Structural Engineering and Mechanics, Vol. 10, No. 5 (2000) 427-440
DOI: http://dx.doi.org/10.12989/sem.2000.10.5.427

Derivation of formulas for perturbation analysis with modes of close eigenvalues

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Abstract. The formulas for the perturbation analysis with modes of close eigenvalues are derived in this paper. Emphasis is made on the consistency of the straightforward perturbation process, given the complete terms of perturbations in the zeroth-order, which is a form of Rayleigh quotient, and in the higher-orders. By dividing the perturbation of eigenvector into two parts, the first-order perturbation with respect to the modes of close eigenvalues is moved into the zeroth-order perturbation. The normality condition is employed to compute the higher-order perturbations of eigenvector. The algorithm can be condensed to a single mode with a distinct eigenvalue, and this can accelerate the convergence of the perturbation analysis. The example confirms that the perturbation approximation obtained from the suggested procedure is in a good accuracy on the eigenvalues, eigenvectors, and normality.

Key words: eigenvalue problem; close eigenvalues; perturbation analysis.

1. Introduction

Deriving the formulas for the perturbation analysis with the modes of close eigenvalues is demanded in two reasons. One is that the conventional method for the perturbation with a distinct eigenvalue may not give a valid result. Another reason is that the sudden change, namely the mode localization and the loci veering, on the modes of close eigenvalues from a perturbation in a dynamic system needs to be well quantitatively measured.

Perturbation analysis with repeated eigenvalues, as a special case of close eigenvalues, has been studied intensively (Courant and Hilbert 1945, Nayfeh 1973, Mills-Curran 1988, Dailey 1989, Hou and Kenny 1992, Lee *et al.* 1996). From the numerical point of view, real numbers are hardly equal and the repeated eigenvalues are the truncated values from close eigenvalues. Therefore, the study is a theoretical guideline for the rarely seen cases of repeated eigenvalues. The analysis with close eigenvalues is indeed much practical in a computational process.

There have been some references devoted to the subject. Hu (1987) applied the zeroth-order eigensolution to Rayleigh quotient to compute the perturbed modes. The quotient is a good approximation to respective modes of close eigenvalues for it is in a second-order error. This procedure can be considered as the technique of subspace, spanning on the modes with close eigenvalues. Chen and Ginsberg (1992) used the approach to examine the eigenvalue loci and the sensitivity of eigenvectors. However, this is not sufficient, as the perturbation terms from the modes

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of distinct eigenvalues need to be added.

Chen *et al.* (1993) presented a perturbation scheme, which employs the formulas of the perturbation analysis for repeated eigenvalues. By transforming the perturbation for close eigenvalues to a problem with zeroth-order repeated eigenvalues, the solution from the perturbation analysis is for the case of repeated eigenvalues. Nevertheless, this is an indirect way of perturbation analysis, requiring setting up a problem with repeated eigenvalues.

Based on Hu's work, Chen *et al.* (1995) added the perturbation terms with the modes of distinct eigenvalues and developed higher-order perturbations. Liu (1995, 1999) used a similar procedure on the reduction of eigenvalue analysis for close eigenvalues.

It is found that the previous works by Chen *et al.* (1995) and Liu (1995, 1999) lack the consistency on using Rayleigh quotient in the perturbation process. In other words, no connection of the quotient with the perturbation analysis is given, and there are jumps on deriving the algorithm. Moreover, the normality condition is not used.

The perturbation procedure given in this paper includes the complete items on the perturbation sequences. Both the eigenvalue equation and the normality condition are employed to derive the formulas. The straightforward process results in the zeroth-order perturbation, in a form of Rayleigh quotient as suggested by Hu (1987), as well as higher-order perturbations. Correction on the higher-order perturbations and more terms are given.

2. Perturbation formulas

Considered in the real state, an eigenvalue equation of a dynamic system is defined as

$$(\boldsymbol{K}^{(0)} - \lambda^{(0)}\boldsymbol{M}^{(0)})\boldsymbol{x}^{(0)}=\boldsymbol{0} \tag{1}$$

where $\boldsymbol{K}^{(0)}$ and $\boldsymbol{M}^{(0)}$ are the matrices of N dimensions, which can be the stiffness and masses in a structural dynamics. The orthonormal relationships for the eigensolution, *i.e.* n eigenpairs $\Lambda^{(0)}$ and $\boldsymbol{X}^{(0)}$, are given as

$$\boldsymbol{X}^{(0)^T}\boldsymbol{K}^{(0)}\boldsymbol{X}^{(0)}=\boldsymbol{\Lambda}^{(0)} \tag{2}$$

$$\boldsymbol{X}^{(0)^T}\boldsymbol{M}^{(0)}\boldsymbol{X}^{(0)}=\boldsymbol{I} \tag{3}$$

where $\boldsymbol{I}$ is the unit matrix.

Taking Eq. (1) as a zeroth-order problem, a small change on the dynamic system may result in a perturbation for the matrices, so that

$$\boldsymbol{K}=\boldsymbol{K}^{(0)}+\varepsilon\boldsymbol{K}^{(1)} \tag{4}$$

$$\boldsymbol{M}=\boldsymbol{M}^{(0)}+\varepsilon\boldsymbol{M}^{(1)} \tag{5}$$

where the parameter ε is small and positive, *i.e.* $0<\varepsilon<<1$, and $\boldsymbol{K}^{(1)}$ and $\boldsymbol{M}^{(1)}$ are the first-order perturbation matrices.

