Structural Engineering and Mechanics, Vol. 10, No. 3 (2000) 277-288
DOI: http://dx.doi.org/10.12989/sem.2000.10.3.277

A simplified dynamic analysis for estimation of the effect of rotary inertia and diaphragmatic operation on the behaviour of towers with additional masses

G.T. Michaltsos† and T.G. Konstantakopoulos‡

Department of Civil Engineering, National Technical University of Athens, 42 Patission Str., Athens, 106 82, Greece

Abstract. The present paper, deals with the dynamic analysis of a thin-walled tower with varying cross-section and additional masses. It, especially, deals with the effect of the rotary inertia of those masses, which have been neglected up to now. Using Galerkin's method, we can find the spectrum of the eigenfrequencies and, also, the shape functions. Finally, we can solve the equations of the problem of the forced vibrations, by using Carson-Laplace's transformation. Applying this method on a tall mast with 2 concentrated masses, we can examine the effect of the rotary inertia and the diaphragmatic operation of the above masses, on the 3 first eigenfrequencies.

Key words: concentrated masses; thin-walled member; tall masts; rotary inertia; diaphragmatic operation.

1. Introduction

In all the world, power-transmission towers, wind-generator towers, even, telecommunication towers, are often made from a cantilever, the tip of which is cut off and it has a circular or polygonal thin-walled cross-section, which varies along the axis of the cantilever by a whatever yet known law. Additionally, very often, there are significant concentrated masses, because of floors, observatories or look-out restaurants. These masses, along with the varying cross-section make a strong non-linear mathematic problem, with serious mathematical difficulties. There are many papers in this field. Rohde (1953) gave a power series solution to this problem. Wang and Lee (1973) extended Rohde's method and Gaines and Voltera (1976) investigated the eigenfrequencies of those constructions. Prathap and Varadan (1976) presented a finite deflection of such cantilever. Bouchet and Biswas (1977) presented a non-linear analysis by means F.E.M. and a vibration analysis, in 1979. Takabatake and others (1990, 1993, and 1995) proposed a solution using Dirac's functions. We are, also, obliged to refer to Kounadis's study (1976) on the dynamic response of a cantilever beam-column, with attached masses, but with unchanged sections along its length. Michaltsos and Konstantakopoulos (1998) presented a non-linear analysis taking into account the effect of the rotational inertia of concentrated masses. In the present paper, assuming linear strain-displacement relations and using Heaviside's and Dirac's functions, we can write the equations of

† Professor
‡ Research Engineer

coupled dynamic motion of the cantilever. At first, we investigate the free vibrations and with the help of Galerkin's method and that of the mode shape of the simple cantilever with constant cross-section and without concentrated masses, we determine the equation, which gives the spectrum of the eigenfrequencies and, furthermore, the shape functions. Then we attempt to solve the problem of the forced vibrations, also by using Galerkin's method and solving the resulting system with Carson-Laplace's transformation.

2. Analysis

2.1. Introductory concepts and determination of the diaphragmatic influence

2.1.1. Assumptions

We consider the model of Fig. 1. This is made from isotropic homogeneous material, having modulus of elasticity E. The relation between height l, diameter of the basis and thickness is suitable for a thin-walled construction. The transverse cross section has geometrical inertia and rigidity parameters varying along the height l. Each of the above parameters can be expressed by the equation:

$$R(x)=R_0 r_R(x)+R_i\delta(x-a_i)=R_0\left[r_R(x)+\frac{R_i}{R_0}\delta(x-a_i)\right]=R_0\rho_R(x) \tag{1}$$

where: $R(x)$ is any of the above parameters, R_0 is its value for $x=0$, R_i is the corresponding parameter of rigidity of the i concentrated mass, $r(x)$ is a given function of x, depending on the law of the variation R of the cantilever and $\delta(x-a_i)$ Dirac's function.

