Structural Engineering and Mechanics, Vol. 10, No. 4 (2000) 337-357
DOI: http://dx.doi.org/10.12989/sem.2000.10.4.337

Analytical solutions using a higher order refined theory for the stability analysis of laminated composite and sandwich plates

T. Kant† and K. Swaminathan‡

Department of Civil Engineering, Indian Institute of Technology Bombay, Powai, Mumbai-400 076, India

Abstract. Analytical formulations and solutions for the first time, to the stability analysis of a simply supported composite and sandwich plates based on a higher order refined theory, developed by the first author and already reported in the literature are presented. The theoretical model presented herein incorporates laminate deformations which account for the effects of transverse shear deformation, transverse normal strain/stress and a nonlinear variation of inplane displacements with respect to the thickness coordinate - thus modelling the warping of transverse cross sections more accurately and eliminating the need for shear correction coefficients. The equations of equilibrium are obtained using the Principle of Minimum Potential Energy (PMPE). The comparison of the results using this higher order refined theory with the available elasticity solutions and the results computed independently using the first order and the other higher order theories developed by other investigators and available in the literature shows that this refined theory predicts the critical buckling load more accurately than all other theories considered in this paper. New results for sandwich laminates are also presented which may serve as a benchmark for future investigations.

Key words: higher order theory; analytical solutions; buckling; shear deformation; sandwich plates; laminated plates; Navier solutions.

1. Introduction

The high values of specific moduli and strength of fibre reinforced composite materials make them attractive for aerospace structural components such as plates and shells. Use of these thin sheet materials in airplane and aerospace structures may prove unstable under the action of forces in their own planes and fail by buckling. Hence the modern use of above structural elements made of fibre reinforced composite material in engineering structures has made elastic instability a problem of great importance. In addition to more accurate and improved methods of analysis to predict the critical buckling load there is also a need to develop consistent refined shear deformation theories for the analysis of these structural elements. The Classical Laminate Plate Theory (Reissner and Stavsky 1961) which is an extension of Classical Plate Theory (Timoshenko and Woinowsky-Krieger 1959, Szilard 1974) neglects the effect of out-of-plane strains. In a composite laminate the in-plane modulus of elasticity is many times larger than that of the matrix material while the transverse shear modulus is largely that of the matrix material. Hence for a given in-plane modulus,

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† Professor
‡ Research Scholar

the plate or shell is very weak in transverse shear resistance and the transverse shear deformation is significant and cannot be neglected. Thus the Classical Laminate Plate Theory (CLPT) which ignores the effect of transverse shear deformation become inadequate for the analysis of multilayer composites. In general the CLPT often underpredicts deflections and overpredicts buckling loads. The first order theories (FSDT) based on Reissner (1945) and Mindlin (1951) assume linear displacement and/or stress variations through the laminate thickness. Since the FSDT accounts for layerwise constant states of transverse shear stress, shear correction coefficients are needed to rectify the unrealistic variation of the shear strain/stress through the thickness.

In order to overcome the limitations of FSDT, higher order shear deformation theories (HSDT) that involve higher order terms in the Taylor's expansions of the displacements in the thickness coordinate were developed. The second and third-order theories involve additional terms in the expression for the in-plane displacements which are parabolic and cubic respectively in thickness direction coordinate. Hildebrand *et al.* (1949) were the first who introduced this approach to derive improved theories of plates and shells. Nelson and Lorch (1974), Librescu (1975) presented higher order displacement based shear deformation theories for the analysis of laminated plates. Lo *et al.* (1977a, 1977b) have presented a closed form solution for a laminated plate with higher order displacement model which also considers the effect of transverse normal deformation. Levinson (1980) and Murthy (1981) presented third order theories neglecting the extension/compression of transverse normal but used the equilibrium equations of the first order theory used by Whitney and Pagano (1970) in the analysis which are variationally inconsistent. Kant (1982) was the first to derive the complete set of variationally consistent governing equations for the flexure of a plate incorporating both distortion of transverse normals and effects of transverse normal stress/strain by utilizing the complete three-dimensional generalized Hookes law. Reddy (1984) derived a set of variationally consistent equilibrium equations for the kinematic models originally proposed by Levinson and Murthy. Using the theory of Reddy, Senthilnathan *et al.* (1987) presented a simplified higher order theory by introducing a further reduction of the functional degrees of freedom by splitting up the transverse displacement into bending and shear contributions. Kant *et al.* (1982) are the first to present a finite element formulation of a higher order flexure theory. This theory considers three-dimensional Hooke's law, incorporates the effect of transverse normal strain in addition to transverse shear deformations. Pandya and Kant (1987, 1988a, 1988b, 1988c, 1988d), Kant and Manjunatha (1988, 1994) and Manjunatha and Kant (1992) have extended this theory for symmetric and unsymmetric laminated composite and sandwich plates and presented only finite element solutions for the various composite and sandwich plates/shells problems using C^0 finite element formulation. Later Kant and Patil (1991) have obtained the buckling loads of a three layered simply supported symmetric sandwich column in closed form and demonstrated the accuracy of the higher order theory in predicting the realistic buckling loads of the conventional sandwich structures. Noor (1975) presented exact elasticity solutions for the stability analysis of symmetric and skew-symmetric multilayered laminated composite plates with large number of layers which serve as a benchmark solution for comparison. Later Noor and Burton (1989) presented a complete list of references of FSDT and HSDT for the static, free vibration and buckling analysis of laminated composites. Eventhough a large number of publications exist on the stability analysis of symmetric and unsymmetric laminated composite plates, Qatu and Leissa (1993) in their paper concluded that the true buckling (i.e., bifurcation) cannot occur in most of the cases of antisymmetric cross-ply and angle-ply laminates reported in the literature. Taking that fact into consideration in this paper analytical formulations and solutions to the buckling analysis of laminated composite and sandwich plates hitherto not reported in the literature are presented using the refined higher order theory developed and already reported in the literature by the first author. Laminated and sandwich plates with stacking sequence that are symmetric with respect to their middle plane are only considered. The parameters considered are varying number of layers, degree of anisotropy, side-to-thickness ratios, thickness of the core to thickness of the flange and aspect ratio of the plate. Numerical results of simply supported composite and sandwich plates are presented and compared with 3D elasticity solutions and with the results computed independently using shear deformation theories already developed by other investigators and reported in the literature to show the improvement in accuracy in predicting the critical buckling loads.

2. Theoretical formulations

2.1. Kinematics

In order to reduce the three-dimensional elasticity problem to a two-dimensional plate problem, the displacement components $u(x, y, z)$, $v(x, y, z)$ and $w(x, y, z)$ at any point in the plate space are expanded in a Taylor's series in terms of the thickness coordinate. The elasticity solution indicates that the transverse shear stress vary parabolically through the plate thickness. This requires the use of a displacement field in which the inplane displacements are expanded as cubic functions of the thickness coordinate. In addition, the transverse normal strain may vary non-linearly through the plate thickness. The displacement field which satisfies the above criteria may be assumed in the form (Kant and Manjunatha 1988)

$$\begin{aligned} u(x,y,z) &= u_o(x,y) + z\theta_x(x,y) + z^2 u_o^*(x,y) + z^3\theta_x^*(x,y) \\ v(x,y,z) &= v_o(x,y) + z\theta_y(x,y) + z^2 v_o^*(x,y) + z^3\theta_y^*(x,y) \\ w(x,y,z) &= w_o(x,y) + z\theta_z(x,y) + z^2 w_o^*(x,y) + z^3\theta_z^*(x,y) \end{aligned} \tag{1}$$

The parameters u_o, v_o are the inplane displacements and w_o is the transverse displacement of a point (x, y) on the middle plane. The functions θ_x, θ_y are rotations of the normal to the middle plane about y and x axes, respectively. The parameters u_o^*, v_o^*, w_o^*, θ_x^*, θ_y^*, θ_z^* and θ_z are the higher-order terms in the Taylor's series expansion and they represent higher-order transverse cross sectional deformation modes. The geometry of a two-dimensional laminated composite plate with positive set of coordinate axes and the physical middle plane displacement terms are shown in Fig. 1. By substitution of these displacement relations into the strain-displacement equations of the classical theory of elasticity, the following relations are obtained.

$$\begin{aligned} \varepsilon_x &= \varepsilon_{xo} + z\kappa_x + z^2\varepsilon_{xo}^* + z^3\kappa_x^* \\ \varepsilon_y &= \varepsilon_{yo} + z\kappa_y + z^2\varepsilon_{yo}^* + z^3\kappa_y^* \\ \varepsilon_z &= \varepsilon_{zo} + z\kappa_z + z^2\varepsilon_{zo}^* + z^3\kappa_y^* \\ \gamma_{xy} &= \varepsilon_{xyo} + z\kappa_{xy} + z^2\varepsilon_{xyo}^* + z^3\kappa_{xy}^* \\ \gamma_{yz} &= \phi_y + z\kappa_{yz} + z^2\phi_y^* + z^3\kappa_{yz}^* \\ \gamma_{xz} &= \phi_x + z\kappa_{xz} + z^2\phi_x^* + z^3\kappa_{xz}^* \end{aligned} \tag{2}$$