The eigenvalue equation of the perturbed system is then in the form of

$$(\boldsymbol{K} - \lambda\boldsymbol{M})\boldsymbol{x}=\boldsymbol{0} \tag{6}$$

The eigenvectors $\boldsymbol{X}$ from Eq. (6) are normalized as

$$X^T M X = I \tag{7}$$

As a usual perturbation technique, the solution of Eq. (6) is assumed to be an asymptotic sequence of the small parameter ε by letting

$$\lambda=\lambda^{(0)}+\varepsilon\lambda^{(1)}+\varepsilon^2\lambda^{(2)}+\varepsilon^3\lambda^{(3)}+\cdots \tag{8}$$

$$x=x^{(0)}+\varepsilon x^{(1)}+\varepsilon^2 x^{(2)}+\varepsilon^3 x^{(3)}+\cdots \tag{9}$$

It is supposed that there is at least one group of close eigenvalues in the zeroth-order eigensolution, that is p close eigenvalues $\Lambda_c^{(0)}=\text{diag}\ \{\lambda_1^{(0)},\ \lambda_2^{(0)},\ \cdots,\ \lambda_p^{(0)}\}$ and their eigenvectors $X_c^{(0)}=\{x_1^{(0)}\ x_2^{(0)}\cdots x_p^{(0)}\}$. $\Lambda_d^{(0)}$ and $X_d^{(0)}$ are the rest of eigensolution. $\Lambda_d^{(0)}$ is the eigenvalues, which are distinct from $\Lambda_c^{(0)}$ in magnitude, but may contain other groups of close eigenvalues.

Because the zeroth-order modes are divided into two parts, the perturbations of the eigenvector in Eq. (9) can then become into two. As the reason, the perturbed eigenvector for one of modes with close eigenvalues is expressed into

$$x=x^{(0)}+\varepsilon(x_c^{(1)}+x_d^{(1)})+\varepsilon^2(x_c^{(2)}+x_d^{(2)})+\varepsilon^3(x_c^{(3)}+x_d^{(3)})+\cdots \tag{10}$$

where the subscripts c and d denote the perturbations with the modes of close eigenvalues and with the modes of distinct eigenvalues.

By introducing

$$\mu=\lambda^{(0)}+\varepsilon\lambda^{(1)} \tag{11}$$

$$\varphi=x^{(0)}+\varepsilon x_c^{(1)} \tag{12}$$

Eqs. (8) and (10) can be rewritten to

$$\lambda=\mu+\varepsilon^2\lambda^{(2)}+\varepsilon^3\lambda^{(3)}+\cdots \tag{13}$$

$$x=\varphi+\varepsilon x_d^{(1)}+\varepsilon^2(x_c^{(2)}+x_d^{(2)})+\varepsilon^3(x_c^{(3)}+x_d^{(3)})+\cdots \tag{14}$$

φ and μ are considered as the perturbed zeroth-order result, for they are in the zeroth-order. Bear in mind that the perturbed eigenpair λ and x is referred to one of modes with close eigenvalues. The subscript i, where i=1, 2, …, p, omitted in Eqs. (13) and (14) for sake of a simpler expression, will be added in appropriate formulas. The perturbations of eigenvector are assumed to be expanded with the respective eigenvectors, that is

$$\varphi=x_c^{(0)}\beta \tag{15}$$

$$x_c^{(2)}=\Phi\beta^{(2)} \tag{16}$$

$$x_c^{(3)}=\Phi\beta^{(3)} \tag{17}$$

where Φ is the collection of φ. β, $\beta^{(2)}$, and $\beta^{(3)}$ are the coefficient vectors, which form the matrices $\{\beta_1,\ \beta_2,\ ...,\ \beta_p\}$, $\{\beta_1^{(2)},\ \beta_2^{(2)},\ \dots,\ \beta_p^{(2)}\}$, and $\{\beta_1^{(3)},\ \beta_2^{(3)},\ \dots,\ \beta_p^{(3)}\}$ for all p modes. Eqs. (15), (16), and (17) can be detailed for ith mode, i=1, 2, …, p, as

$$\varphi_i=\sum_{j=1}^{p}\beta_{ji}\boldsymbol{x}_j^{(0)} \tag{18}$$

$$\boldsymbol{x}_{c_i}^{(2)}=\sum_{j=1}^{p}\beta_{ji}^{(2)}\varphi_j \tag{19}$$

$$\boldsymbol{x}_{c_i}^{(3)}=\sum_{j=1}^{p}\beta_{ji}^{(3)}\varphi_j \tag{20}$$

The expansion of $\boldsymbol{x}_d^{(1)}$, $\boldsymbol{x}_d^{(2)}$, and $\boldsymbol{x}_d^{(3)}$ is given in two cases. For the expansion with the zeroth-order eigenvectors of distinct eigenvalues

$$\boldsymbol{x}_d^{(1)}=\boldsymbol{X}_d^{(0)}\gamma^{(1)} \tag{21}$$

$$\boldsymbol{x}_d^{(2)}=\boldsymbol{X}_d^{(0)}\gamma^{(2)} \tag{22}$$

$$\boldsymbol{x}_d^{(3)}=\boldsymbol{X}_d^{(0)}\gamma^{(3)} \tag{23}$$

With the modes of other groups of close eigenvalues, though distinct from the considered close eigenvalues, they have their perturbed zeroth-order eigenvectors $\boldsymbol{\Phi}_d$, so that the expansion is

$$\boldsymbol{x}_d^{(1)}=\boldsymbol{\Phi}_d\xi^{(1)} \tag{24}$$

$$\boldsymbol{x}_d^{(2)}=\boldsymbol{\Phi}_d\xi^{(2)} \tag{25}$$

$$\boldsymbol{x}_d^{(3)}=\boldsymbol{\Phi}_d\xi^{(3)} \tag{26}$$

Note that Eqs. (21) through (26) can be detailed into a form as Eqs. (18), (19), and (20).