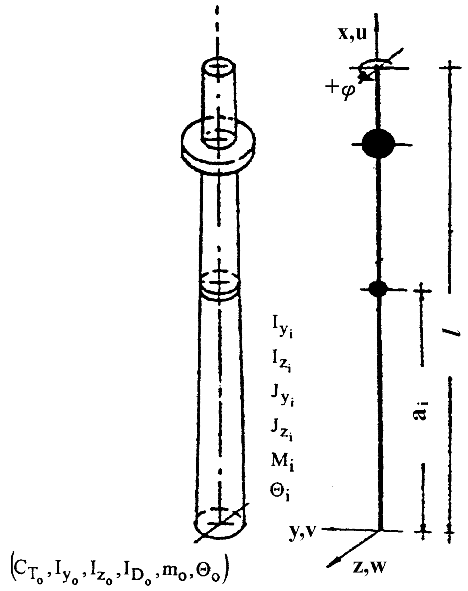

Fig. 1 A tower model

2.1.2. The diaphragmatic operation

Now, we consider an infinitesimal part dx of the cantilever, in which a diaphragm with thickness δ_i, which has a distance a_i from the origin is included.

The deformation $u(x, s)$ of the beam (which is its warping) is given by the form:

$$u(x, s) = -\frac{d\varphi}{dx} \cdot \dot{\omega}(s) \tag{2}$$

where $\dot{\omega}$ is the sectional co-ordinate.

On the other hand, the deformations of the plate-diaphragm must be (at the border) equal to those of the beam:

$$u(a_i, s) = u_\delta(y, z) \tag{3}$$

The work of the internal forces of the plate is given by the form:

$$\varepsilon_1 = \int_F \int D_i(1-\mu)\left(\frac{\partial^2 u_\delta}{\partial y \partial z}\right)^2 dydz \tag{4a}$$

where:

$$D_i = \frac{E_p \cdot \delta_i^3}{12 \cdot (1-\mu)^2} \tag{4b}$$

E_p is the modulus of elasticity of the plate and μ Poisson's ratio.

The integral of Eqs. (4a) extends on the total surface of the diaphragm. We note that the diaphragm may be extended out of the border of the beam (see Fig. 1). Then the integral of the Eqs. (4a) extends on the total surface of the plate-diaphragm.

If $\sigma_1(s)$, $\sigma_2(s)$ are the stresses of the sections 1 and 2 respectively (see Fig. 2), the work of the beam in the neighbourhood of the diaphragm, because of the above σ_1 and σ_2, is given by the form:

$$\varepsilon_2 = \frac{1}{2} \cdot t \cdot \left(\int_{s_2} \sigma_2(s) u_2(a_2, s) ds - \int_{s_1} \sigma_1(s) u_1(a_1, s) ds \right) \tag{5}$$

The above integral (5) extends on the cross-section of the beam only, and expresses the work of the external (for the part dx) forces $\sigma_1(s)$, $\sigma_2(s)$ because of the longitudinal displacements

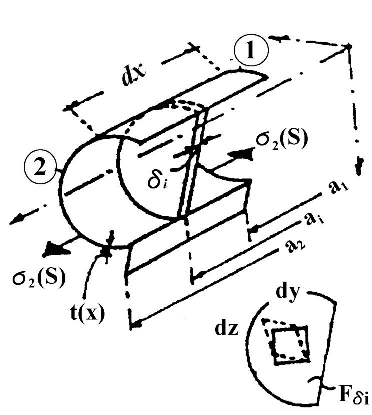

Fig. 2 Infinitesimal part *dx* including a diaphragm

$u_1(a_1,\ s),\ u_2(a_2,\ s)$.

Taking into account that the bimoment $B(x)$ is given by:

$$B = \int_s \sigma_x(s) \cdot \dot{\omega}(s) \cdot t \cdot ds \tag{6}$$

we can write (because of Eqs. 2):

$$\int_s \sigma_1(s) u_1(a_1,\ s) t ds = -\int_s \sigma_1(s) \cdot t \cdot \frac{d\varphi(a_1)}{dx} \cdot \dot{\omega}(s) ds = -B(a_1) \cdot \varphi'(a_1)$$

and in a similar way:

$$\int_s \sigma_2(s) u_2(a_2,\ s) t ds = -B(a_2) \cdot \varphi'(a_2)$$

Then Eq. (5) becomes:

$$\varepsilon_2 = -\frac{1}{2}[\varphi'(a_2) \cdot B(a_2) - \varphi'(a_1) \cdot B(a_1)]$$