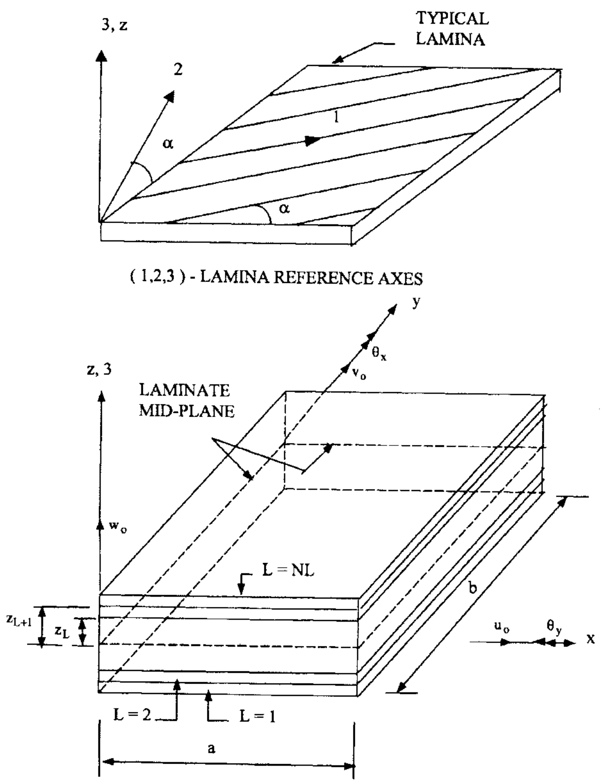

Fig. 1 Laminate geometry with positive set of lamina/laminate reference axes, displacement components and fibre orientation

where,

$$(\varepsilon_{xo}, \varepsilon_{yo}, \varepsilon_{xyo}) = \left(\frac{\partial u_o}{\partial x}, \frac{\partial v_o}{\partial y}, \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right)$$

$$(\varepsilon^*_{xo}, \varepsilon^*_{yo}, \varepsilon^*_{xyo}) = \left(\frac{\partial u^*_o}{\partial x}, \frac{\partial v^*_o}{\partial y}, \frac{\partial u^*_o}{\partial y} + \frac{\partial v^*_o}{\partial x}\right)$$

$$(\varepsilon_{zo}, \varepsilon^*_{zo}) = (\theta_z, 3\theta^*_z)$$

$$(\kappa_x, \kappa_y, \kappa_z, \kappa_{xy}) = \left(\frac{\partial \theta_x}{\partial x}, \frac{\partial \theta_y}{\partial y}, 2w^*_o, \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right)$$

$$(\kappa^*_x, \kappa^*_y, \kappa^*_{xy}) = \left(\frac{\partial \theta^*_x}{\partial x}, \frac{\partial \theta^*_y}{\partial y}, \frac{\partial \theta^*_x}{\partial y} + \frac{\partial \theta^*_y}{\partial x}\right)$$

$$(\kappa_{xz}, \kappa_{yz}) = \left(2u^*_o, \frac{\partial \theta_z}{\partial x}, 2v^*_o + \frac{\partial \theta_z}{\partial y}\right)$$

$$(\kappa^*_{xz}, \kappa^*_{yz}) = \left(\frac{\partial \theta^*_z}{\partial x}, \frac{\partial \theta^*_z}{\partial y}\right)$$

$$(\phi_x, \phi_x^*, \phi_y, \phi_y^*) = \left(\theta_x + \frac{\partial w_o}{\partial x}, 3\theta_x^* + \frac{\partial w_o^*}{\partial x}, \theta_y + \frac{\partial w_o}{\partial y}, 3\theta_y^* + \frac{\partial w_o^*}{\partial y}\right) \tag{3}$$

2.2. Constitutive equations

Each lamina in the laminate is assumed to be in a three-dimensional stress state so that the constitutive relation for a typical lamina L with reference to the fibre-matrix coordinate axes (1-2-3) can be written as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{Bmatrix}^L = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix}^L \tag{4}$$

where $(\sigma_1, \sigma_2, \sigma_3, \tau_{12}, \tau_{23}, \tau_{13})$ are the stresses and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{12}, \gamma_{23}, \gamma_{13})$ are the linear strain components referred to the lamina coordinates (1-2-3) and the C_{ij}'s are the elastic constants or the elements of stiffness matrix of the Lth lamina with reference to the fibre axes (1-2-3). In the laminate coordinates (x, y, z) the stress strain relations for the Lth lamina can be written as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^L = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & 0 \\ & Q_{22} & Q_{23} & Q_{24} & 0 & 0 \\ & & Q_{33} & Q_{34} & 0 & 0 \\ & & & Q_{44} & 0 & 0 \\ & \text{symmetric} & & & Q_{55} & Q_{56} \\ & & & & & Q_{66} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^L \tag{5}$$

where $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$ are the stresses and $(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{xz})$ are the strains with respect to the laminate axes. Q_{ij}'s are the transformed elastic constants or stiffness matrix with respect to the laminate axes x, y, z. The elements of matrices $[C]$ and $[Q]$ are defined in Appendices A and B.

2.3. Governing equations of equilibrium

The equations of equilibrium can be obtained using the Principle of Minimum Potential Energy (PMPE). In analytical form it can be written as (Reddy 1984b, 1996);

$$\int_{vol} (\delta U - \delta W) dV + \int_{vol} (\sigma_x' \, \delta\varepsilon_x' + \sigma_y' \, \delta\varepsilon_y' + \tau_{xy}' \, \delta\gamma_{xy}') dV = 0 \tag{6}$$

where U is the total strain energy due to deformation, W is the work done by external loads, the second term in the above equation is the potential energy due to applied in-plane compressive and shear loads (i.e., $\sigma_x', \sigma_y', \tau_{xy}'$). $\varepsilon_x', \varepsilon_y', \gamma_{xy}'$ are the midplane strain caused by transverse displacement and δ denotes the variational symbol. Substituting the appropriate energy expression in the above equation, the final expression can thus be written as

$$\left[\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_A (\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \sigma_z\delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{yz})dAdz - \int_A p_z^+\delta w^+ dA\right]$$

$$+\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_A (\sigma_x'\delta\varepsilon_x' + \sigma_y'\delta\varepsilon_y' + \tau_{xy}'\delta\gamma_{xy}')dAdz = 0 \qquad (7)$$

where $w^+ = w_o + (h/2)\theta_z + (h^2/4)w_o^* + (h^3/8)\theta_z^*$ is the transverse displacement of any point on the top surface of the plate and is the transverse load applied at the top surface of the plate. Using Eqs. (1), (2) and (3) in Eq. (7) and integrating the resulting expression by parts, and collecting the coefficients of $\delta u_o, \delta v_o, \delta w_o, \delta\theta_x, \delta\theta_y, \delta\theta_z, \delta u_o^*, \delta v_o^*, \delta w_o^*, \delta\theta_x^*, \delta\theta_y^*, \delta\theta_z^*$ the following equations of equilibrium are obtained :

