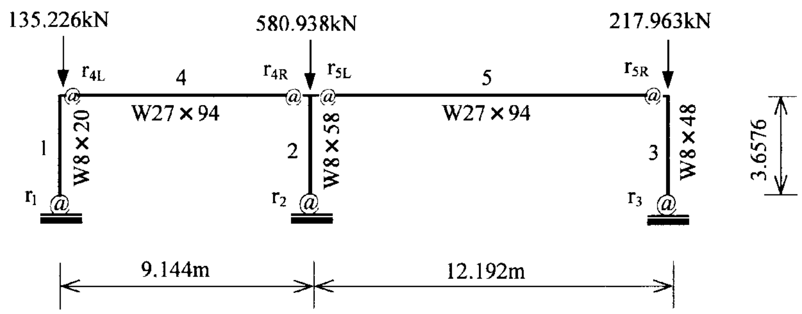

Fig. 5 Example 1: Two-bay and one story frame

Table 1 Example 1: Calculating values of coefficients for Case 1

Column	r_l	r_u	β_0	β_1	a_i	b_i	P_{ei} (kN)
1	0	0.9742	0.2435	0.0995	3.6773	28.7540	4262
2	0	0.9504	0.2376	0.0990	15.721	91.7606	13940
3	0	0.9049	0.2260	0.0980	5.8424	70.7529	11300

for solving the critical load multiplier λ_{cr} under proportional loading.

For the proportional loading case, the value of the critical load multiplier λ_{cr} solved by Eq. (25) is 7.61 and the value obtained by LeMessurier is 7.57. For the non-proportional loading case, by setting the lower bounds of loads to be zero, Eqs. (27) yields

Minimize/Maximize: $Z=P_1+P_2+P_3$

Constraint: $699.58-0.0095P_1-0.0990P_2-0.0980P_3=0$

$$0 \le P_1 \le 4262$$
$$0 \le P_2 \le 13940$$
$$0 \le P_3 \le 11300$$

The results for both proportional and non-proportional loading cases are summarized in Table 2. It can be seen from Table 2 that the critical load under the proportional loading case is between the maximum and minimum values of the non-proportional loading cases as expected. For this example, the difference of the critical load between proportional and non-proportional loading is relatively small, only about 0.24%. The difference between the maximum and the minimum critical loads under non-proportional loading is 1.07%.

To investigate effects of pinned and rigid connections of columns and beams on the critical loads of the frame and the associated load patterns under non-proportional loading, the following case considers the members of the frame with four different combinations of pinned and rigid connections.

Case 2: Four different combinations of connections for the frame are shown in Fig. 6. The maximum and minimum loads associated with each combination under non-proportional loading are presented in Table 3. Frame Type-1 has all column-bases that are fully rigid and the beam-to-column connections are pin-connected. It is found that there is no difference between the maximum and minimum critical loads. The column-bases and the beam-to-column connections are all rigid in frame Type-2. The results show that the difference between the maximum and minimum critical loads is only 0.05%. For frame Type-3, the left exterior column has pinned connection for both ends; therefore, it is a lean-on column. The difference between the maximum and minimum critical loads under non-proportional loading is 2.67%. In frame Type-4, the interior column is a lean-on column and the difference between the maximum and minimum critical load reaches as high as 13.58%.

Table 2 Example 1: Critical loads of Case 1

Loading case		P_1 (kN)	P_2 (kN)	P_3 (kN)	$\sum P_i$ (kN)
Proportional loading		1028	4415	1657	7100
Non-proportional loading	Max.	0	0	7158	7158
	Min.	4262	2821	0	7083

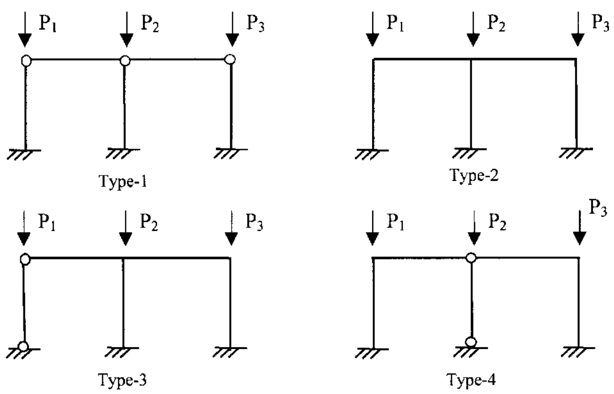

Fig. 6 Example 1: Four typical frames in case 2

As there is a considerable disparity observed between the maximum and minimum critical loads under non-proportional loading in the four frames above, a further exploration of the possible maximum difference between the maximum and minimum critical loads under non-proportional loading with respect to the end rotational condition of the column in the frame is carried out in the following case.