If the quantity dx has zero as its limit, the values of a_1 and a_2 coincide with each other.
If the diaphragm did not exist, we would have:

$$B(a_2) - B(a_1) \Rightarrow 0$$

But because of the existence of the diaphragm the difference $[B(a_2) - B(a_1)]$ must have a real (existing) value B, which is caused by the diaphragmatic operation.
Then:

$$\varepsilon_2 = -\frac{1}{2} \cdot \lim_{a_2 \to a_1} \{[\varphi'(a_2) - \varphi'(a_1)] \cdot B\} = -\frac{1}{2} \cdot \varphi'(a_i) \cdot B \tag{7}$$

The work of the internal forces must be equal to that of the external forces:

$$-\frac{1}{2} \cdot \varphi'(a_i) \cdot B = \int_F\int D_i(1-\mu)\left(\frac{\partial^2 u_\delta}{\partial y \partial z}\right)^2 dydz \tag{8}$$

From the theory of thin plates, we know that for pure torsion we have:

$$u_\delta(y, z) = C \cdot y \cdot z \tag{9}$$

where C is a constant, which will be determined from the boundary conditions.

We consider, as a first approximation, that the warping is varying linearly along the axes oy and oz. Then from Eq. (2) we have:

$$u(a_i, s) = -\varphi'(a_i) \cdot \dot{\omega}(s) = -\varphi'(a_i) \cdot y \cdot z \tag{10}$$

Because of the displacements, which must be equal at the points of touching between beam and diaphragm (see Eq. 3), and also of Eqs. (9) and (10) we have:

$$C = -\varphi'(a_i),$$

and then:

$$u_\delta(y, z) = -\varphi'(a_i) \cdot y \cdot z \tag{11}$$

Therefore, Eq. (8) is written:

$$-\varphi'(a_i)\cdot B=2\cdot\int_F\int D_i(1-\mu)\cdot[-\varphi'(a_i)]^2dydz=2\cdot D_i\cdot(1-\mu)\cdot[\varphi'(a_i)]^2\cdot F_{\delta i}$$

As long as there is deformation, it must be $\varphi'(a_i)\neq 0$, and then we get:

$$B(a_i)=-2\cdot D_i\cdot(1-\mu)\cdot F_{\delta i}\cdot\varphi'(a_i) \tag{12}$$

The above Eq. (12), shows that the influence of a diaphragm can be expressed with an external loading, which consists of the concentrated bimoment $B(a_i)$

2.2. The equations of the problem

The part of the load due to the torsional vibration of the masses i round the y axis is:

$$q=m_\sigma''=[J_y(x)\ddot{w}'(x,t)+J_{y_i}\ddot{w}'(x,t)H(x-a_i)]''$$

where $H(x-a_i)$ is Heaviside's function.

Neglecting the very small influence of the torsional inertia of the beam (compared to that of the concentrated masses) we can write:

$$q=[J_{y_i}\ddot{w}'(x,t)H(x-a_i)]''=[J_{y_i}\ddot{w}'(a_i,t)H(x-a_i)]''$$
$$=[J_{y_i}\ddot{w}'(a_i,t)\delta(x-a_i)]'=J_{y_i}\ddot{w}'(a_i,t)\delta'(x-a_i)$$

Then, neglecting the longitudinal and torsional inertia forces associated with warping and, also, the transverse shear deformation, the equations of the problem are:

$$\left.\begin{aligned}&[EI_y(x)w''(x,t)]''+m(x)\ddot{w}(x,t)+c_w\dot{w}(x,t)+\sum_{i=0}^{\kappa}J_{y_i}\ddot{w}'(x,t)\cdot\delta'(x-a_i)=p_z(x,t)-m_y'(x,t)\\&[EC_T(x)\varphi''(x,t)]''-[GI_D(x)\varphi']'+c_\varphi\dot{\varphi}(x,t)+\Theta(x)\ddot{\varphi}(x,t)=M_x(x,t)\end{aligned}\right\} \tag{13}$$