The orthonormality relationships in Eqs. (2) and (3) can be extended to

$$\varphi^T\boldsymbol{K}^{(0)}\boldsymbol{x}_d^{(0)}=0 \tag{27}$$

$$\varphi^T\boldsymbol{M}^{(0)}\boldsymbol{x}_d^{(0)}=0 \tag{28}$$

$$\varphi^T\boldsymbol{K}^{(0)}\boldsymbol{x}_d^{(1)}=0 \tag{29}$$

$$\varphi^T\boldsymbol{M}^{(0)}\boldsymbol{x}_d^{(1)}=0 \tag{30}$$

$$\varphi_d^T\boldsymbol{K}^{(0)}\varphi=0 \tag{31}$$

$$\varphi_d^T\boldsymbol{M}^{(0)}\varphi=0 \tag{32}$$

φ is also orthogonal to the second or third-order perturbation of $\boldsymbol{x}_d$.

2.1. Zeroth-order perturbation and perturbation of eigenvalue

Substituting Eqs. (13) and (14) into Eq. (6) and expanding the equation, then inserting Eqs. (4) and (5), collecting the terms with like power of ε, and premultiplying the equation by φ^T, the equation becomes to

$$\begin{aligned}
&\varphi^T(\boldsymbol{K}-\mu\boldsymbol{M})\varphi \\
&+\varepsilon\varphi^T(\boldsymbol{K}^{(0)}-\mu\boldsymbol{M}^{(0)})\boldsymbol{x}_d^{(1)} \\
&+\varepsilon^2\varphi^T[(\boldsymbol{K}^{(1)}-\mu\boldsymbol{M}^{(1)})\boldsymbol{x}_d^{(1)}+(\boldsymbol{K}-\mu\boldsymbol{M})\boldsymbol{x}_c^{(2)}+(\boldsymbol{K}^{(0)}-\mu\boldsymbol{M}^{(0)})\boldsymbol{x}_d^{(2)}-\lambda^{(2)}\boldsymbol{M}\varphi] \\
&+\varepsilon^3\varphi^T[(\boldsymbol{K}^{(1)}-\mu\boldsymbol{M}^{(1)})\boldsymbol{x}_d^{(2)}+(\boldsymbol{K}-\mu\boldsymbol{M})\boldsymbol{x}_c^{(3)}+(\boldsymbol{K}^{(0)}-\mu\boldsymbol{M}^{(0)})\boldsymbol{x}_d^{(3)} \\
&-\lambda^{(2)}\boldsymbol{M}^{(0)}\boldsymbol{x}_d^{(1)}-\lambda^{(3)}\boldsymbol{M}\varphi] \\
&+\cdots=0
\end{aligned} \tag{33}$$

As the equation is an identity of the parameter ε, each coefficient of ε vanishes independently. After applying the normality relationships to Eq. (33) and equating the coefficients of ε, the equation splits to

$$\varphi^T(\boldsymbol{K}-\mu\boldsymbol{M})\varphi=0 \tag{34}$$

$$\varphi^T[(\boldsymbol{K}^{(1)}-\mu\boldsymbol{M}^{(1)})\boldsymbol{x}_d^{(1)}+(\boldsymbol{K}-\mu\boldsymbol{M})\boldsymbol{x}_c^{(2)}-\lambda^{(2)}\boldsymbol{M}\varphi]=0 \tag{35}$$

$$\varphi^T[(\boldsymbol{K}^{(1)}-\mu\boldsymbol{M}^{(1)})\boldsymbol{x}_d^{(2)}+(\boldsymbol{K}-\mu\boldsymbol{M})\boldsymbol{x}_c^{(3)}-\lambda^{(3)}\boldsymbol{M}\varphi]=0 \tag{36}$$

Inserting Eq. (15) for φ into Eq. (34) yields the eigenvalue

$$\mu=\frac{\boldsymbol{\beta}^T\boldsymbol{X}_c^{(0)^T}\boldsymbol{K}\boldsymbol{X}_c^{(0)}\boldsymbol{\beta}}{\boldsymbol{\beta}^T\boldsymbol{X}_c^{(0)^T}\boldsymbol{M}\boldsymbol{X}_c^{(0)}\boldsymbol{\beta}} \tag{37}$$

This is clearly Rayleigh quotient with respect to β, a vector to be determined. It is known that the quotient is a minimum if it is minimized on the vector β (Courant and Hilbert 1945). The equivalence is $\partial\mu/\partial\beta=0$, which yields an eigenvalue equation

$$(\boldsymbol{X}_c^{(0)^T}\boldsymbol{K}\boldsymbol{X}_c^{(0)}-\mu\boldsymbol{X}_c^{(0)^T}\boldsymbol{M}\boldsymbol{X}_c^{(0)})\boldsymbol{\beta}=0 \tag{38}$$

The solution of the equation is p pairs of μ and β. The normality of the vector β is

$$\boldsymbol{\beta}^T\boldsymbol{X}_c^{(0)^T}\boldsymbol{M}\boldsymbol{X}_c^{(0)}\boldsymbol{\beta}=1 \tag{39}$$

For simplicity on computation, Eqs. (38) and (39) can be rewritten to

$$[\boldsymbol{\Lambda}_c^{(0)}+\varepsilon\boldsymbol{X}_c^{(0)^T}\boldsymbol{K}^{(1)}\boldsymbol{X}_c^{(0)}-\mu(\boldsymbol{I}+\varepsilon\boldsymbol{X}_c^{(0)^T}\boldsymbol{M}^{(1)}\boldsymbol{X}_c^{(0)})]\boldsymbol{\beta}=0 \tag{40}$$

$$\boldsymbol{\beta}^T(\boldsymbol{I}+\boldsymbol{X}_c^{(0)^T}\boldsymbol{M}^{(1)}\boldsymbol{X}_c^{(0)})\boldsymbol{\beta}=1 \tag{41}$$

The orthonormality is observed as

$$\varphi_i^T\boldsymbol{K}\varphi_j=\mu_i\delta_{ij} \tag{42}$$

$$\varphi_i^T\boldsymbol{M}\varphi_j=\delta_{ij} \tag{43}$$

where $i, j = 1, 2, \ldots, p$ and $\delta_{ij}=0$ if $i\neq j$, $\delta_{ij}=1$ if $i=j$.