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + (\bar{N}_x)\frac{\partial^2 w_o}{\partial x^2} + (\bar{M}_x)\frac{\partial^2 \theta_z}{\partial x^2} + (\bar{N}_x^*)\frac{\partial^2 w_o^*}{\partial x^2} + (\bar{M}_x^*)\frac{\partial^2 \theta_z^*}{\partial x^2} + (\bar{N}_y^*)\frac{\partial^2 w_o}{\partial y^2}$$

$$+(\bar{M}_y)\frac{\partial^2 \theta_z}{\partial y^2} + (\bar{N}_y^*)\frac{\partial^2 w_o^*}{\partial y^2} + (\bar{M}_y^*)\frac{\partial^2 \theta_z^*}{\partial y^2} + 2(\bar{N}_{xy})\frac{\partial^2 w_o}{\partial x\partial y} + 2(\bar{M}_{xy})\frac{\partial^2 \theta_z}{\partial x\partial y} + 2(\bar{N}_{xy}^*)\frac{\partial^2 w_o^*}{\partial x\partial y}$$

$$+2(\bar{M}_{xy}^*)\frac{\partial^2 \theta_z^*}{\partial x\partial y} + (p_z^+) = 0$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0$$

$$\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} - N_z + (\bar{N}_x^*)\frac{\partial^2 \theta_z}{\partial x^2} + (\bar{M}_x)\frac{\partial^2 w_o}{\partial x^2} + (\bar{M}_x^*)\frac{\partial^2 w_o^*}{\partial x^2} + (\hat{N}_x^*)\frac{\partial^2 \theta_z^*}{\partial x^2} + (\bar{N}_y^*)\frac{\partial^2 \theta_z}{\partial y^2} + (\bar{M}_y)\frac{\partial^2 w_o^*}{\partial y^2} +$$

$$(\bar{M}_y^*)\frac{\partial^2 w_o^*}{\partial y^2} + (\hat{N}_y^*)\frac{\partial^2 \theta_z^*}{\theta y^2} + 2(\bar{N}_{xy}^*)\frac{\partial^2 \theta_z}{\partial x\partial y} + 2(\bar{M}_{xy})\frac{\partial^2 w_o}{\partial x\partial y} + 2(\bar{M}_{xy}^*)\frac{\partial^2 w_o^*}{\partial x\partial y} + 2(\hat{N}_{xy}^*)\frac{\partial^2 \theta_z^*}{\partial x\partial y} + \frac{h}{2}(p_z^+) = 0$$

$$\frac{\partial N_x^*}{\partial x} + \frac{\partial N_{xy}^*}{\partial y} - 2S_x = 0$$

$$\frac{\partial N_y^*}{\partial y} + \frac{\partial N_{xy}^*}{\partial x} - 2S_y = 0$$

$$\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} - 2M_z^* + (\hat{N}_x^*)\frac{\partial^2 w_o^*}{\partial x^2} + (\bar{N}_x^*)\frac{\partial^2 w_o}{\partial x^2} + (\bar{M}_x^*)\frac{\partial^2 \theta_z}{\partial x^2} + (\hat{M}_x^*)\frac{\partial^2 \theta_z^*}{\partial x^2} + (\hat{N}_y^*)\frac{\partial^2 w_o^*}{\partial y^2} + (\bar{N}_y^*)\frac{\partial^2 w_o}{\partial y^2} +$$
$$(\bar{M}_y^*)\frac{\partial^2 \theta_z}{\partial y^2} + (\hat{M}_y^*)\frac{\partial^2 \theta_z^*}{\partial y^2} + 2(\hat{N}_{xy}^*)\frac{\partial^2 w_o^*}{\partial x \partial y} + 2(\bar{N}_{xy}^*)\frac{\partial^2 w_o}{\partial x \partial y} + 2(\bar{M}_{xy}^*)\frac{\partial^2 \theta_z}{\partial x \partial y} + 2(\hat{M}_{xy}^*)\frac{\partial^2 \theta_z^*}{\partial x \partial y} + \frac{h}{4}(p_z^+) = 0$$

$$\frac{\partial M_x^*}{\partial x} + \frac{\partial M_{xy}^*}{\partial y} - 3Q_x^* = 0$$

$$\frac{\partial M_y^*}{\partial y} + \frac{\partial M_{xy}^*}{\partial x} - 3Q_y^* = 0$$

$$\frac{\partial S_x^*}{\partial x} + \frac{\partial S_y^*}{\partial y} - 3N_z^* + (\bar{N}_x^{**})\frac{\partial^2 \theta_z^*}{\partial x^2} + (\bar{M}_x^*)\frac{\partial^2 w_o}{\partial x^2} + (\hat{N}_x^*)\frac{\partial^2 \theta_z}{\partial x^2} + (\hat{M}_x^*)\frac{\partial^2 w_o^*}{\partial x^2} + (\bar{N}_y^{**})\frac{\partial^2 \theta_z^*}{\partial y^2} + (\bar{M}_y^*)\frac{\partial^2 w_o}{\partial y^2} +$$
$$(\bar{N}_y^*)\frac{\partial^2 \theta_z}{\partial y^2} + (\hat{M}_y^*)\frac{\partial^2 w_o^*}{\partial y^2} + 2(\bar{N}_{xy}^{**})\frac{\partial^2 \theta_z^*}{\partial x \partial y} + 2(\bar{M}_{xy}^*)\frac{\partial^2 w_o}{\partial x \partial y} + 2(\hat{N}_{xy}^*)\frac{\partial^2 \theta_z}{\partial x \partial y} + 2(\hat{M}_{xy}^*)\frac{\partial^2 w_o^*}{\partial x \partial y} + \frac{h^3}{8}(p_z^+) = 0 \tag{8}$$

and the boundary conditions are of the form

On the edge x = constant

$$\begin{aligned}
u_o &= \bar{u}_o \text{ or } N_x = \bar{N}_x & \theta_x &= \bar{\theta}_x \text{ or } M_x = \bar{M}_x \\
v_o &= \bar{v}_o \text{ or } N_{xy} = \bar{N}_{xy} & \theta_y &= \bar{\theta}_y \text{ or } M_{xy} = \bar{M}_{xy} \\
w_o &= \bar{w}_o \text{ or } Q_x = \bar{Q}_x & \theta_z &= \bar{\theta}_z \text{ or } S_x = \bar{S}_x \\
u_o^* &= \bar{u}_o^* \text{ or } N_x^* = \bar{N}_x^* & \theta_x^* &= \bar{\theta}_x^* \text{ or } M_x^* = \bar{M}_x^* \\
v_o^* &= \bar{v}_o^* \text{ or } N_{xy}^* = \bar{N}_{xy}^* & \theta_y^* &= \bar{\theta}_y^* \text{ or } M_{xy}^* = \bar{M}_{xy}^* \\
w_o^* &= \bar{w}_o^* \text{ or } Q_x^* = \bar{Q}_x^* & \theta_z^* &= \bar{\theta}_z^* \text{ or } S_x^* = \bar{S}_x^*
\end{aligned} \tag{9}$$

On the edge y = constant

$$\begin{aligned}
u_o &= \bar{u}_o \text{ or } N_{xy} = \bar{N}_{xy} & \theta_x &= \bar{\theta}_x \text{ or } M_{xy} = \bar{M}_{xy} \\
v_o &= \bar{v}_o \text{ or } N_y = \bar{N}_y & \theta_y &= \bar{\theta}_y \text{ or } M_y = \bar{M}_y \\
w_o &= \bar{w}_o \text{ or } Q_y = \bar{Q}_y & \theta_z &= \bar{\theta}_z \text{ or } S_y = \bar{S}_y \\
u_o^* &= \bar{u}_o^* \text{ or } N_{xy}^* = \bar{N}_{xy}^* & \theta_x^* &= \bar{\theta}_x^* \text{ or } M_{xy}^* = \bar{M}_{xy}^* \\
v_o^* &= \bar{v}_o^* \text{ or } N_y^* = \bar{N}_y^* & \theta_y^* &= \bar{\theta}_y^* \text{ or } M_y^* = \bar{M}_y^*
\end{aligned}$$

$$w_o^* = \bar{w}_o^* \text{ or } Q_y^* = \bar{Q}_y^* \qquad \theta_z^* = \bar{\theta}_z^* \text{ or } S_y^* = \bar{S}_y^* \tag{10}$$

where the stress resultants are defined by

$$\begin{bmatrix} M_x & M_x^* \\ M_y & M_y^* \\ M_z & 0 \\ M_{xy} & M_{xy}^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{Bmatrix} [z \quad z^3] dz \tag{11}$$

$$\begin{bmatrix} Q_x & Q_x^* \\ Q_y & Q_y^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} [1 \quad z^2] dz \tag{12}$$

$$\begin{bmatrix} N_x & N_x^* \\ N_y & N_y^* \\ N_z & N_z^* \\ N_{xy} & N_{xy}^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \end{Bmatrix} [1 \quad z^2] dz \tag{13}$$

$$\begin{bmatrix} S_x & S_x^* \\ S_y & S_y^* \end{bmatrix} = \sum_{L=1}^{NL} \int_{z_L}^{z_{L+1}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix} [z \quad z^3] dz \tag{14}$$

where

$$\begin{bmatrix} \bar{N}_x & \bar{N}_y & \bar{N}_{xy} \\ \bar{M}_x & \bar{M}_y & \bar{M}_{xy} \\ \bar{N}_x^* & \bar{N}_y^* & N_{xy}^* \\ \bar{M}_x^* & \bar{M}_y^* & \bar{M}_{xy}^* \\ \hat{N}_x^* & \hat{N}_y^* & \hat{N}_{xy}^* \\ \hat{M}_x^* & \hat{M}_y^* & \hat{M}_{xy}^* \\ \bar{N}_x^{**} & \bar{N}_y^{**} & \bar{N}_{xy}^{**} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \\ z^4 \\ z^5 \\ z^6 \end{Bmatrix} [\sigma'_x \quad \sigma'_y \quad \sigma'_{xy}] dz \tag{15}$$