Case 3: Generally, if column bases and beam to column connections of a single-storey frame can be an either pinned connection or rigid connection, then there will result in $2^{(m+2n)}-1$ combinations of the frame which are geometrically stable, where m is the number of the column and n is the number of the beam. For this example, the total number of possible combinations is 127 because m is 3 and n is 2.

After searching for all 127 combinations, it is found that the maximum difference between the maximum and minimum critical loads for non-proportional loading is 20%. There are 11 combinations that result in the same maximum difference of 20% although the magnitudes and load patterns of the maximum and minimum critical loads among the 11 combinations are not identical. All of the 11 combinations are shown in Fig. 7, and it is found that there are at least one or two lean-on columns in each of the combinations. As demonstrated in Fig. 8, 54.3% of the total 127 combinations yields a difference between the maximum and minimum critical loads of less than 5% and 25.2% of the total combinations has the difference between 15% and 20%. In general, it is found that the difference between the maximum and minimum critical loads decreases as the stiffness of the column-base connection increases and so does that for the beam-to-column connection.

Table 3 Example 1: Critical loads of Case 2

Frame	Maximum (kN)				Minimum (kN)				$\dfrac{\sum_{max} P_i - \sum_{min} P_i}{\sum_{min} P_i}$ (%)
Type	P_1	P_2	P_3	$\sum_{max} P_i$ (kN)	P_1	P_2	P_3	$\sum_{min} P_i$ (kN)	
1	2491	2491	2491	7473	2491	2491	2491	7473	0
2	3440	13940	11300	28680	4262	13940	10464	28666	0.05
3	4262	9329	11300	24891	0	13940	10304	24244	2.67
4	0	13940	2313	16253	4262	0	10048	14310	13.58

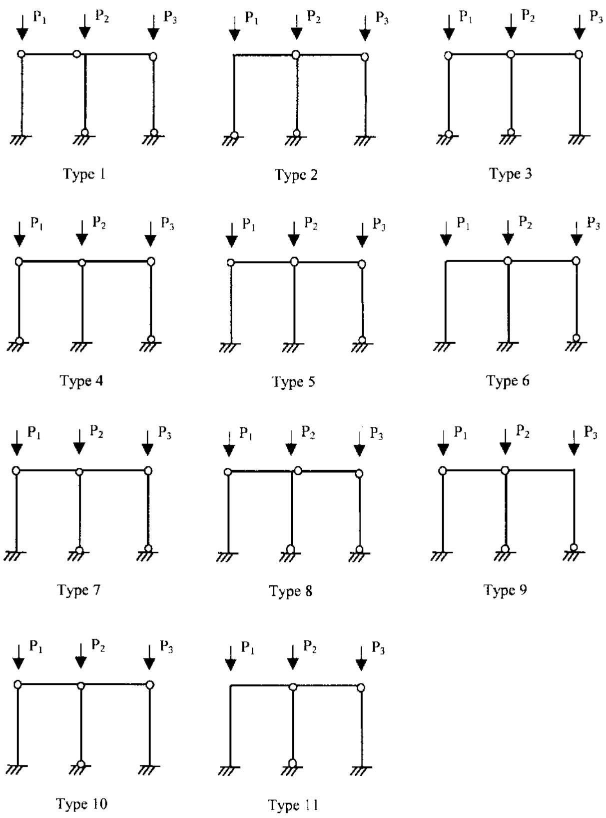

Fig. 7 Example1: 11 frames of case 3

5.2. Example 2

The stability of the four-bay single-storey building shown in Fig. 9 was studied by Yura and Helwig (1996) under the proportional loading case. The moment of inertia for exterior columns are

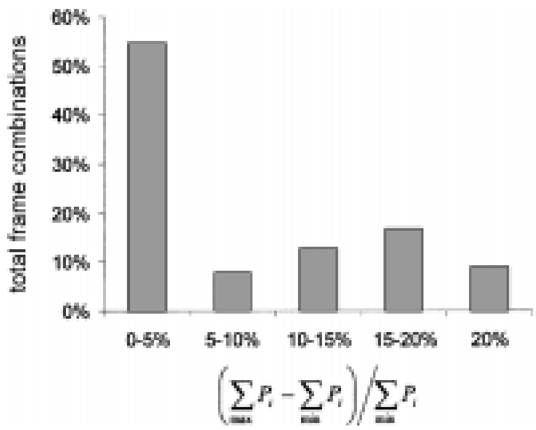

Fig. 8 Example 1: Distribution of difference between the max. and min. critical loads