By inserting Eq. (16) for $x_c^{(2)}$ and Eq. (17) for $x_c^{(3)}$ into Eqs. (35) and (36) and applying the orthonormality in Eqs. (42) and (43), the second and third-order perturbations of eigenvalue are obtained as

$$\lambda_i^{(2)}=\varphi_i^T(K^{(1)}-\mu_i M^{(1)})x_{d_i}^{(1)} \tag{44}$$

$$\lambda_i^{(3)}=\varphi_i^T(K^{(1)}-\mu_i M^{(1)})x_{d_i}^{(2)} \tag{45}$$

for i=1, 2, ..., p.

2.2. Perturbation with zeroth-order eigenvector of distinct eigenvalue

The perturbation of eigenvector has been assumed to be the expansion of the known eigenvectors. In this section and the following one, the formulas to compute coefficients γ and ξ for the perturbations $x_d^{(1)}$, $x_d^{(2)}$, and $x_d^{(3)}$ will be given.

Substituting Eqs. (4), (5), (13), and (14) into Eq. (6), collecting the terms with the like power of the parameter ε, and premultiplying $x_d^{(0)^T}$, yields

$$\begin{aligned}&x_d^{(0)^T}(K^{(0)}-\mu M^{(0)})\varphi\\&\quad+\varepsilon x_d^{(0)^T}(K^{(1)}-\mu M^{(1)})\varphi+(K^{(0)}-\mu M^{(0)})x_d^{(1)}\\&\quad+\varepsilon^2 x_d^{(0)^T}[(K^{(1)}-\mu M^{(1)})x_d^{(1)}+(K^{(0)}-\mu M^{(0)})(x_c^{(2)}+x_d^{(2)})-\lambda^{(2)}M^{(0)}\varphi]\\&\quad+\varepsilon^3 x_d^{(0)^T}[(K^{(1)}-\mu M^{(1)})(x_c^{(2)}+x_d^{(2)})+(K^{(0)}-\mu M^{(0)})(x_c^{(3)}+x_d^{(3)})\\&\quad-\lambda^{(2)}M^{(1)}\varphi-\lambda^{(2)}M^{(0)}x_d^{(1)}-\lambda^{(3)}M^{(0)}\varphi]\\&\quad+\cdots=0\end{aligned} \tag{46}$$

which leads to

$$x_d^{(0)^T}[(K^{(1)}-\mu M^{(1)})\varphi+(K^{(0)}-\mu M^{(0)})x_d^{(1)}]=0 \tag{47}$$

$$x_d^{(0)^T}[(K^{(1)}-\mu M^{(1)})x_d^{(1)}+(K^{(0)}-\mu M^{(0)})x_d^{(2)}]=0 \tag{48}$$

$$x_d^{(0)^T}[(K^{(1)}-\mu M^{(1)})(x_c^{(1)}+x_d^{(2)})+(K^{(0)}-\mu M^{(0)})x_d^{(3)}-\lambda^{(2)}M^{(1)}\varphi-\lambda^{(2)}M^{(1)}x_d^{(1)}]=0 \tag{49}$$

Inserting Eqs. (21), (22), and (23) for the expansion of $x_{d_i}^{(1)}$, $x_{d_i}^{(2)}$, and $x_{d_i}^{(3)}$ into the equations and applying the orthonormality, yields the coefficients

$$\gamma_{ji}^{(1)}=\frac{x_j^{(0)^T}(K^{(1)}-\mu_i M^{(1)})\varphi_i}{\mu_i-\lambda_j^{(0)}} \tag{50}$$

$$\gamma_{ji}^{(2)}=\frac{x_j^{(0)^T}(K^{(1)}-\mu_i M^{(1)})x_{d_i}^{(1)}}{\mu_i-\lambda_j^{(0)}} \tag{51}$$

$$\gamma_{ji}^{(3)}=\frac{x_j^{(0)^T}[(K^{(1)}-\mu_i M^{(1)})(x_{c_i}^{(2)}+x_{d_i}^{(2)})-\lambda_i^{(2)}M^{(1)}\varphi_i-\lambda_i^{(2)}M^{(0)}x_{d_i}^{(1)}]}{\mu_i-\lambda_j^{(0)}} \tag{52}$$

where i=1, 2, . . ., p and j is for any mode with a distinct eigenvalue.

2.3. Perturbation with multi groups of close eigenvalues

If there are multi groups of close eigenvalues, a likely case with two or more groups of close eigenvalues, a different perturbation needs to be considered. As the same procedure is also used, $\boldsymbol{\Lambda}_d$ and $\boldsymbol{\Phi}_d$, the perturbed zeroth-order eigenpairs for the modes of those groups, are obtained. Substituting Eqs. (4), (5), (13), and (14) into Eq. (6) and premultiplying the equation with φ_d^T, yields