The resultants in Eqs. (11) to (14) can be related to the total strains in Eq. (2) by the following equations;

$$\begin{Bmatrix} N_x \\ N_y \\ N_x^* \\ N_y^* \\ N_z \\ N_z^* \\ M_x \\ M_y \\ M_x^* \\ M_y^* \\ M_z^* \end{Bmatrix} = [A]\begin{Bmatrix} \varepsilon_{xo} \\ \varepsilon_{yo} \\ \varepsilon_{xo}^* \\ \varepsilon_{yo}^* \\ \varepsilon_{zo} \\ \varepsilon_{zo}^* \\ \kappa_x \\ \kappa_y \\ \kappa_x^* \\ \kappa_y^* \\ \kappa_z \end{Bmatrix} + [A']\begin{Bmatrix} \varepsilon_{xyo} \\ \varepsilon_{xyo}^* \\ \kappa_{xy} \\ \kappa_{xy}^* \end{Bmatrix} \qquad \begin{Bmatrix} N_{xy} \\ N_{xy}^* \\ M_{xy} \\ M_{xy}^* \end{Bmatrix} = [B']\begin{Bmatrix} \varepsilon_{xo} \\ \varepsilon_{yo} \\ \varepsilon_{xo}^* \\ \varepsilon_{yo}^* \\ \varepsilon_{zo} \\ \varepsilon_{zo}^* \\ \kappa_x \\ \kappa_y \\ \kappa_x^* \\ \kappa_y^* \\ \kappa_z \end{Bmatrix} + [B]\begin{Bmatrix} \varepsilon_{xyo} \\ \varepsilon_{xyo}^* \\ \kappa_{xy} \\ \kappa_{xy}^* \end{Bmatrix} \tag{16}$$

$$\begin{Bmatrix} Q_x \\ Q_x^* \\ S_x \\ S_x^* \end{Bmatrix} = [D]\begin{Bmatrix} \phi_x \\ \phi_x^* \\ \kappa_{xz} \\ \kappa_{xz}^* \end{Bmatrix} + [D']\begin{Bmatrix} \phi_y \\ \phi_y^* \\ \kappa_{yz} \\ \kappa_{yz}^* \end{Bmatrix} \qquad \begin{Bmatrix} Q_y \\ Q_y^* \\ S_y \\ S_y^* \end{Bmatrix} = [E']\begin{Bmatrix} \phi_x \\ \phi_x^* \\ \kappa_{xz} \\ \kappa_{xz}^* \end{Bmatrix} + [E]\begin{Bmatrix} \phi_y \\ \phi_y^* \\ \kappa_{yz} \\ \kappa_{yz}^* \end{Bmatrix} \tag{17}$$

where the matrices $[A]$, $[A']$, $[B]$, $[B']$, $[D]$, $[D']$, $[E]$, $[E']$ are the matrices of plate stiffnesses whose elements are defined in Appendix C.

3. Exact solutions for simply supported rectangular plates

Here the exact solution of Eqs. (8)-(17) for cross-ply rectangular plates are considered. Assuming that the plate is simply supported in such a manner that normal displacement is admissible, but the tangential displacement is not, the following boundary conditions are appropriate:

At edges $x = 0$ and $x = a$:

$$\begin{gathered} v_o = 0; \quad w_o = 0; \quad \theta_y = 0; \quad \theta_z = 0; \quad M_x = 0; \\ v_o^* = 0; \quad w_o^* = 0; \quad \theta_y^* = 0; \quad \theta_z^* = 0; \quad M_x^* = 0; \\ N_x = 0; \quad N_x^* = 0. \end{gathered} \tag{18}$$

At edges $y = 0$ and $y = b$:

$$\begin{gathered} u_o = 0; \quad w_o = 0; \quad \theta_x = 0; \quad \theta_z = 0; \quad M_y = 0; \\ u_o^* = 0; \quad w_o^* = 0; \quad \theta_x^* = 0; \quad \theta_z^* = 0; \quad M_y^* = 0; \\ N_y = 0; \quad N_y^* = 0. \end{gathered} \tag{19}$$

Following Navier's solution procedure (Timoshenko and Woinowsky-Krieger 1959, Szilard 1974, Reddy 1996) the solution to the displacement variables satisfying the above boundary conditions can be expressed in the following forms:

$$u_o = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{o_{mn}} \cos\alpha x \ \sin\beta y$$

$$v_o = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{o_{mn}} \sin\alpha x \ \cos\beta y$$

$$w_o = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{o_{mn}} \sin\alpha x \ \sin\beta y$$

$$\theta_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{x_{mn}} \cos\alpha x \ \sin\beta y$$

$$\theta_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{y_{mn}} \sin\alpha x \ \cos\beta y$$

$$\theta_z = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{z_{mn}} \sin\alpha x \ \sin\beta y$$

$$u_o^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{o_{mn}}^* \cos\alpha x \ \sin\beta y$$

$$v_o^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{o_{mn}}^* \sin\alpha x \ \cos\beta y$$

$$w_o^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{o_{mn}}^* \sin\alpha x \ \sin\beta y$$

$$\theta_x^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{x_{mn}}^* \cos\alpha x \ \sin\beta y$$

$$\theta_y^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{y_{mn}}^* \sin\alpha x \ \cos\beta y$$

$$\theta_z^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{z_{mn}}^* \sin\alpha x \ \sin\beta y$$

$$P_z^+ = 0 \tag{20}$$

where $\alpha = \dfrac{m\pi}{a}, \beta = \dfrac{n\pi}{b}$

Substituting Eqs. (18), (19) and (20) into Eq. (8) and collecting the coefficients one obtains

$$([X]_{12\times 12}-\lambda[G]_{12\times 12})\begin{Bmatrix} u_o \\ v_o \\ w_o \\ \theta_x \\ \theta_y \\ \theta_z \\ u_o^* \\ v_o^* \\ w_o^* \\ \theta_x^* \\ \theta_y^* \\ \theta_z^* \end{Bmatrix}_{12\times 1} = \{0\} \text{ where } \lambda\text{=buckling load factor} \tag{21}$$

for any fixed values of m and n. The matix $[G]$ refers to the stiffness matrix due to inplane forces. The elements of coefficient matrix $[X]$ and $[G]$ are given in Appendix D. The above equation can be solved and the lowest eigenvalue ($\lambda_{\min} = \lambda_{cr}$) is the critical buckling load factor and the critical uniaxial buckling load is given by

$$[\bar{N}_x]_{cr} = \lambda_{cr}[\bar{N}_x] \tag{22}$$

4. Numerical results and discussions

The various shear deformation theories considered for comparison is given in Table 1. Three, four and ten layer symmetric cross-ply laminated plates are considered for the present study. The orthotropic material properties of individual layers in all the above laminate considered are E_1/E_2 open, $E_2=E_3$, $G_{12}=G_{13}=0.6E_2$, $G_{23}=0.5E_2$, $\upsilon_{12}=\upsilon_{13}=\upsilon_{23}=0.25$. The numerical results of three, five and ten layer laminates are shown in Table 2 and are compared with the three-dimensional elasticity solution given by Noor (1975). For all the laminate types considered E_1/E_2, at lower range of ratios that is 3 and 10, Kant-Manjunatha theory gives better accurate results of critical buckling load

Table 1 Displacement models (Shear deformation theories) compared

Author	Theory	Year (Ref.)	Degrees of freedom	Transverse normal deformation
Kant-Manjunatha	HSDT	1988	12	Considered
Pandya-Kant	HSDT	1988(d)	9	Not considered
Reddy	HSDT	1984(a)	5	Not considered
Senthilnathan *et al.*	HSDT	1987	4	Not considered
Whitney-Pagano	FSDT	1970	5	Not considered

Table 2 Nondimensionalized critical buckling coefficients $\bar{N}_x = N_x b^2/(E_2 h^3)$ for simply supported cross-ply square laminated plates with a/h=10, E_2=E_3, G_{12}=G_{13}=0.6E_2, G_{23}=0.5E_2, ν_{12}=ν_{13}=ν_{23}=0.25