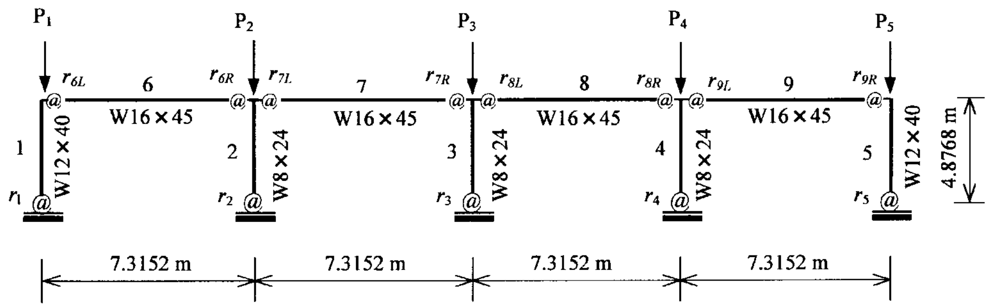

Fig. 9 Example 2: Four-bay and one-storey frame

$I_1=I_5=129\times10^6$ mm^4 while $I_2=I_3=I_4=34.1\times10^6$ mm^4 for interior columns. The moment of inertia of beams are $I_6=I_7=I_8=I_9=245\times10^6$ mm^4. Youngs Modulus is E=200,000 Mpa. The end-fixity factors r_1 to r_5 are associated with columns while r_6 to r_9 are associated with beams.

Case 1: Considering the frame studied by Yura and Helwig (1996), the column-bases are all pinned and the beam-to-column connections for exterior columns and interior columns are rigid and pinned, respectively. The initial loads are $P_1=P_5$=311.4 kN, $P_2=P_3=P_4$=444.8 kN. The coefficients associated with Eqs. (27) are

$$\sum_{1}^{5}\frac{EI_i}{L_i^2}\beta_{0i}=303.05,\ \beta_{11}=\beta_{15}=0.0916;\ \beta_{12}=\beta_{13}=\beta_{14}=0.08333\,,$$

The Eulerbuckling loads of the columns are

$$P_{e1}=P_{e5}=\frac{\pi^2 EI}{L^2}=10707\text{ kN},\ P_{e2}=P_{e3}=P_{e4}=\frac{\pi^2 EI}{L^2}=2830\text{ kN}\,.$$

For the proportional loading case, the critical load multiplier λ_{cr} solved by Eq. (25) is 1.8, which yields the total critical load of 3523 kN for the frame. For the non-proportional loading case, the maximum and minimum critical loads are obtained by solving the following problem:

Minimize/Maximize: $Z=P_1+P_2+P_3+P_4+P_5$

Subject to: $303.05–0.0916P_1–0.0833P_2–0.0833P_3–0.0833P_4–0.0916P_5=0$

$$0\le P_1\le 10707$$
$$0\le P_2\le 2830$$
$$0\le P_3\le 2830$$
$$0\le P_4\le 2830$$
$$0\le P_5\le 10707$$

The results for both proportional and non-proportional loading cases are summarized in Table 4. The critical load of the proportional loading case lies between the maximum and minimum critical loads of the non-proportional loading case. The difference between the proportional and non-proportional loading case is about 6.3%, while the difference between the maximum and the minimum critical loads under the non-proportional loading case is 9.9%. For non-proportional loading, three more cases are investigated:

Case 2: Four frames shown in Fig. 10 are investigated. For frame Type-1, there is no difference between the maximum and minimum critical loads. The linear programming result shows that the solutions are unique, which indicates that Type-1 is the only combination that has the same magnitude

Table 4 Example 2: Critical loads of Case 1

Loading Case		P_1 (kN)	P_2 (kN)	P_3 (kN)	P_4 (kN)	P_5 (kN)	$\sum P_i$ (kN)
Proportional		560	800	800	800	560	3520
Non-proportional	Max.	0	1213	1213	1213	0	3639
	Min.	1655	0	0	0	1656	3311

