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Thick laminated circular plates on elastic foundation subjected to a concentrated load

Hongyu Sheng†

Department of Architectural Engineering, Hefei University of Technology, Hefei, Anhui 230009, China

Abstract. In this study, the state equation for axisymmetric bending of laminated transversely isotropic circular plates on elastic foundation is established on the basis of three-dimensional elasticity. By using the expansions of Bessel functions, an analytical solution of the problem is presented. As a result, all the fundamental equations of three-dimensional elasticity can be satisfied exactly and all the independent elastic constants can be fully taken into account. Furthermore, the continuity conditions at the interfaces of plies can also be satisfied.

Key words: laminated circular plate; state equation; elastic foundation; analytical solution.

1. Introduction

As it is well known, the analysis of plates and shells on elastic foundations is becoming increasingly important in engineering. The majority of published work in this area used either the thin plate theory (Timoshenko and Woinowsky 1959) or some improved thick plate theories (Frederick 1956). Svec (1976) investigated thick plates by using FEM. Henwood *et al.* (1982) developed a Fourier series solution for rectangular thick plates. Katsikadelis and Armenàkas (1984) presented a Boundary Integral Element Method for thin circular plates. Reddy (1981, 1984) developed some new approaches to study linear, nonlinear problems of rectangular plates. Celep (1988) solved an isotropic circular thin plate resting freely on Winkler foundation. The free vibration problem was investigated by Prathap and Varadan (1976). Nath (1982) discussed large amplitude response of circular plates on elastic foundations.

In the above studies, the formulas were developed based on some hypotheses, e.g. by assuming that the mechanical quantities were polynomials of a certain coordinate variable. It can be shown that the exact solution of the problems cannot be in the form of polynomials. If the form of a polynomial is adopted, the incompatibility among fundamental equations of elasticity must appear in the deductive process and only some of the elastic constants can be taken into account. The errors caused by these hypotheses increased rapidly as plate thickness increased.

With no assumptions regarding to displacement models and stress distributions, Fan and Ye (1990), Fan (1996), and Sheng and Fan (1997) used the Fourier series as basic solutions and introduced the theory of state space. The state equations for laminated rectangular plates and cylindrical shells having arbitrary thickness and general boundary conditions were established. The

† Associate Professor

solution obtained satisfies all the fundamental equations of elasticity, as well as the continuity conditions between plies of the laminates.

In the present study, the state equation for laminated transversely isotropic circular plates on elastic foundations in the cylindrical coordinate system is established. The concentrated load acting on the plate surface is expanded into Bessel's series. Meanwhile, the Bessel functions are used to form the solution of the problem. Numerical results are obtained and compared with those of BIEM and FEM.

2. Establishment and solution of the state equation

Consider a circular plate of transversely isotopic materials. The principal elastic directions of the plate coincide with coordinate axes. The coordinate origin is located at the center of the upper surface, and the z-axis is directed vertically downward. Let U and W denote displacement along radius r and z directions, respectively. Hence, strain-stress relations and the equilibrium equation of the circular plate can be shown as follows

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 \\ C_{12} & C_{11} & C_{13} & 0 \\ C_{13} & C_{13} & C_{33} & 0 \\ 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \partial U/\partial r \\ U/r \\ \partial W/\partial z \\ \partial W/\partial r + \partial U/\partial z \end{Bmatrix} \tag{1}$$

$$\begin{cases} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{\partial \tau_{rz}}{\partial z} + \dfrac{\sigma_r - \sigma_\theta}{r} = 0 \\ \dfrac{\partial \tau_{rz}}{\partial r} + \dfrac{\partial \sigma_z}{\partial z} + \dfrac{\tau_{rz}}{r} = 0 \end{cases} \tag{2}$$

Let $\alpha=\partial/\partial r$, $C_1=-C_{13}/C_{33}$, $C_2=C_{11}-C_{13}^2/C_{33}$, $C_3=C_{12}-C_{13}^2/C_{33}$, $C_4=1/C_{33}$, $C_5=1/C_{44}$, $R=\tau_{rz}$ and $Z=\sigma_z$, the third row of Eg. (1) gives

$$\frac{\partial W}{\partial z} = C_1\left(\alpha + \frac{1}{r}\right)U + C_4 Z \tag{3}$$