$$\begin{aligned}&\varphi_d^T(\boldsymbol{K}^{(0)}-\mu\boldsymbol{M}^{(0)})\varphi\\&\quad+\varepsilon\varphi_d^T(\boldsymbol{K}^{(1)}-\mu\boldsymbol{M}^{(1)})\varphi+(\boldsymbol{K}-\mu\boldsymbol{M})\boldsymbol{x}_d^{(1)}\\&\quad+\varepsilon^2\varphi_d^T[(\boldsymbol{K}^{(0)}-\mu\boldsymbol{M}^{(0)})\boldsymbol{x}_c^{(2)}+(\boldsymbol{K}-\mu\boldsymbol{M})\boldsymbol{x}_d^{(2)}-\lambda^{(2)}\boldsymbol{M}^{(0)}\varphi]\\&\quad+\varepsilon^3\varphi_d^T[(\boldsymbol{K}^{(1)}-\mu\boldsymbol{M}^{(1)})\boldsymbol{x}_c^{(2)}+(\boldsymbol{K}^{(0)}-\mu\boldsymbol{M}^{(0)})\boldsymbol{x}_c^{(3)}+(\boldsymbol{K}-\mu\boldsymbol{M})\boldsymbol{x}_d^{(3)}\\&\quad-\lambda^{(2)}\boldsymbol{M}^{(1)}\varphi-\lambda^{(2)}\boldsymbol{M}^{(0)}\boldsymbol{x}_d^{(1)}-\lambda^{(3)}\boldsymbol{M}^{(0)}\varphi]\\&\quad+\cdots=0\end{aligned}\tag{53}$$

Equating the coefficients of ε leads to

$$\varphi_d^T[(\boldsymbol{K}^{(1)}-\mu\boldsymbol{M}^{(1)})\varphi+(\boldsymbol{K}-\mu\boldsymbol{M})\boldsymbol{x}_d^{(1)}]=0\tag{54}$$

$$\varphi_d^T(\boldsymbol{K}-\mu\boldsymbol{M})\boldsymbol{x}_d^{(2)}=0\tag{55}$$

$$\varphi_d^T[(\boldsymbol{K}^{(1)}-\mu\boldsymbol{M}^{(1)})\boldsymbol{x}_c^{(2)}+(\boldsymbol{K}-\mu\boldsymbol{M})\boldsymbol{x}_d^{(3)}-\lambda^{(2)}\boldsymbol{M}^{(1)}\varphi]=0\tag{56}$$

Inserting $\boldsymbol{x}_{d_i}^{(1)}$, $\boldsymbol{x}_{d_i}^{(2)}$, and $\boldsymbol{x}_{d_i}^{(3)}$, which are the expansion of φ_{d_j} on the respective group of close eigenvalues in Eqs. (24), (25), and (26), into the equations, applying the othornormal relationship, yields the coefficients

$$\xi_{ji}^{(1)}=\frac{\varphi_{d_j}^T(\boldsymbol{K}^{(1)}-\mu_i\boldsymbol{M}^{(1)})\varphi_i}{\mu_i-\mu_j}\tag{57}$$

$$\xi_{ji}^{(2)}=0\tag{58}$$

$$\xi_{ji}^{(3)}=\frac{\varphi_{d_j}^T[(\boldsymbol{K}^{(1)}-\mu_i\boldsymbol{M}^{(1)})\boldsymbol{x}_{c_i}^{(2)}-\lambda_i^{(2)}\boldsymbol{M}^{(1)}\varphi_i]}{\mu_i-\mu_j}\tag{59}$$

where i = 1, 2, ..., p and j is for any of other groups of close eigenvalues.

2.4. Perturbation for $\boldsymbol{x}_c^{(2)}$ and $\boldsymbol{x}_c^{(3)}$

Note that $\boldsymbol{x}_c^{(2)}$ and $\boldsymbol{x}_c^{(3)}$ have not been determined in the perturbation process, based on the eigenvalue Eq. (6). They can be computed from the orthonormal relationship, a condition for which an eigenvector must be met. The orthonormality for ith and jth eigenvectors in Eq. (7) becomes

$$x_i^T M x_j = \delta_{ij} \tag{60}$$

Inserting Eqs. (14) and (5) into the equation, the normality is then

$$\begin{aligned}
&\varphi_i^T M \varphi_j \\
&+\varepsilon(\varphi_i^T M^{(0)} x_{d_j}^{(1)} + x_{d_i}^{(1)^T} M^{(0)} \varphi_j) \\
&+\varepsilon^2(\varphi_i^T M^{(1)} x_{d_j}^{(1)} + x_{d_i}^{(1)^T} M^{(1)} \varphi_j + x_{d_i}^{(1)^T} M^{(0)} x_{d_j}^{(1)} + \varphi_i^T M x_{c_j}^{(2)} + x_{c_i}^{(2)^T} M \varphi_j) \\
&+\varepsilon^3(\varphi_i^T M^{(1)} x_{d_j}^{(2)} + x_{d_i}^{(2)^T} M^{(1)} \varphi_j + x_{d_i}^{(1)^T} M^{(1)} x_{d_j}^{(1)} + x_{d_j}^{(1)^T} M^{(0)} x_{d_j}^{(2)} + x_{d_i}^{(2)^T} M^{(0)} x_{d_j}^{(1)} + \varphi_i^T M x_{c_j}^{(3)} + x_{c_i}^{(3)^T} M \varphi_j) \\
&+\cdots = \delta_{ij}
\end{aligned} \tag{61}$$

Applying the orthonormality and equating the terms with like power of ε, yields

$$\varphi_i^T M \varphi_j = \delta_{ij} \tag{62}$$

$$\varphi_i^T M^{(1)} x_{d_j}^{(1)} + x_{d_i}^{(1)^T} M^{(1)} \varphi_j + \varphi_i^T M x_{c_j}^{(2)} + x_{d_i}^{(1)^T} M^{(0)} x_{d_j}^{(1)} + x_{c_i}^{(2)^T} M \varphi_j = 0 \tag{63}$$

$$\varphi_i^T M^{(1)} x_{d_j}^{(2)} + x_{d_i}^{(2)^T} M^{(1)} \varphi_j + x_{d_i}^{(1)^T} M^{(1)} x_{d_j}^{(1)} + x_{d_i}^{(1)^T} M^{(0)} x_{d_j}^{(2)} + x_{d_i}^{(2)^T} M^{(0)} x_{d_j}^{(1)} + x_{c_i}^{(3)^T} M \varphi_j + \varphi_i^T M x_{c_j}^{(3)} = 0 \tag{64}$$