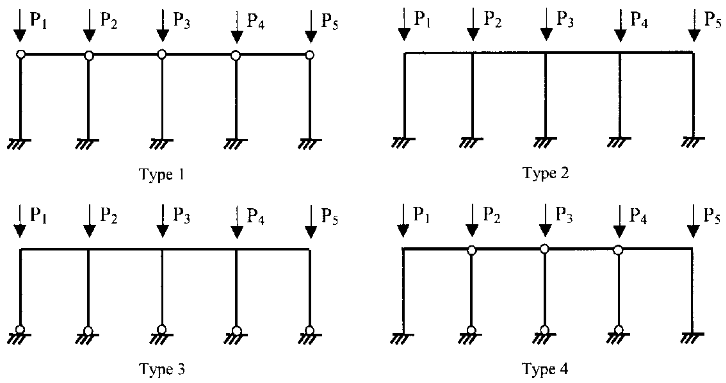

Fig. 10 Example 2: Four typical frames in case 2

for the maximum and minimum critical loads under non-proportional loading. The difference for frame Type-2 is only 0.748%, and the solutions are not unique. For frames Type-3 and Type-4, the difference between the maximum and minimum value of critical loads are 4.8% and 6.6%, respectively, and there are no unique solutions. The results of the four frames are listed in Table 5.

Case 3: As that has been done in Example 1, an investigation is carried out to find the maximum difference between the maximum and minimum critical loads with all possible combinations of

Table 5 Example 2: Critical loads of Case 2

Structure		Type 1	Type 2	Type 3	Type 4
Minimum (kN)	P_1	1515	10707	0	7120
	P_2	1515	2830	1998	0
	P_3	1515	2830	1998	0
	P_4	1515	2830	1998	0
	P_5	1515	6151	0	7120
	$\sum P_i$	7575	25340	5994	14240
Maximum (kN)	P_1	1515	10707	3140	3346
	P_2	1515	0	0	2830
	P_3	1515	1295	0	2830
	P_4	1515	2830	0	2830
	P_5	1515	10707	3140	3346
	$\sum P_i$	7575	25539	6280	15182
$\left(\sum_{\max} P_i - \sum_{\min} P_i\right) / \sum_{\min} P_i$	(%)	0	0.75	4.8	6.6

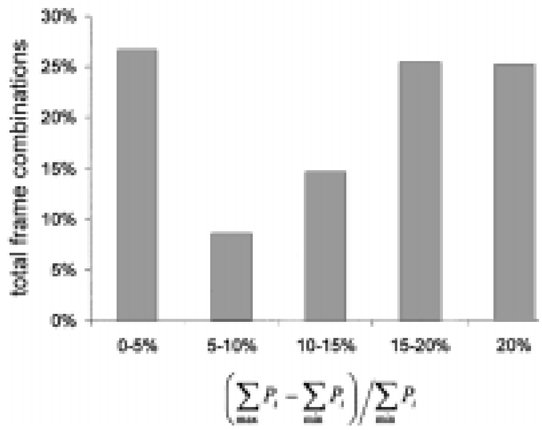

Fig. 11 Example 2: Distribution of difference between the max. and min. critical loads

pinned and rigid connection for columns and beams. With all of $2^{(m+2n)}-1=2^{(5+2\times4)}-1=8191$ combinations, the obtained results are somewhat similar to that of Example 1. The maximum difference between the maximum and minimum critical loads under non-proportional loading is 20.0%. As that demonstrated in Fig. 11, 26.6% of the total 8191 combinations yield a difference between the maximum and minimum critical loads of less than 5%, while 50.4% of the total combinations have the difference between 15% and 20%.

Case 4: Consider that in practice, the dead loads are always exiting on columns and only live loads vary from time to time. Based on the assumption that the spacing of the frame is 6 m, the tributary areas for the interior and exterior columns are 43.89 m^2 and 21.45 m^2, respectively, with an estimated dead load of 3.4 kPa (125-mm concrete slab: 3.065 kPa, sub-flooring: 0.050 kPa, ceiling and lights, etc: 0.095 kPa and moveable partitions: 0.190 kPa). The dead loads for the interior and

Table 6 Example 2: Critical loads of Case 4

Structure		Type 1	Type 2	Type 3	Type 4
Minimum (kN)	P_1	1515	10707	100	10707
	P_2	1515	2830	2830	150
	P_3	1515	2830	2823	150
	P_4	1515	2830	150	150
	P_5	1515	6151	100	3133
	$\sum P_i$	7575	25348	6003	14290
Maximum (kN)	P_1	1515	10707	5710	6592
	P_2	1515	150	150	2830
	P_3	1515	1145	150	2830
	P_4	1515	2830	150	2830
	P_5	1515	10707	100	100
	$\sum P_i$	7575	25539	6260	15182
$\left(\sum_{max} P_i - \sum_{min} P_i\right) / \sum_{min} P_i$ (%)		0	0.75	4.11	6.25

exterior columns are 150 kN and 100 kN (considering some attachment to the exterior columns), respectively.