The expressions of σ_r and σ_θ can be obtained from Eqs. (1) and (3) as follows:

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix} = \begin{bmatrix} C_2\alpha + C_3/r & -C_1 \\ C_3\alpha + C_2/r & -C_1 \end{bmatrix} \begin{Bmatrix} U \\ Z \end{Bmatrix} \tag{4}$$

Substituting Eq. (4) into Eq. (2) and considering Eq. (3) and the fourth row of Eq. (1) yield the following state equation

$$\frac{\partial}{\partial z} \begin{Bmatrix} U \\ W \\ R \\ Z \end{Bmatrix} = \begin{bmatrix} 0 & -\alpha & C_5 & 0 \\ C_1(\alpha + 1/r) & 0 & 0 & C_4 \\ -C_2\alpha(\alpha + 1/r) & 0 & 0 & C_1\alpha \\ 0 & 0 & -(\alpha + 1/r) & 0 \end{bmatrix} \begin{Bmatrix} U \\ W \\ R \\ Z \end{Bmatrix} \tag{5}$$

Let

$$\begin{cases} U=\sum_m U_m(z)J_1(\xi_m r)+f(r)\bar{U}(z), & W=\sum_m W_m(z)J_0(\xi_m r) \\ R=\sum_m R_m(z)J_1(\xi_m r), & Z=\sum_m Z_m(z)J_0(\xi_m r) \end{cases} \tag{6}$$

in which $\bar{U}(z)$ is an unknown function of z and $f(r)$ is a given function of r which satisfies $f(0)=0$. In $\xi_m=K_m/b$, b is the radius of the circular plate and K_m (m=1, 2, $\cdots$) are zeros of Bessel function of zero order. It can be seen that Eq. (6) satisfies $U|_{r=0}=0$ and $W|_{r=b}=0$ for axisymmetric bending of simply supported or clamped plates. The unknown function $\bar{U}(z)$ can be determined from the remaining boundary conditions; for example, the condition for simply supported circular plate is $\sigma_r|_{r=b}=0$. Substituting Eq. (6) into Eq.(4) and simplifying it by using Bessel function's properties, the following is obtained.

$$\sum_m U_m(z)J_1(K_m)+\frac{C_3f(b)+C_2bf'(b)}{C_3-C_2}\bar{U}(z)=0 \tag{7}$$

In order to represent the state equation in the form of ordinary differential equations, the following series expansions are used.

$$f(r)=\sum_m A_mJ_1(\xi_m r),\ \left(\alpha+\frac{1}{r}\right)f(r)=\sum_m B_mJ_0(\xi_m r) \tag{8}$$

The coefficients A_m and B_m can be determined accordingly, e.g. when $f(r)=r/b$, one has

$$A_m=\frac{4}{K_m^2J_1(K_m)},\qquad B_m=\frac{4}{bK_mJ_1(K_m)} \tag{9}$$

Introducing Eqs.(6) and (8) into Eq.(5) and using the properties of Bessel functions yield the following equation for each m

$$\frac{d}{dz}\boldsymbol{S}(z)=\boldsymbol{D}\boldsymbol{S}(z)+\boldsymbol{B}(z) \tag{10}$$

Eq. (10) is called a non-homogeneous state equation with constant coefficients, in which

$$\boldsymbol{S}(z)=[U_m(z)\quad W_m(z)\quad R_m(z)\quad Z_m(z)]^T \tag{11}$$

$$\boldsymbol{D}=\begin{bmatrix} 0 & \xi_m & C_5 & 0 \\ C_1\xi_m & 0 & 0 & C_4 \\ C_2\xi_m^2 & 0 & 0 & -C_1\xi_m \\ 0 & 0 & -\xi_m & 0 \end{bmatrix},\qquad \boldsymbol{B}(z)=\begin{Bmatrix} -A_m\bar{U}'(z) \\ C_1B_m\bar{U}(z) \\ 0 \\ 0 \end{Bmatrix} \tag{12}$$