Eq. (62) is the normality in Eq. (44). Inserting the expansion of $x_c^{(2)}$ and $x_c^{(3)}$ from Eqs. (18) and (19) into Eqs. (63) and (64), the coefficients from the equations are

$$\beta_{ij}^{(2)} + \beta_{ji}^{(2)} = -\varphi_i^T M^{(1)} x_{d_j}^{(1)} - x_{d_i}^{(1)^T} M^{(1)} \varphi_j - x_{d_i}^{(1)^T} M^{(0)} x_{d_j}^{(1)} \tag{65}$$

$$\beta_{ij}^{(3)} + \beta_{ji}^{(3)} = -\varphi_i^T M^{(1)} x_{d_j}^{(2)} - x_{d_i}^{(2)^T} M^{(1)} \varphi_j - x_{d_i}^{(1)^T} M^{(1)} x_{d_j}^{(1)} - x_{d_i}^{(1)^T} M^{(0)} x_{d_j}^{(2)} - x_{d_i}^{(2)^T} M^{(0)} x_{d_j}^{(1)} \tag{66}$$

It is found that the right hand of the equations is symmetrical, and Eq. (60) is also symmetrical to the ith and jth vectors. As the reason, the matrices $\beta^{(2)}$ and $\beta^{(3)}$ are taken to be symmetrical, so that the coefficients are obtained as

$$\beta_{ij}^{(2)} = \beta_{ji}^{(2)} = -\frac{1}{2}(\varphi_i^T M^{(1)} x_{d_j}^{(1)} + x_{d_i}^{(1)^T} M^{(1)} \varphi_j + x_{d_i}^{(1)^T} M^{(0)} x_{d_j}^{(1)}) \tag{67}$$

$$\beta_{ij}^{(3)} = \beta_{ji}^{(3)} = -\frac{1}{2}(\varphi_i^T M^{(1)} x_{d_j}^{(2)} + x_{d_i}^{(2)^T} M^{(1)} \varphi_j + x_{d_i}^{(1)^T} M^{(1)} x_{d_j}^{(1)} + x_{d_i}^{(1)^T} M^{(0)} x_{d_j}^{(2)} + x_{d_i}^{(2)^T} M^{(0)} x_{d_j}^{(1)}) \tag{68}$$

The perturbations up to the third-order are completed. The first, second, and third-order approximations can be obtained by inserting appropriate perturbations into Eqs. (13) and (14).

3. Discussion

The suggested perturbation procedure has two features, which are different from the conventional perturbation:

(a) The division of eigenvector perturbation into two parts, that is x_c for close eigenvalues and x_d for distinct eigenvalues;

(b) The zeroth-order perturbation, that is μ and φ.

The zeroth-order perturbation is derived by inserting the first-order perturbation of eigenvalue and some of first-order perturbation of eigenvector into the zeroth-order solution as given in Eqs. (11) and (12). This leads to an eigenvalue equation (38), which is in the form of Rayleigh quotient. The algorithm is based on the eigenvalue equation and the normality condition of the perturbed system, two conditions in the eigenvalue analysis.

3.1. Condensation of perturbation to a single mode

As suggested by Chen *et al.* (1995), the analysis for close eigenvalues is applicable to any mode with a distinct eigenvalue. A single mode can act as one of "a close eigenvalue" and this is the condensation of the algorithm. Taking p=1, Eqs. (38) and (39) of the zeroth-order perturbation can be reduced to

$$\mu_1=\frac{\boldsymbol{x}_1^{(0)^T}\boldsymbol{K}\boldsymbol{x}_1^{(0)}}{\boldsymbol{x}_1^{(0)^T}\boldsymbol{M}\boldsymbol{x}_1^{(0)}} \tag{69}$$

$$\beta_1=\frac{1}{\sqrt{\boldsymbol{x}_1^{(0)^T}\boldsymbol{M}\boldsymbol{x}_1^{(0)}}} \tag{70}$$

In this case, there are multi groups of close eigenvalues and the equations for the perturbation of eigenvector in Section 2.3 should be used. Due to the improved zeroth-order solution, the convergence of the perturbation can be accelerated.

3.2. Comparison to other algorithms

The early algorithms used by Chen *et al.* (1995) and Liu (1995, 1999) are unable to give the derivation of the zeroth-order perturbation, as Eq. (38), but the result μ and φ from the equation are directly used in the formulas of the late perturbation.

Some differences are found in the algorithm by Chen *et al.* (1995), in which the second and third-order perturbations of eigenvalue are given as

$$\lambda_i^{(2)}=\frac{\varphi_i^T(\boldsymbol{K}^{(1)}-\mu_i\boldsymbol{M}^{(1)})\boldsymbol{x}_{d_i}^{(1)}}{\beta_i^T\beta_i} \tag{71}$$

$$\lambda_i^{(3)}=\frac{\varphi_i^T[(\boldsymbol{K}^{(1)}-\mu_i\boldsymbol{M}^{(1)})\boldsymbol{x}_{d_i}^{(2)}-\lambda_i^{(2)}\boldsymbol{M}^{(1)}\varphi_i]}{\beta_i^T\beta_i} \tag{72}$$

with the vector β at the denominators. By the comparison to Eqs. (44) and (45), the differences are found on the term $\lambda_i^{(2)}\varphi_i^T\boldsymbol{M}^{(1)}\varphi_i$ in Eq. (72) for the third-order perturbation, which appears in Eq. (44) for the second-order perturbation.