Lamination and Number of Layers	Source	E_1/E_2				
		3	10	20	30	40
$(0°/\bar{90}°)_s$	3D Elasticity	5.3044	9.7621	15.0191	19.3040	22.8807
	Kant-Manjunatha	5.3745 (1.32)†	9.8066 (0.46)	14.8522 (-1.11)	18.8313 (-2.45)	22.0671 (-3.56)
	Pandya-Kant	5.3896 (1.61)	9.8319 (0.72)	14.8882 (-0.87)	18.8750 (-2.22)	22.1163 (-3.34)
	Reddy	5.3899 (1.61)	9.8325 (0.72)	14.8896 (-0.86)	18.8776 (-2.21)	22.1207 (-3.32)
	Senthilnathan *et al.*	5.4142 (2.07)	10.2133 (4.62)	16.2309 (8.07)	21.4288 (11.0)	25.9651 (13.48)
	Whitney-Pagano	5.3961 (1.73)	9.8711 (1.12)	14.9846 (-0.23)	19.0265 (1.44)	22.3151 (-2.47)
$(0°/90°/\bar{0}°)_s$	3D Elasticity	5.3255	9.9603	15.6527	20.4663	24.5929
	Kant-Manjunatha	5.3911 (1.23)	10.0552 (0.92)	15.7152 (0.40)	20.4584 (-0.04)	24.5026 (-0.37)
	Pandya-Kant	5.4063 (1.52)	10.0811 (1.21)	15.7529 (0.64)	20.5047 (0.19)	24.5553 (-0.15)
	Reddy	5.4066 (1.52)	10.0897 (1.30)	15.7879 (0.86)	20.5781 (0.55)	24.6755 (0.34)
	Senthilnathan *et al.*	5.4142 (1.67)	10.2133 (2.54)	16.2309 (3.69)	21.4288 (4.70)	25.9651 (5.58)
	Whitney-Pagano	5.4068 (1.53)	10.0762 (1.16)	15.7362 (0.53)	20.4847 (0.09)	24.5465 (-0.19)
$(0°/90°/0°/90°/\bar{0}°)_s$	3D Elasticity	5.3352	10.0417	15.9153	20.9614	25.3436
	Kant-Manjunatha	5.3966 (1.15)	10.1451 (1.03)	16.0390 (0.78)	21.0821 (0.58)	25.4511 (0.42)
	Pandya-Kant	5.4177 (1.55)	10.1714 (1.29)	16.0778 (1.02)	21.1304 (0.81)	25.5068 (0.64)
	Reddy	5.4120 (1.44)	10.1772 (1.35)	16.1007 (1.16)	21.1779 (1.03)	25.5840 (0.95)
	Senthilnathan *et al.*	5.4142 (1.48)	10.2133 (1.71)	16.2309 (1.98)	21.4288 (2.23)	25.9651 (2.45)
	Whitney-Pagano	5.4116 (1.43)	10.1682 (1.26)	16.0680 (0.96)	21.1168 (0.74)	25.4940 (0.59)

†Numbers in the parenteses are the percentage error with respect to 3D elasticity values.

Table 3 Variation of critical buckling coefficients $\bar{N}_x = N_x b^2/(E_2 h^3)$ with a/h for for simply supported cross-ply square laminated plate $E_1 E_2$=40, E_2=E_3, G_{12}=G_{13}=0.6E_2, G_{23}=0.5E_2, ν_{12}=ν_{13}=ν_{23}=0.25

Lamination and Number of Layers	Source	a/h					
		2	4	10	20	50	100
(0°/90°/0°)	Kant-Manjunatha	2.8065	8.0554	22.0671	31.0541	35.2248	35.9211
	Pandya-Kant	2.9065	8.1381	22.1164	31.0742	35.2287	35.9211
	Reddy	3.0433	8.1752	22.1207	31.0767	35.2293	35.9211
	Senthilnathan *et al.*	3.6802	10.6504	25.9651	32.9173	35.5981	36.0176
	Whitney-Pagano	2.8540	8.1631	22.3151	31.1959	35.2552	35.9290
(0°/90°/90°/0°)	Kant-Manjunatha	2.9495	8.8148	23.2527	31.6278	35.3409	35.9511
	Pandya-Kant	3.0342	8.8901	23.3026	31.6481	35.3448	35.9521
	Reddy	3.1514	8.9822	23.3400	31.6596	35.3467	35.9526
	Senthilnathan *et al.*	3.6802	10.6504	25.9651	32.9173	35.5981	36.0176
	Whitney-Pagano	3.1099	9.1138	23.4529	31.7071	35.3560	35.9550

compared to all other theories. At higher range of E_1/E_2 equal to 20, 30 and 40, the theory of Kant-Manjunatha predicts the critical buckling load more accurately than all other theories in the case of five and ten layer laminate whereas the first order theory of Whitney-Pagano gives most accurate results in the case of three-layer laminate. The results of critical buckling load with respect various plate side-to-thickness a/h ratio of a three and four layer symmetric laminate are shown in Table 3. For a/h equal to 2, the theories of Reddy and Senthilnathan *et al.* differ from Kant-Manjunatha theory by 8.42% and 31.13% in the case of three-layer laminate and by 6.85% and 24.77% in the case of four-layer laminate. The percentage difference between all the theories is minimum in the case very thin laminate which shows that the effect of shear deformation is quite significant on the buckling parameter for thick and relatively thick laminates only. The results also confirm the fact that as the number of layers increases the percentage difference between the theories and hence the effect of shear deformation tend to decrease.

The variation of critical buckling load with respect to plate aspect ratio (a/b), side-to-thickness ratio (a/h) and the ratio of core thickness to face sheet thickness (t_c/t_f) of a five layer symmetric (0°/90°/core/90°/0°) sandwich plate is shown in Tables 4-6. The following material properties are used for the face sheets and the core.

Face sheets (Graphite-Epoxy T300/934)

$$E_1 = 19\times10^6 \text{ psi (131 GPa)} \quad E_2 = 1.5\times10^6 \text{ psi (10.34 Gpa)} \quad E_2 = E_3$$

Table 4 Critical buckling coefficients $\bar{N}_x = N_x b^2/[(E_2)_f h^3]$ of a symmetric (0°/90°/core/90°/0°) sandwich plate with $a/b=1$ and $t_c/t_f=10$

a/h	Kant-Manjunatha	Pandya-Kant	Reddy	Senthilnathan *et al.*	Whitney-Pagano
2	0.0315	0.0305	0.0583	0.0627	0.5995
4	0.0972	0.0963	0.2115	0.2159	1.8339
10	0.5181	0.5175	1.0909	1.0951	4.3197
20	1.6220	1.6221	2.7913	2.7946	5.3628
30	2.6932	2.6937	3.9213	3.9235	5.5983
40	3.5256	3.5261	4.5695	4.5714	5.6899
50	4.1139	4.1144	4.9634	4.9646	5.7495
60	4.5323	4.5327	5.2100	5.2110	5.7855
70	4.8091	4.8094	5.3553	5.3560	5.7907
80	5.0164	5.0166	5.4610	5.4615	5.8013
90	5.1657	5.1659	5.5328	5.5332	5.8053
100	5.2794	5.2795	5.5862	5.5866	5.8097

Table 5 Critical buckling coefficients $\bar{N}_x = N_x b^2/[(E_2)_f h^3]$ of a symmetric (0°/90°/core/90°/0°) sandwich plate with $a/b=1$ and $a/h=10$

t_c/t_f	Kant-Manjunatha	Pandya-Kant	Reddy	Senthilnathan *et al.*	Whitney-Pagano
4	3.0763	3.0783	4.4039	4.4302	7.5160
10	0.5184	0.5178	1.0915	1.0957	4.3206
20	0.1454	0.1453	0.2815	0.2824	2.5007
30	0.0974	0.0973	0.1455	0.1459	1.7577
40	0.0845	0.0845	0.1057	0.1058	1.3330
50	0.0796	0.0796	0.0976	0.0908	1.1041
100	0.0726	0.0726	0.0734	0.0737	0.5728