It can be proved from Eq. (10) that no mechanical quantity can be a polynomial of coordinate z. If $W_m(z)$ and $R_m(z)$ are assumed to be the polynomials of degree n of variable z, from the first and fourth rows of Eq. (10), $U_m(z)$ and $Z_m(z)$ would be the polynomials of degree n+1 of z. As a result, from the second and third rows of Eq. (10), $W_m(z)$ and $R_m(z)$ would have to be the polynomials of degree n+2 of z, which contradicts what has been assumed. The solution of Eq. (10) is (Fan 1996)

$$\boldsymbol{S}(z)=\boldsymbol{G}(z)\boldsymbol{S}(0)+\boldsymbol{C}(z) \tag{13}$$

$$\boldsymbol{C}(z)=\int_0^z e^{D(z-\tau)}\boldsymbol{B}(\tau)d\tau \tag{14}$$

Letting λ_1, λ_2, λ_3, and λ_4 be the eigenvalues of matrix $\boldsymbol{D}$ and $\boldsymbol{P}$ the matrix composed of the corresponding eigenvectors. From the linear algebra one has.

$$\boldsymbol{G}(z)=e^{Dz}=\boldsymbol{P}\begin{bmatrix} e^{\lambda_1 z} & & \\ & \ddots & \\ & & e^{\lambda_4 z} \end{bmatrix}\boldsymbol{P}^{-1} \tag{15}$$

3. Solution of the laminated circular plate on an elastic foundation

Consider a p-piled laminated circular plate composed of transversely isotropic layers. The thickness of layer j is h_j. After dividing the jth layer into K_j thin plies, the thickness of the ply is $d_j=h_j/K_j$. If the thin ply is thin enough, it is reasonable to assume that the unknown function $\overline{U}(z)$ within the thin ply is linearly distributed in z direction, i.e.

$$\overline{U}_{ji}(z)=E_{ji}\left(1-\frac{z}{d_j}\right)+E_{j,i+1}\frac{z}{d_j} \qquad z\in[0,\ d_j],\ i=1,\ 2,\ \ldots,\ K_j \tag{16}$$

where the E_{ji} are end values of the linear function and subscripts ji denote ith thin ply in the jth layer of the plate. If a layer of the laminated plate is very thin, division is not needed. If some layers are relatively thick, the number of division depends on the accuracy required. Therefore, the errors caused by assumption (16) are controllable. According to Eqs. (10) and (16), the state equation for any thin ply can be written as

$$\frac{d}{dz}\boldsymbol{S}_{ji}(z)=\boldsymbol{D}_j\boldsymbol{S}_{ji}(z)+\boldsymbol{B}_{ji}(z) \tag{17}$$

$$\boldsymbol{B}_{ji}(z)=\left[A_m\frac{E_{ji}-E_{j,i+1}}{d_j} \quad C_1B_m\overline{U}_{ji}(z)\ 0\ \ 0\right]^T \tag{18}$$

From Eqs. (14) -(16), the solution of Eq. (17) is

$$\boldsymbol{S}_{ji}(z)=\boldsymbol{G}_{ji}(z)\boldsymbol{S}_{ji}(0)+\boldsymbol{C}_{ji}(z) \tag{19}$$

Introducing continuity conditions between thin plies, the mechanical quantities at the bottom surface of the K_jth thin ply and the top surface of the first thin ply are linked by followings

$$\boldsymbol{S}_{jK_j}(d_j)=\boldsymbol{H}_{jK_j}S_{j1}(0)+\overline{\boldsymbol{H}}_{jK_j}=[G_j(d_j)]^{K_j}S_{j1}(0)+\overline{\boldsymbol{H}}_{jK_j} \tag{20}$$

$$\bar{\boldsymbol{H}}_{jK_j}=[\boldsymbol{G}_j(d_j)]^{K_j-1}\boldsymbol{C}_{j1}(d_j)+\cdots+\boldsymbol{G}_j(d_j)\boldsymbol{C}_{j,K_j-1}(d_j)+C_{jK_j}(d_j) \tag{21}$$