In the perturbation of eigenvector with a distinct eigenvalue, the algorithm by Chen *et al* is different at the third-order perturbation of eigenvector, given as

$$\lambda_{ji}^{(3)}=\frac{\boldsymbol{x}_j^{(0)^T}[(\boldsymbol{K}^{(1)}-\mu_i\boldsymbol{M}^{(1)})\boldsymbol{x}_{d_i}^{(2)}-\lambda_i^{(2)}\boldsymbol{M}^{(1)}\varphi_i-\lambda_i^{(2)}\boldsymbol{M}^{(0)}\boldsymbol{x}_{d_i}^{(1)}]}{\mu_i-\lambda_j^{(0)}} \tag{73}$$

which is as same as Eq. (52) except without $\boldsymbol{x}_{c_i}^{(2)}$.

These algorithms lack the perturbation terms $\boldsymbol{x}_c^{(2)}$ and $\boldsymbol{x}_c^{(3)}$, which are computed from the normality condition with mass matrices $\boldsymbol{M}^{(0)}$ and $\boldsymbol{M}^{(1)}$, so that the analysis would affect the eigenvectors from the neglect of them and it would be even more if $\boldsymbol{M}^{(1)}$ is not null. The error will be reflected on the normality, since it is not met, and on the eigenvectors as well.

No formulation is given for the perturbation with multi groups of close eigenvalues in these algorithms.

4. Example

To examine the suggested formulas, a simple problem with four unknowns is chosen as the example. The numerical results from the perturbation computations by the suggested formulas, the condensation of the formulas on all modes, and Chen *et al.* (1995) will be compared.

The zeroth-order matrices are given by

$$\boldsymbol{K}^{(0)}=\begin{bmatrix} 3 & -2 & -1 & 0 \\ -2 & 22 & 0 & 0 \\ -1 & 0 & 3 & 2 \\ 0 & 0 & 2 & 22 \end{bmatrix}$$

$$\boldsymbol{M}^{(0)}= \text{diag}\ \{1, 1, 1, 1\}$$

The matrices are assumed to take the perturbation

$$\varepsilon\boldsymbol{K}^{(1)}=\begin{bmatrix} 0.2 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$\varepsilon\boldsymbol{M}^{(1)}= \text{diag}\ \{0.02, 0.02, -0.05, -0.05\}$$

The zeroth-order eigensolution and the perturbed eigensolution (referred as the exact one) are given in Table 1. The mode shapes are illustrated in Fig. 1. The third and forth eigenvalues in the zeroth-order eigensolution are close eigenvalues, that is 22.19804 and 22.21954 respectively. In Fig. 1, their modal shapes are noted to be far from the perturbed ones. The conventional method for a distinct eigenvalue would be unsuccessful to develop the perturbations of eigenvectors due to the big changes.

Three computations are considered.

Perturbation I: The conventional method for a distinct eigenvalue is used in the first and second modes, while the suggested perturbation is applied to last two modes. The approximations up to the third-order are listed in Table 1. The convergence with the perturbation order is observed. The modal shapes from the perturbation analysis, shown in Fig. 1, are approaching to the exact ones. However, the convergence in first and second modes is not as good as that in last two modes.

Perturbation II: The condensation is considered in first and second modes, so that there are three groups with close eigenvalues. Formulas in Section 2.3 are used in the perturbations of eigenvector.

Perturbation III: The third computation uses the algorithm from Chen *et al.* (1995). The condensation is applied to first and second modes.

To examine the overall errors for all modes, special norms for eigenvalues, eigenvectors as well as normality are defined as

$$E(\lambda)=\left\{\sum_i [(\lambda_i-\lambda_{e_i})/\lambda_{e_i}]^2\right\}^{1/2} \tag{74}$$

Table 1 Comparison of eigenvalues and normality in Perturbation I

Mode		Exact	Zeroth	First	Second	Third
	Eigenvalue	1.83205	1.80196	1.83385	1.83174	1.83212
1	Error (%)		−1.6425	0.0978	−0.0173	0.0038
	Normality		0.9850	1.0035	0.9992	0.9993
	Eigenvalue	4.25526	3.78046	4.23835	4.25464	4.25517
2	Error (%)		−11.1581	−0.3974	−0.0148	−0.0021
	Normality		0.98500	0.99970	0.99956	0.99962
	Eigenvalue	21.77572	22.19804	21.77564	21.77572	21.77572
3	Error (%)		1.93940	−0.00037	−0.00002	0.00000
	Normality		0.9850000	1.0000029	1.0000002	1.0000000
	Eigenvalue	24.42179	22.21954	24.42145	24.42177	24.42179
4	Error (%)		−9.01756	−0.00142	−0.00009	−0.00001
	Normality		0.9850000	1.0000143	1.0000011	1.0000001

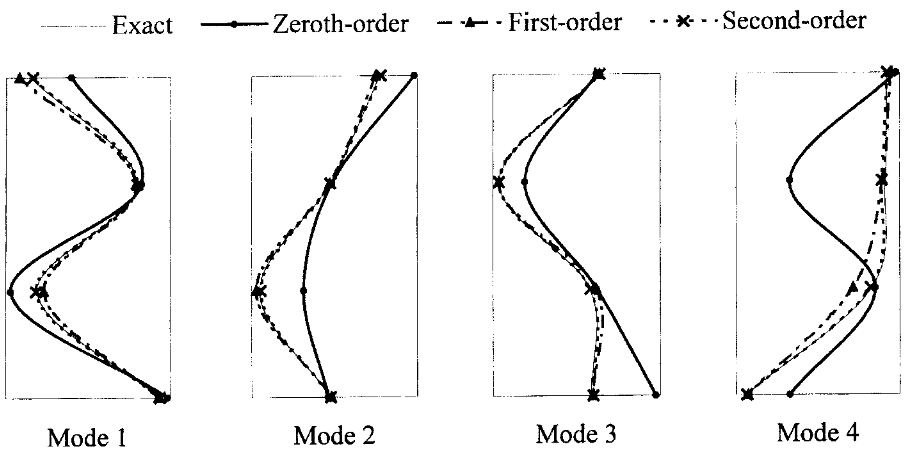

Fig. 1 Comparison of modal shapes in Perturbation I

$$E(x)=\left[\sum_i (x_i^T x_i - x_{e_i}^T x_{e_i})^2\right]^{1/2} \tag{75}$$

$$E(n)=\left[\sum_i (x_i^T M x_i - 1)^2\right]^{1/2} \tag{76}$$

where λ_i and x_i are the ith eigensolution from a respective perturbation order and the subscript e denotes the exact solution. Obviously, norms in Eq. (74) and Eq. (76) are relative errors.