Table 6 Critical buckling coefficients $\bar{N}_x = N_x b^2 / [(E_2)_f h^3]$ of a symmetric (0°/90°/core/90°/0°) sandwich plate with a/h=10 and t_c/t_f=10

a/b	Kant-Manjunatha	Pandya-Kant	Reddy	Senthilnathan *et al.*	Whitney-Pagano
0.5	1.2997	1.2991	2.6361	2.7124	9.11
1.0	0.5181	0.5175	1.0909	1.0951	4.3197
1.5	0.3803	0.3795	0.8327	0.8328	4.4597
2.0	0.3360	0.3348	0.7485	0.7494	5.1479
2.5	0.3194	0.3177	0.7124	0.7131	5.8048
3.0	0.3144	0.3122	0.6952	0.6955	6.3310
5.0	0.3386	0.3337	0.6892	0.6892	7.459

$$G_{12}=1\times10^6 \text{ psi } (6.895 \text{ GPa}) \quad G_{13}=0.90\times10^6 \text{ psi } (6.205 \text{ GPa})$$

$$G_{23}=1\times10^6 \text{ psi } (6.895 \text{ GPa})$$

$$\upsilon_{12}=0.22 \quad \upsilon_{13}=0.22 \quad \upsilon_{23}=0.49$$

Core Properties (Isotropic)

$$E_1=E_2=E_3=2G=1000 \text{ psi } (6.89\times10^{-3} \text{ GPa})$$

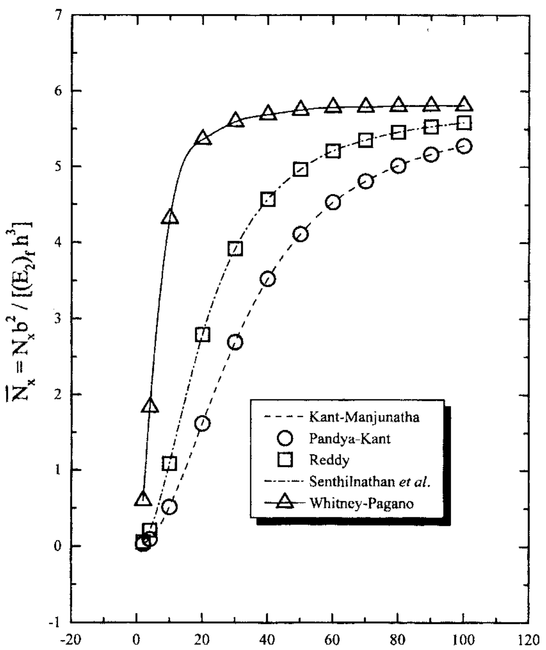

Fig. 2 Nondimensionalized buckling load ($\bar{N}_x$) versus side-to-thickness ratio (a/h) of a five-layer sandwich plate subjected to in-plane compressive load along the edges x=0, a

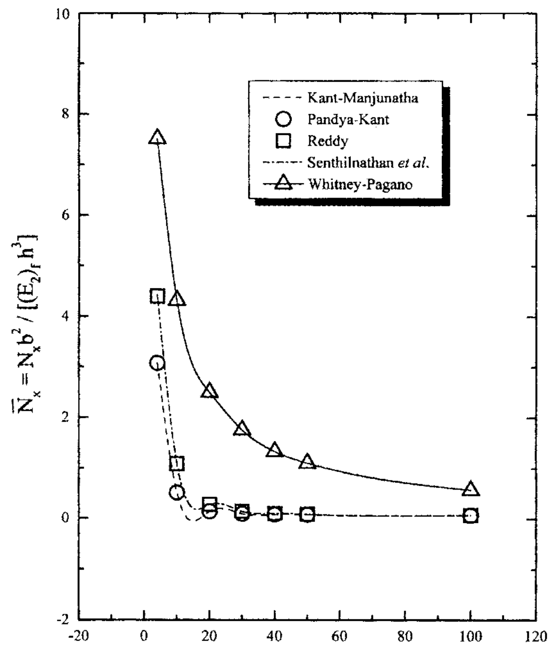

Fig. 3 Nondimensionalized buckling load ($\bar{N}_x$) versus thickness of core-to-thickness of face sheet ratio (t_c/t_f) of a five-layer sandwich plate subjected to in-plane compressive load along the edges x=0, a

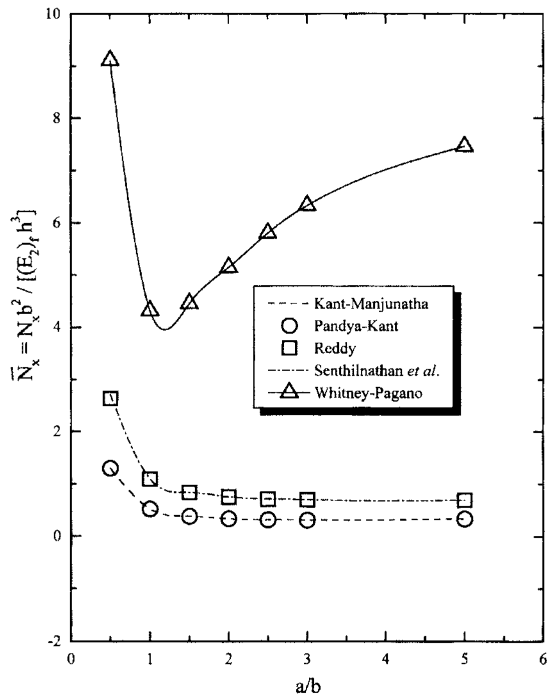

Fig. 4 Nondimensionalized buckling load ($\bar{N}_x$) versus side-to-thickness ratio (a/b) of a five-layer sandwich plate subjected to in-plane compressive load along the edges x = 0, a.

$$G_{12}=G_{13}=G_{23}=500 \text{ psi } (3.45\times10^{-3} \text{ GPa})$$

$$\upsilon_{12}=\upsilon_{13}=\upsilon_{23}=0$$

It can be observed that the results of Kant-Manjunatha and Pandya-Kant are in good agreement for all the above parameters considered. The theories of Reddy and Senthilnathan *et al.* over-predicts the critical buckling load in all the above cases. The results of of various parametric studies clearly indicate that the first order theory very much overpredicts the critical buckling load compared to other higher order theories. The variation of critical buckling load with respect to various parameters, that is a/h, t_c/t_f, a/b are shown in graphical form in Figs. 2-4.

5. Conclusions

A higher order refined shear deformation theory developed by the first author and already reported in the literature has been used for the stability analysis of laminated composite and sandwich plates. The displacement field of this theory takes into account both the transverse shear and normal deformation thus making it more accurate than the first order and other higher order theories considered in this study. Analytical formulations and solutions using this refined theory are presented for the first time. The comparison of results shows that for laminated composite plates the solution of this higher order refined theory are found to be in excellent agreement with the three-dimensional elasticity solution and the percentage error with respect to 3D elasticity solution is very much less compared to other theories. For sandwich plates the results of Kant-Manjunatha and Pandya-Kant theories are in good agreement whereas the first order theory and the theories of Reddy and Senthilnathan *et al.* overestimates the critical buckling loads for all the parameters considered.

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Appendix A

Coefficients of $[C]$ matrix

$$C_{11}=\frac{E_1(1-\upsilon_{23}\upsilon_{32})}{\Delta};\quad C_{12}=\frac{E_1(\upsilon_{21}+\upsilon_{31}\upsilon_{23})}{\Delta}$$

$$C_{13}=\frac{E_1(\upsilon_{31}+\upsilon_{21}\upsilon_{32})}{\Delta};\quad C_{22}=\frac{E_2(1-\upsilon_{13}\upsilon_{31})}{\Delta}$$

$$C_{23}=\frac{E_2(\upsilon_{32}+\upsilon_{12}\upsilon_{31})}{\Delta};\quad C_{33}=\frac{E_3(1-\upsilon_{12}\upsilon_{21})}{\Delta}$$

$$C_{44}=G_{12};\quad C_{55}=G_{23};\quad C_{66}=G_{13}$$

where

$$\Delta=(1-\upsilon_{12}\upsilon_{21}-\upsilon_{23}\upsilon_{32}-\upsilon_{31}\upsilon_{13}-2\upsilon_{12}\upsilon_{23}\upsilon_{31})$$

and

$$\varepsilon_1=\frac{\sigma_1}{E_1}-\upsilon_{21}\frac{\sigma_2}{E_2}-\upsilon_{31}\frac{\sigma_3}{E_3}$$

$$\varepsilon_2=\frac{\sigma_2}{E_2}-\upsilon_{32}\frac{\sigma_3}{E_3}-\upsilon_{12}\frac{\sigma_1}{E_1}$$

$$\varepsilon_3=\frac{\sigma_3}{E_3}-\upsilon_{13}\frac{\sigma_1}{E_1}-\upsilon_{23}\frac{\sigma_2}{E_2}$$

$$\gamma_{12}=\frac{\tau_{12}}{G_{12}};\quad \gamma_{23}=\frac{\tau_{23}}{G_{23}};\quad \gamma_{13}=\frac{\tau_{13}}{G_{13}}$$

$$\frac{\upsilon_{12}}{E_1}=\frac{\upsilon_{21}}{E_2};\quad \frac{\upsilon_{31}}{E_3}=\frac{\upsilon_{13}}{E_1};\quad \frac{\upsilon_{32}}{E_3}=\frac{\upsilon_{23}}{E_2}$$