Again, according to the continuity condition between the layers, the mechanical quantities at the bottom surface of the laminated plate can be finally expressed as

$$\boldsymbol{S}_{pK_p}=\Pi S_{11}(0)+\bar{\Pi}=\boldsymbol{H}_{pK_p}\cdots\boldsymbol{H}_{2K_2}\boldsymbol{H}_{1K_1}\boldsymbol{S}_{11}(0)+\bar{\Pi} \tag{22}$$

$$\bar{\Pi}=\boldsymbol{H}_{pK_p}\ldots\boldsymbol{H}_{2K_2}\bar{\boldsymbol{H}}_{1K_1}+\boldsymbol{H}_{pK_p}\ldots\boldsymbol{H}_{3K_3}\bar{\boldsymbol{H}}_{2K_2}+\cdots+\boldsymbol{H}_{pK_p}\bar{\boldsymbol{H}}_{p-1,K_{p-1}}+\bar{\boldsymbol{H}}_{pK_p} \tag{23}$$

$\boldsymbol{S}_{11}(0)$ in Eq. (22) is the mechanical quantity on the top surface of the plate and is called initial value. From the second row of Eq. (22), one has

$$W_m(d_p)=\pi_{21}U_m(0)+\pi_{22}W_m(0)+\pi_{23}R_m(0)+\pi_{24}Z_m(0)+\bar{\pi}_2 \tag{24}$$

If the plate is loaded by a concentrated force P at centre of the top surface and e denotes the spring constant of the Winkler elastic foundation, we have $R_m(0)=R_m(d_p)=0$, $Z_m(d_p)=-eW_m(d_p)$. Using the third and forth rows in matrix Eq. (22) and considering Eq. (24), one has

$$\begin{Bmatrix} U_m(0) \\ W_m(0) \end{Bmatrix}=-\begin{bmatrix} \pi_{31} & \pi_{32} \\ \pi_{41}+e\pi_{21} & \pi_{42}+e\pi_{22} \end{bmatrix}^{-1}\left[Z_m(0)\begin{Bmatrix} \pi_{34} \\ \pi_{44}+e\pi_{24} \end{Bmatrix}+\begin{Bmatrix} \bar{\pi}_3 \\ \bar{\pi}_4+e\bar{\pi}_2 \end{Bmatrix}\right] \tag{25}$$

To obtain $Z_m(0)$, the concentrated force P must be expanded into Bessel's series. For this purpose, we construct an auxiliary function as follows:

$$\bar{\sigma}_z(r,\,0,\,\varepsilon)=\begin{cases} 0 & \text{When } r>\varepsilon \\ -\dfrac{P}{\pi\varepsilon^2} & \text{When } r\le\varepsilon \end{cases} \tag{26}$$

and the true distribution of σ_z on the top surface is

$$\sigma_z(r,\,0)=\lim_{\varepsilon\to 0}\bar{\sigma}_z(r,\,0,\,\varepsilon) \tag{27}$$

Expanding $\bar{\sigma}_z$ into Bessel's series, we obtain

$$\bar{\sigma}_z(r,\,0,\,\varepsilon)=\sum_m C_m(\varepsilon)J_0(\xi_m r), \qquad C_m(\varepsilon)=-2\frac{PJ_1(\xi_m\varepsilon)}{\pi b^2\xi_m\varepsilon J_1^2(K_m)} \tag{28}$$

Comparing Eq. (28) with Eq. (6) yields

$$Z_m(0)=\lim_{\varepsilon\to 0}C_m(\varepsilon)=-2\frac{P}{\pi b^2 J_1^2(K_m)} \tag{29}$$

Thus, $\boldsymbol{S}_{11}(0)$ can be determined from Eqs. (25) and (29). According to the deductive process of Eq. (22), it is easy to express the mechanical quantities of the ith thin ply in jth layer of the plate as follows

$$S_{ji}(z)=\Pi_{ji}(z)S_{11}(0)+\overline{\Pi}_{ji}(z) \tag{30}$$

The expressions for matrix $\Pi_{ji}(z)$ and vector $\overline{\Pi}_{ji}(z)$ are not difficult to obtain. It should be explained here that Eqs. (25) and (30) contain unknown coefficients $E_{ji}(i=1, 2, \cdots, K_j+1, j=1, 2, \cdots, p)$. Consider the continuity of displacement $\overline{U}(z)$, the following relationships are obtained.