Norms from three computations, denoted as Perturbation I, Perturbation II, and Perturbation III, are illustrated in Fig. 2, Fig. 3, and Fig. 4. Now, the examination of the convergence can be clearly

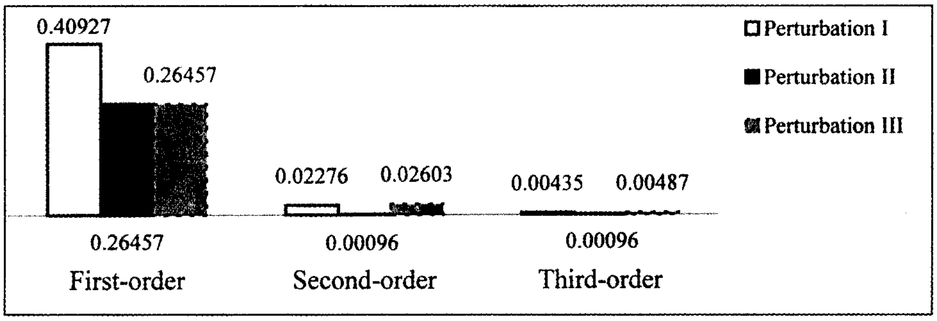

Fig. 2 Norm of perturbed eigenvalues (%)

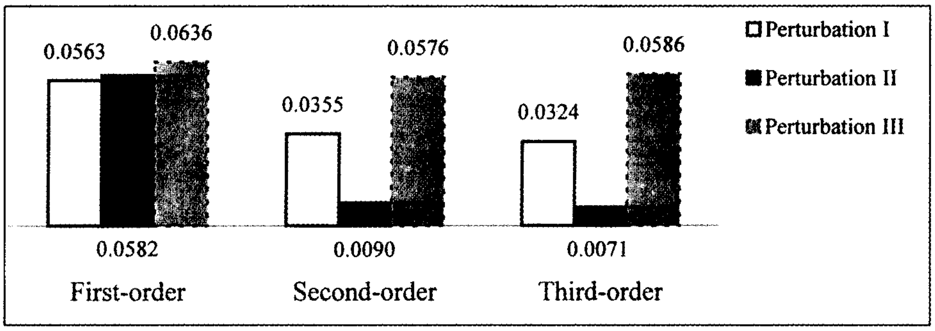

Fig. 3 Norm of perturbed eigenvectors

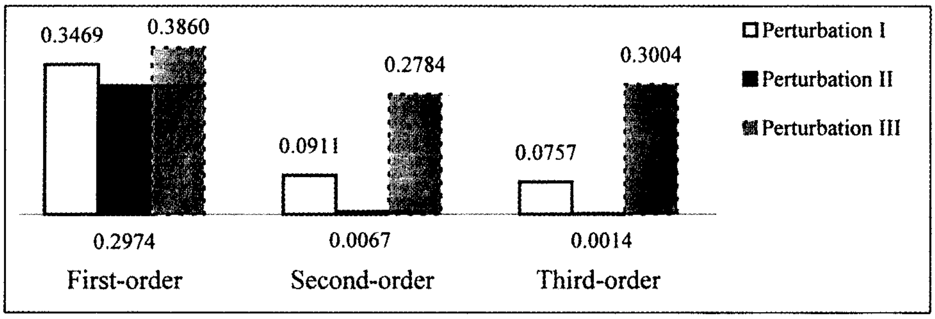

Fig. 4 Norm of normality (%)

made.

Perturbation I is convergent on all three examined norms.

With the help of the condensation in Perturbation II, the convergence is much improved, with the norms at 0.00096%, 0.0071, and 0.0014% for the third-order.

The worst approximation is found in Perturbation III. Though the first-order eigenvalues are the same as Perturbation II, the norm from third-order eigenvalues is 0.00487, which cannot be compared to 0.00096 from the second-order in Perturbation II. The eigenvectors and normality are not convergent and are far from the excellent agreement as in Perturbation II. In other words, the second-order and third-order perturbations of eigenvectors are hardly improved from the algorithm by Chen *et al.* (1995). This demonstrates that the ignoring of the perturbation terms, solved from the normality condition, has a big effect on the higher-order perturbations.

5. Conclusions

The algorithm for the perturbation analysis with modes of close eigenvalues has been presented in the paper. All perturbation terms of the eigensolution are obtained from the straightforward process. The zeroth-order perturbation contributed from the first-order perturbation leads to an eigenvalue equation in a form of Rayleigh quotient. The corrected terms on higher-order perturbations and the use of normality can help the convergence of perturbation approximation, which is shown in the example.

The condensation of the analysis to a mode of distinct eigenvalue also forces the perturbation at the zeroth-order and it is an advantage on generating a better approximation over the conventional perturbation. It suggests the potential use of the procedure to all modes.

The perturbation analysis can be a numerical tool on the quantitative computation of the mode localization and the loci veering, since the prediction of the dramatic change resulted from close eigenvalues is demanded.

Acknowledgement

This work has been partially supported by the Portuguese National Foundation "Fundação Para A Ciência E A Tecnologia", under the grant No. PRAXIS XXI/BD/19804/99.

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