Appexdix B

Coefficients of $[Q]$ matrix

$$
\begin{aligned}
Q_{11} &= C_{11}c^4 + 2(C_{12} + 2C_{44})s^2c^2 + C_{22}s^4 \\
Q_{12} &= C_{12}(c^4 + s^4) + (C_{11} + C_{22} - 4C_{44})s^2c^2 \\
Q_{13} &= C_{13}c^2 + C_{23}s^2 \\
Q_{14} &= (C_{11} - C_{12} - 2C_{44})s\,c^3 + (C_{12} - C_{22} + 2C_{44})cs^3 \\
Q_{22} &= C_{11}s^4 + C_{22}c^4 + (2C_{12} + 4C_{44})s^2c^2 \\
Q_{23} &= C_{13}s^2 + C_{23}c^2 \\
Q_{24} &= (C_{11} - C_{12} - 2C_{44})s^3c + (C_{12} - C_{22} + 2C_{44})c^3s \\
Q_{33} &= C_{33} \\
Q_{34} &= (C_{31} - C_{32})sc \\
Q_{44} &= (C_{11} - 2C_{12} + C_{22} - 2C_{44})c^2s^2 + C_{44}(c^4 + s^4) \\
Q_{55} &= C_{55}c^2 + C_{66}s^2 \\
Q_{56} &= (C_{66} - C_{55})cs \\
Q_{66} &= (C_{55}s^2 + C_{66}c^2)
\end{aligned}
$$

and $$Q_{ij} = Q_{ji}, \quad i, j = 1 \text{ to } 6$$

where $$c = \cos\alpha \quad \text{and} \quad s = \sin\alpha$$

Appendix C

Elements of $[A]$, $[A']$, $[B]$, $[B']$, $[D]$, $[D']$, $[E]$, $[E']$ matrices

$$
[A] = \sum_{L=1}^{NL} \begin{bmatrix}
Q_{11}H_1 & Q_{12}H_1 & Q_{11}H_3 & Q_{12}H_3 & Q_{13}H_1 & 3Q_{13}H_3 & Q_{11}H_2 & Q_{12}H_2 & Q_{11}H_4 & Q_{12}H_4 & 2Q_{13}H_3 \\
Q_{12}H_1 & Q_{22}H_1 & Q_{12}H_3 & Q_{22}H_3 & Q_{23}H_1 & 3Q_{23}H_3 & Q_{12}H_2 & Q_{22}H_2 & Q_{12}H_4 & Q_{22}H_4 & 2Q_{23}H_2 \\
Q_{11}H_3 & Q_{12}H_3 & Q_{11}H_5 & Q_{12}H_5 & Q_{13}H_3 & 3Q_{13}H_5 & Q_{11}H_4 & Q_{12}H_4 & Q_{11}H_6 & Q_{12}H_6 & 2Q_{13}H_4 \\
Q_{12}H_3 & Q_{22}H_3 & Q_{12}H_5 & Q_{22}H_5 & Q_{23}H_3 & 3Q_{23}H_5 & Q_{12}H_4 & Q_{22}H_4 & Q_{12}H_6 & Q_{22}H_6 & 2Q_{23}H_4 \\
Q_{13}H_1 & Q_{23}H_1 & Q_{13}H_3 & Q_{23}H_3 & Q_{33}H_1 & 3Q_{33}H_3 & Q_{13}H_2 & Q_{23}H_2 & Q_{13}H_4 & Q_{23}H_4 & 2Q_{33}H_2 \\
Q_{13}H_3 & Q_{23}H_3 & Q_{13}H_5 & Q_{23}H_5 & Q_{33}H_3 & 3Q_{33}H_5 & Q_{13}H_4 & Q_{23}H_4 & Q_{13}H_6 & Q_{23}H_6 & 2Q_{33}H_4 \\
Q_{11}H_2 & Q_{12}H_2 & Q_{11}H_4 & Q_{12}H_4 & Q_{13}H_2 & 3Q_{13}H_4 & Q_{11}H_3 & Q_{12}H_3 & Q_{11}H_5 & Q_{12}H_5 & 2Q_{13}H_3 \\
Q_{12}H_2 & Q_{22}H_2 & Q_{12}H_4 & Q_{22}H_4 & Q_{23}H_2 & 3Q_{23}H_4 & Q_{12}H_3 & Q_{22}H_3 & Q_{12}H_5 & Q_{22}H_5 & 2Q_{23}H_3 \\
Q_{11}H_4 & Q_{12}H_4 & Q_{11}H_6 & Q_{12}H_6 & Q_{13}H_4 & 3Q_{13}H_6 & Q_{11}H_5 & Q_{12}H_5 & Q_{11}H_7 & Q_{12}H_7 & 2Q_{13}H_5 \\
Q_{12}H_4 & Q_{22}H_4 & Q_{12}H_6 & Q_{22}H_6 & Q_{23}H_4 & 3Q_{23}H_6 & Q_{12}H_5 & Q_{22}H_5 & Q_{12}H_7 & Q_{22}H_7 & 2Q_{23}H_5 \\
Q_{13}H_2 & Q_{23}H_2 & Q_{13}H_4 & Q_{23}H_4 & Q_{33}H_2 & 3Q_{33}H_4 & Q_{13}H_3 & Q_{23}H_3 & Q_{13}H_5 & Q_{23}H_5 & 2Q_{33}H_3
\end{bmatrix}
$$

$$
[B] = \sum_{L=1}^{NL} \begin{bmatrix}
Q_{44}H_1 & Q_{44}H_3 & Q_{44}H_2 & Q_{44}H_4 \\
Q_{44}H_3 & Q_{44}H_5 & Q_{44}H_4 & Q_{44}H_6 \\
Q_{44}H_2 & Q_{44}H_4 & Q_{44}H_3 & Q_{44}H_5 \\
Q_{44}H_4 & Q_{44}H_6 & Q_{44}H_5 & Q_{44}H_7
\end{bmatrix}
$$

$$[A'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{14}H_1 & Q_{14}H_3 & Q_{14}H_2 & Q_{14}H_4 \\ Q_{24}H_1 & Q_{24}H_3 & Q_{24}H_2 & Q_{24}H_4 \\ Q_{14}H_3 & Q_{14}H_5 & Q_{14}H_4 & Q_{14}H_6 \\ Q_{24}H_3 & Q_{24}H_5 & Q_{24}H_4 & Q_{24}H_6 \\ Q_{34}H_1 & Q_{34}H_3 & Q_{34}H_2 & Q_{34}H_4 \\ Q_{34}H_3 & Q_{34}H_5 & Q_{34}H_4 & Q_{34}H_6 \\ Q_{14}H_2 & Q_{14}H_4 & Q_{14}H_3 & Q_{14}H_5 \\ Q_{24}H_2 & Q_{24}H_4 & Q_{24}H_3 & Q_{24}H_5 \\ Q_{14}H_4 & Q_{14}H_6 & Q_{14}H_5 & Q_{14}H_7 \\ Q_{24}H_4 & Q_{24}H_6 & Q_{24}H_5 & Q_{24}H_7 \\ Q_{34}H_2 & Q_{34}H_4 & Q_{34}H_3 & Q_{34}H_5 \end{bmatrix}$$

$$[B'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{14}H_1 & Q_{24}H_1 & Q_{14}H_3 & Q_{24}H_3 & Q_{34}H_1 & 3Q_{34}H_3 & Q_{14}H_2 & Q_{24}H_2 & Q_{14}H_4 & Q_{24}H_4 & 2Q_{34}H_2 \\ Q_{14}H_3 & Q_{24}H_3 & Q_{14}H_5 & Q_{24}H_5 & Q_{34}H_3 & 3Q_{34}H_5 & Q_{14}H_4 & Q_{24}H_4 & Q_{14}H_6 & Q_{24}H_6 & 2Q_{34}H_4 \\ Q_{14}H_2 & Q_{24}H_2 & Q_{14}H_4 & Q_{24}H_4 & Q_{34}H_2 & 3Q_{34}H_4 & Q_{14}H_3 & Q_{24}H_3 & Q_{14}H_5 & Q_{24}H_5 & 2Q_{34}H_3 \\ Q_{14}H_4 & Q_{24}H_4 & Q_{14}H_6 & Q_{24}H_6 & Q_{34}H_4 & 3Q_{34}H_6 & Q_{14}H_5 & Q_{24}H_5 & Q_{14}H_7 & Q_{24}H_7 & 2Q_{34}H_5 \end{bmatrix}$$

$$[D] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{66}H_1 & Q_{66}H_3 & Q_{66}H_2 & Q_{66}H_4 \\ Q_{66}H_3 & Q_{66}H_5 & Q_{66}H_4 & Q_{66}H_6 \\ Q_{66}H_2 & Q_{66}H_4 & Q_{66}H_3 & Q_{66}H_5 \\ Q_{66}H_4 & Q_{66}H_6 & Q_{66}H_5 & Q_{66}H_7 \end{bmatrix}$$