$$E_{j+1,1}=E_{j,Kj+1} \qquad (j=1, 2, \cdots, p-1) \tag{31}$$

Hence, there are $K_1+K_2+\cdots+K_p+1$ coefficients altogether. To solve these coefficients by using Eq.(7), the coordinate z in Eq. (30) should take the values of K_j. For example, let $z=d_j$, $2d_j$, $\cdots$, h_j, calculate the corresponding mechanical quantities respectively and substitute $[U_m(z)]_{ji}$ into Eq. (7). Let $j=1, 2, ..., p$ and consider the initial value $S_{11}(0)$ in Eq. (25), we can obtain $K_1+K_2+\cdots+K_p+1$ algebraic equations which are used to solve the same number of unknowns. Thus the initial value $S_{11}(0)$ can be found by Eq. (25) and the problem is solved.

4. Numerical results

4.1. Example 1

In order to examine the effectiveness of the present method, a simply supported isotropic circular plate on elastic foundation was considered first. The plate is subjected to a concentrated force P at the top centre surface of the plate. Numerical results are shown in Table 1, where $D=Eh^3/[12(1-$

Table 1. Stress and deflections for isotopic circular plate on elastic foundation (λ=7)

β	K	z	$\overline{\overline{W}}$ (r=0.0)	$\overline{\overline{W}}$ (r=0.2b)		$\overline{W}$ (r=0.6b)		$\overline{\sigma}_\theta$ (r=b)	
			Present ($\times10^{-2}$)	Present ($\times10^{-2}$)	BEIM ($\times10^{-2}$)	Present ($\times10^{-4}$)	BIEM ($\times10^{-4}$)	Present ($\times10^{-3}$)	BIEM ($\times10^{-3}$)
0.01	4	0.0	0.2573	0.1175	0.1175	-0.2216	-0.2217	0.8805	0.8748
		0.5h	0.2567	0.1175	0.1175	-0.2218	-0.2217	0.0094	0.0000
		1.0h	0.2562	0.1175	0.1175	-0.2216	-0.2217	-0.8611	-0.8748
0.05	8	0.0	0.4627	0.1172	0.1175	-0.2206	-0.2217	1.0921	0.8748
		0.5h	0.2812	0.1174	0.1175	-0.2262	-0.2217	0.2398	0.0000
		1.0h	0.2694	0.1172	0.1175	-0.2198	-0.2217	-0.6047	-0.8748
0.2	12	0.0	13.441	0.1282	0.1175	-0.1836	-0.2217	4.4581	0.8748
		0.5h	0.4722	0.1331	0.1175	-0.2429	-0.2217	3.8470	0.0000
		1.0h	0.2961	0.1158	0.1175	-0.1493	-0.2217	3.2922	-0.8748
0.4	16	0.0	106.41	0.4396	0.1175	1.3950	-0.2217	13.672	0.8748
		05h	1.0201	0.4001	0.1175	0.7975	-0.2217	15.861	0.0000
		1.0h	0.1774	0.1022	0.1175	0.3501	-0.2217	17.740	-0.8748
0.6	20	0.0	359.44	1.7630	0.1175	17.801	-0.2217	14.152	0.8748
		0.5h	2.1041	1.2551	0.1175	14.162	-0.2217	38.562	0.0000
		1.0h	0.0922	0.0693	0.1175	1.1651	-0.2217	53.301	-0.8748

$\nu^2)]$, $\bar{\sigma}_\theta=\sigma_\theta h^2/p$, $\bar{W}=WD/(Pb^2)$, $\lambda=b\sqrt[4]{e/D}$, e=foundation's spring constant, E=Young's modulus, ν=Poisson's ratio, β=h/b, and K is the number of divided thin plies. The numerical results were calculated with m=1, 2, ..., 200 and compared with those of BIEM (Katiskadelis and Armenakas 1984).