Reanalyze the four frames that have been discussed in Case 2 in this example by setting the lower bounds in Eq. (27c) as the dead loads 100 kN and 150 kN for exterior columns and interiors columns, respectively. The obtained maximum and minimum critical loads under non-proportional live loads are listed in Table 6. Compared with Table 5, there is a slight reduction in the difference between the maximum and the minimum critical loads.

6. Conclusions

This paper discusses the elastic stability of unbraced frames under non-proportional loading based on the concept of storey-based buckling. With the derivation of the lateral stiffness of an axially by loaded column in an unbraced frame, the procedure of evaluation of the critical load of the frame under proportional loading is described. For the case of non-proportional loading, which takes into consideration the volatility of live loads, the problem of lateral buckling of unbraced frames is presented as a minimization problem subject to stability constraints on applied loads. To overcome the difficulty associated with non-proportional loading, a linear programming method was adopted to determine the most critical loads and the associated load patterns for the frame. In addition to determining the most critical loads which correspond to the worst load patterns and the minimum critical load to cause lateral buckling of the frame, the same procedure was adopted to determine the maximum critical load and the most associated favourable load pattern for the frame by revising the problem to a maximization problem. These minimum and maximum loads represent the lower and upper bounds for the frame buckling under non-proportional loading and the solution obtained from proportional loading is to be within the lower and upper bounds.

To illustrate the characteristics of frame buckling under non-proportional loading, two examples are presented to compare the critical loads of the frame with those of the proportional loading case. Numerical studies are also carried out to investigate the effects of pinned and rigid connections of columns and beams to the lower and upper bounds of the frame and their associated load patterns under non-proportional loading.

From this study, it is found that both the maximum and minimum critical loads are directly related to load patterns. In both examples, the critical loads associated with proportional loading are always between the maximum and the minimum loads under non-proportional loading. To examine the buckling behaviour of different frames under non-proportional loading, the maximum difference between the maximum and minimum critical loads and their associated load patterns are determined through searching all combinations with variations of end connections of columns and beams to be either rigid or pinned.

With given frame dimensions, beam and column sizes, the magnitudes of the critical loads are dependent on the connection rigidity, i.e., the beam to column rotational restraint. The differences between the maximum and minimum critical loads, however, are related to the distribution of the connection rigidity and flexibility of the frame. For example, for conventional rigid frames in which both end-connections of beams and columns are rigid connections, the end-fixity factors all of connections are one. Therefore, the connection rigidity is evenly distributed and the differences between the maximum and minimum critical loads for both examples presented herein are found to be within 1%. The marginal difference indicates that the load capacity of the frame is almost invariant regardless of the variation of load patterns. Therefore, the current proportional loading approaches are practically sufficient to evaluate the critical loads of such frames. In fact, one type of unbraced frame, in which column bases are all rigidly connected and beam-to-column connections are all pinned, has no difference between the maximum and minimum critical loads since the beam to column rotational restraint is zero. This has also been found through the uniqueness of the linear programming solutions herein for both the maximum and minimum critical loads. Therefore, theoretically speaking, this is the only case that the critical load of the frame is independent to the variation of load patterns. In other words, this is the only case that the proportional loading solution is fully validated.

However, large differences between the maximum and minimum critical loads are found in flexible frames with considerable number of pinned connections. For example, for frames with pinned column bases, the differences between the critical loads exceed 10% for almost all of the combinations in which the beam-to-column connection can be either pinned or rigid. For frames with at least one column base rigidly connected, large differences occur between the critical loads when there is only one or no rigid beam-to-column connection. The maximum difference between the maximum and minimum critical loads is 20.0% for both examples presented herein. Frames that are associated with the maximum difference between the maximum and minimum critical loads all have at least one lean-on column. This may suggest that evaluating the critical load based on proportional loading may not be adequate to assess the load capacity of flexible frames with lean-on columns. Therefore, the non-proportional loading approach is recommended for such frames in design practice.

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