$$[D'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{56}H_1 & Q_{56}H_3 & Q_{56}H_2 & Q_{56}H_4 \\ Q_{56}H_3 & Q_{56}H_5 & Q_{56}H_4 & Q_{56}H_6 \\ Q_{56}H_2 & Q_{56}H_4 & Q_{56}H_3 & Q_{56}H_5 \\ Q_{56}H_4 & Q_{56}H_6 & Q_{56}H_5 & Q_{56}H_7 \end{bmatrix}$$

$$[E] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{55}H_1 & Q_{55}H_3 & Q_{55}H_2 & Q_{55}H_4 \\ Q_{55}H_3 & Q_{55}H_5 & Q_{55}H_4 & Q_{55}H_6 \\ Q_{55}H_2 & Q_{55}H_4 & Q_{55}H_3 & Q_{55}H_5 \\ Q_{55}H_4 & Q_{55}H_6 & Q_{55}H_5 & Q_{55}H_7 \end{bmatrix}$$

$$[E'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{56}H_1 & Q_{56}H_3 & Q_{56}H_2 & Q_{56}H_4 \\ Q_{56}H_3 & Q_{56}H_5 & Q_{56}H_4 & Q_{56}H_6 \\ Q_{56}H_2 & Q_{56}H_4 & Q_{56}H_3 & Q_{56}H_5 \\ Q_{56}H_4 & Q_{56}H_6 & Q_{56}H_5 & Q_{56}H_7 \end{bmatrix}$$

Appendix D

Coefficients of matrix [X]

$X_{1,1}=A_{1,1}\alpha^2+B_{1,1}\beta^2$ $X_{1,2}=A_{1,2}\alpha\beta+B_{1,2}\alpha\beta$

$X_{1,3}=0$ $X_{1,4}=A_{1,7}\alpha^2+B_{1,5}\beta^2$ $X_{1,5}=A_{1,8}\alpha\beta+B_{1,6}\alpha\beta$

$X_{1,6}=-A_{1,5}\alpha$ $X_{1,7}=A_{1,3}\alpha^2+B_{1,3}\beta^2$

$X_{1,8}=A_{1,4}\alpha\beta+B_{1,4}\alpha\beta$ $X_{1,9}=-A_{1,11}\alpha$

$X_{1,10}=A_{1,9}\alpha^2+B_{1,7}\beta^2$ $X_{1,11}=A_{1,10}\alpha\beta+B_{1,8}\alpha\beta$

$X_{1,12}=-A_{1,6}\alpha$

$X_{2,2}=A_{2,2}\beta^2+B_{1,2}\alpha^2$ $X_{2,3}=0$ $X_{2,4}=A_{2,7}\alpha\beta+B_{1,5}\alpha\beta$

$X_{2,5}=A_{2,8}\beta^2+B_{1,6}\alpha^2$ $X_{2,6}=-A_{2,5}\beta$

$X_{2,7}=A_{2,3}\alpha\beta+B_{1,3}\alpha\beta$ $X_{2,8}=A_{2,4}\beta^2+B_{1,4}\alpha^2$

$X_{2,9}=-A_{2,11}\beta$ $X_{2,10}=A_{2,9}\alpha\beta+B_{1,7}\alpha\beta$

$X_{2,11}=A_{2,10}\beta^2+B_{1,8}\alpha^2$ $X_{2,12}=-A_{2,6}\beta$

$X_{3,3}=D_{1,2}\alpha^2+E_{1,2}\beta^2$ $X_{3,4}=D_{1,1}\alpha$

$X_{3,5}=E_{1,1}\beta$ $X_{3,6}=D_{1,6}\alpha^2+E_{1,6}\beta_2$

$X_{3,7}=D_{1,5}\alpha$ $X_{3,8}=E_{1,5}\beta$

$X_{3,9}=D_{1,4}\alpha^2+E_{1,4}\beta^2$ $X_{3,10}=D_{1,3}\alpha$

$X_{3,11}=E_{1,3}\beta$ $X_{3,12}=D_{1,7}\alpha^2+E_{1,7}\beta^2$

$X_{4,4}=A_{7,7}\alpha^2+B_{3,5}\beta^2+D_{1,1}$ $X_{4,5}=A_{7,8}\alpha\beta+B_{3,6}\alpha\beta$

$X_{4,6}=-A_{7,5}\alpha+D_{1,6}\alpha$ $X_{4,7}=A_{7,3}\alpha^2+B_{3,3}\beta^2+D_{1,5}$

$X_{4,8}=A_{7,4}\alpha\beta+B_{3,4}\alpha\beta$ $X_{4,9}=-A_{7,11}\alpha+D_{1,4}\alpha$

$X_{4,10}=A_{7,9}\alpha^2+B_{3,7}\beta^2+D_{1,3}$ $X_{4,11}=A_{7,10}\alpha\beta+B_{3,8}\alpha\beta$

$X_{4,12}=-A_{7,6}\alpha+D_{1,7}\alpha$

$X_{5,5}=A_{8,8}\beta^2+B_{3,6}\alpha^2+E_{1,1}$ $X_{5,6}=-A_{8,5}\beta+E_{1,6}\beta$

$X_{5,7}=A_{8,3}\alpha\beta+B_{3,3}\alpha\beta$ $X_{5,8}=A_{8,4}\beta^2+B_{3,4}\alpha^2+E_{1,5}$

$X_{5,9}=-A_{8,11}\beta+E_{1,4}\beta$ $X_{5,10}=A_{8,9}\alpha\beta+B_{3,7}\alpha\beta$

$X_{5,11}=A_{8,10}\beta^2+B_{3,8}\alpha^2+E_{1,3}$ $X_{5,12}=-A_{8,6}\beta+E_{1,7}\beta$

$X_{6,6}=D_{3,6}\alpha^2+E_{3,6}\beta^2+A_{5,5}$ $X_{6,7}=D_{3,5}\alpha-A_{5,3}\alpha$

$X_{6,8}=E_{3,5}\beta-A_{5,4}\beta$ $X_{6,9}=D_{3,4}\alpha^2+E_{3,4}\beta^2+A_{5,11}$

$X_{6,10}=D_{3,3}\alpha-A_{5,9}\alpha$ $X_{6,11}=E_{3,3}\beta-A_{5,10}\beta$

$X_{6,12}=D_{3,7}\alpha^2+E_{3,7}\beta^2+A_{5,6}$

$X_{7,7}=A_{3,3}\alpha^2+B_{2,3}\beta^2+2D_{3,5}$ $X_{7,8}=A_{3,4}\alpha\beta+B_{2,4}\alpha\beta$

$X_{7,9}=-A_{3,11}\alpha+2D_{3,4}\alpha$ $X_{7,10}=A_{3,9}\alpha^2+B_{2,7}\beta^2+2D_{3,3}$

$X_{7,11}=A_{3,10}\alpha\beta+B_{2,8}\alpha\beta$ $\quad$ $X_{7,12}=-A_{3,6}\alpha+2D_{3,7}\alpha$

$X_{8,8}=A_{4,4}\beta^2+B_{2,4}\alpha^2+2E_{3,5}$ $\quad$ $X_{8,9}=-A_{4,11}\beta+2E_{3,4}\beta$

$X_{8,10}=A_{4,9}\alpha\beta+B_{2,7}\alpha\beta$ $\quad$ $X_{8,11}=A_{4,10}\beta^2+B_{2,8}\alpha^2+2E_{3,3}$

$X_{8,12}=-A_{4,6}\beta+2E_{3,7}\beta$

$X_{9,9}=D_{2,4}\alpha^2+E_{2,4}\beta^2+2A_{11,11}$ $\quad$ $X_{9,10}=D_{2,3}\alpha-2A_{11,9}\alpha$

$X_{9,11}=E_{2,3}\beta-2A_{11,10}\beta$ $\quad$ $X_{9,12}=D_{2,7}\alpha^2+E_{2,7}\beta^2+2A_{11,6}$

$X_{10,10}=A_{9,9}\alpha^2+B_{4,7}\beta^2+3D_{2,3}$ $\quad$ $X_{10,11}=A_{9,10}\alpha\beta+B_{4,8}\alpha\beta$

$X_{10,12}=-A_{9,6}\alpha+3D_{2,7}\alpha$

$X_{11,11}=A_{10,10}\beta^2+B_{4,8}\alpha^2+3E_{2,3}$ $\quad$ $X_{11,12}=-A_{10,6}\beta+3E_{2,7}\beta$

$X_{12,12}=D_{2,4}\alpha^2+E_{4,7}\beta^2+3A_{6,6}$ and $X_{i,j}=X_{j,i}$ $\quad (i, j=1, 12)$

Coefficients of matrix $[G]$

$G_{3,3}=\alpha^2$ and $G_{i,j}=0$, for all $i, j=1, 2,.....12.$ $(i \neq j \neq 3)$