4.2. Example 2

A three-plied laminated transversely isotropic circular plate is simply supported on a Winkler elastic foundation. The plate is subjected to a concentrated force P at the center of top surface of the plate. The materials of the upper and lower layers of the plate are identical. Each layer has the same ratios of elastic constants: C_{12}/C_{11}=0.246269, C_{13}/C_{11}=0.0831715, C_{33}/C_{11}=0.530172, C_{44}/C_{11}= 0.266810, $C_{11}^{(1)}/C_{11}^{(2)}$=5, $C_{11}^{(2)}h/e$=10, where $C_{11}^{(1)}$ and $C_{11}^{(2)}$ denote the values of C_{11} corresponding to the upper and middle layers respectively. The geometric parameters are h_1=h_3=0.1h and h_2=0.8h, where h is the thickness of the laminated plate. The convergence rate using different K_j and series terms m are illustrated in Tables 2 and 3, in which $\bar{U}=UC_{11}^{(2)}h/p$, $\bar{\sigma}_r=\sigma_r h^2/p$ *et al.* are

Table 2. Convergence for the present solution with different number of thin plies (m=200, z=0.0, $\beta=0.4$)

K_1	K_2	K_3	$\bar{U}$ $(r=0.2b)$	$\bar{W}$ $(r=0.0)$	$\bar{\sigma}_r$ $(r=0.4b)$	$\bar{\sigma}_\theta$ $(r=0.4b)$
2	10	2	-0.25796	31.2582	1.35325	-0.42044
3	10	3	-0.25800	31.2586	1.35319	-0.42050
4	10	4	-0.25801	31.2587	1.35317	-0.42053
4	12	4	-0.25801	31.2587	1.35316	-0.42053

Table 3. Convergence for the present solution with different series terms m (K_1=K_3=4, K_2=12, $\beta=0.4$)

m		$\bar{U}$ $(r=0.2b)$	$\bar{W}$ $(r=0.0)$	$\bar{\sigma}_r$ $(r=0.4b)$	$\bar{\sigma}_\theta$ $(r=0.4b)$
160	1–	-0.25747	26.7453	1.24043	-0.46560
	1+	-0.03775	5.60112	-0.32798	-0.58331
	3+	0.13726	1.29908	0.33206	0.91696
180	1–	-0.25776	29.5020	1.32982	-0.44245
	1+	-0.03773	5.60109	-0.32797	-0.58332
	3+	0.13726	1.29908	0.33206	0.91696
190	1–	-0.25789	30.8804	1.33895	-0.43184
	1+	-0.03773	5.60107	-0.32796	-0.58332
	3+	0.13726	1.29908	0.33206	0.91696
195	1–	-0.25798	31.1982	1.34921	-0.42136
	1+	0.03773	5.60107	-0.32796	-0.58332
	3+	0.13726	1.29908	0.33206	0.91696
200	1–	-0.25801	31.2587	1.35316	-0.42053
	1+	-0.03773	5.60107	-0.32796	-0.58332
	3+	0.13726	1.29908	0.33206	0.91696

Note: j (j = 1, 2, 3) denotes jth layer, "–" and "+" denote it's top and bottom surface, respectively.

Table 4. Stresses and displacements of laminated circular plate on elastic foundation

		β=0.1, K_1=K_3=2, K_2=8		β=0.2, K_1=K_3=3, K_2=10		β=0.4, K_1=K_3=4, K_2=12	
		present	FEM	present	FEM	present	FEM
$\bar{\sigma}_r$ (r=0.4b)	1−	0.18266	0.09894	0.52823	0.19720	1.35316	0.30484
	1+	0.07459	0.07523	0.09938	0.10371	-0.32796	-0.30829
	2−	0.01494	0.01503	0.01974	0.02060	-0.06557	-0.06146
	2+	-0.01640	-0.01619	-0.02293	-0.02391	0.07051	0.06848
	3−	-0.08078	-0.07970	-0.09768	-0.10252	0.39423	0.38676
	3+	-0.10191	-0.10181	-0.18519	-0.19442	0.33206	0.31326
$\bar{\sigma}_\theta$ (r=0.4b)	1−	-0.02461	-0.00281	-0.06722	-0.17661	-0.42053	-0.86391
	1+	-0.00345	-0.00330	-0.13506	-0.13386	-0.58332	-0.57811
	2−	-0.00067	-0.00067	-0.02715	-0.02691	-0.11664	-0.11543
	2+	0.00193	0.00199	0.02873	0.02849	0.11719	0.11624
	3−	0.01089	0.01116	0.16509	0.15950	0.62759	0.62505
	3+	0.01106	0.01058	0.20404	0.19805	0.91696	0.90412
$\bar{\tau}_{rz}$ (r=0.2b)	1−	0.00000	0.00505	0.00000	0.09305	0.00000	0.41695
	1+	-0.01837	-0.00537	-0.07764	0.04350	-0.34524	-0.05594
	2−	-0.01837	-0.01647	-0.07764	-0.05484	-0.34524	-0.35334
	2+	-0.02297	-0.02007	-0.11986	-0.13039	-0.19333	-0.19509
	3−	-0.02297	-0.01190	-0.11986	-0.15160	-0.19333	-0.23370
	3+	0.00000	0.00467	0.00000	-0.03492	0.00000	-0.04608
$\bar{U}$ (r=0.2b)	1−	-0.09537	-0.09547	-0.15209	-0.15176	-0.25801	-0.26027
	1+	-0.06836	-0.06865	-0.08353	-0.08458	-0.03773	-0.03884
	2−	-0.06836	-0.06865	-0.08353	-0.08458	-0.03773	-0.03884
	2+	0.07932	0.07959	0.08985	0.08943	0.07209	0.07262
	3−	0.07932	0.07959	0.08985	0.08943	0.07209	0.07262
	3+	0.10571	0.10492	0.14746	0.14774	0.13726	0.13688
$\bar{W}$ (r=0.0)	1−	10.5522	10.6342	17.4304	16.5513	31.2587	27.5207
	1+	5.54556	5.19949	5.57250	5.28355	5.60107	5.34148
	2−	5.54556	5.19949	5.57250	5.28355	5.60107	5.34148
	2+	1.27956	1.26967	1.28662	1.22441	1.31283	1.25531
	3−	1.27956	1.26967	1.28662	1.22441	1.31283	1.25531
	3+	1.26685	1.25578	1.27388	1.30921	1.29908	1.35832

dimensionless displacements and stresses, K_1, K_2, K_3 are the thin-ply number of the three layers, respectively. Numerical results are given and compared with those of FEM in Table 4. The mesh of FEM is 4 (in z-direction) $\times$ 10 (in r-direction). Because of axial symmetry, 40 rectangular isoparametric ring elements with 8 nodes are employed in the calculations.

5. Conclusions

From the above analyses and numerical examples, we can draw the following conclusions:

(1) It can be seen from Table 1 that the numerical results of thin plate theory (h/b=0.01) and the present paper are almost identical. As β increases, the differences of the two solutions increase rapidly. Also, there exists a local central area where larger deformation occurs, and the local area expands outward in r-direction and downward in z-direction as β increases.

(2) It can be seen from Table 2 that the convergence of calculation against K_j is fast. Hence, the assumption (16) is reasonable and the present method is efficient. In Table 3, the convergence of the mechanical quantities against m is fast at the bottom surface and in the interior, but it is slower at the upper surface which is subjected to the concentrated load.

(3) It can be seen from Table 4 that the displacements and stresses in the interior of the plate calculated by present method and FEM are almost identical. However, some differences exist in the area close to the upper surface of the plate. The differences are small for displacements and significant for stresses. This is because the FEM calculates stresses by using interpolation. In addition, the stresses at the interface calculated by FEM do not satisfy the continuity conditions of the laminates.

(4) All the mechanical quantities appearing in the state equation are the compatible quantities at the interfaces, thus, the present study is extremely useful for solving laminate problems.

(5) The present method can be used to check the accuracy of approximate theories.

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