We know that: $M_{XT}=-B'(x)$, therefore:

$$M_x(x,t)=m(x,t)+\sum_{i=1}^{\kappa}M_{XT}(x,t)\cdot\delta(x-a_i)=m_x(x,t)+\sum_{i=1}^{\kappa}B'(a_i,t)\cdot\delta(x-a_i)$$

Then Eq. (13) become:

$$\left.\begin{aligned}&[EI_y(x)w''(x,t)]''+m(x)\ddot{w}(x,t)+c_w\dot{w}(x,t)+\sum_{i=1}^{\kappa}J_{y_i}\ddot{w}'(x,t)\cdot\delta'(x-a_i)=p_z(x,t)-m_y'(x,t)\\&[EC_T(x)\varphi''(x,t)]''-[GI_D(x)\varphi']'+c_\varphi\dot{\varphi}(x,t)+\Theta(x)\ddot{\varphi}(x,t)\\&\qquad=m_x(x,t)+2\sum_{i=1}^{\kappa}D_i(1-\mu)F_{\delta i}\varphi''(x,t)\delta(x-a_i)\end{aligned}\right\} \tag{14}$$

2.3. The free vibration

2.1.1. Free flexural vibration

The equation of the free flexural vibration, if the damping $c_w = 0$, is:

$$[EI_y(x)\cdot w''(x,t)]''+\sum_{i=1}^{\kappa} J_{y_i}\ddot{w}'(x,t)\delta'(x-a_i)+m(x)\ddot{w}(x,t)=0 \tag{15}$$

which, if we take into account $w(x,t)=X(x)\cdot T(t)$ gives:

$$[EI_y(x)\cdot X''(x)]''\cdot T(t)+\ddot{T}(t)\sum_{i=1}^{\kappa} J_{y_i}X'(x)\cdot\delta'(x-a_i)+m(x)\cdot X(x)\ddot{T}(t)=0$$

or

$$\frac{[EI(x)\cdot X''(x)]''}{\sum_{i=1}^{\kappa} J_{y_i}X'(x)\cdot\delta'(x-a_i)+m(x)\cdot X(x)}=-\frac{\ddot{T}(t)}{T(t)}=\omega_\omega^2$$

or

$$[EI(x)\cdot X''(x)]''-\omega_\omega^2\left[\sum_{i=1}^{\kappa} J_{y_i}X'(x)\cdot\delta'(x-a_i)+m(x)\cdot X(x)\right]=0 \tag{16}$$

Then Eqs. (16), because of Eqs. (1), can be written:

$$\left.\begin{aligned}&\rho_{I_y}(x)\cdot X''''(x)+2\rho'_{I_y}(x)\cdot X'''(x)+\rho''_{I_y}(x)\cdot X''(x)\\&-\lambda\cdot\left[\sum_{i=1}^{\kappa}\frac{J_{y_i}}{m_0}\cdot X'(x)\cdot\delta'(x-a_i)+\rho_m(x)\cdot X(x)\right]=0\\&\text{with:}\qquad \lambda=\frac{\omega_w^2 m_0}{EI_{y_0}}\end{aligned}\right\} \tag{17}$$

In order to apply Galerkin's method, we set:

$$X(x)=c_1\Psi_1(x)+c_2\Psi_2(x)+\ldots+c_n\Psi_n(x) \tag{18}$$

where: c_i are unknown coefficients, which will be determined and $\Psi_i(x)$ are arbitrarily chosen functions of x, which satisfy the boundary conditions. As such functions, we choose the shape functions of the simple cantilever (without varying cross-section) given by Eqs. (a) of the appendix.

Introducing Eqs. (18) into (17), multiplying the outcome successively by Ψ_1, Ψ_2, ... Ψ_n and integrating the results from 0 to l, we obtain the following homogeneous, linear system without second member of n equations, with unknowns c_1, c_2, ..., c_n.

$$c_1(A_{i_1}-\lambda\cdot B_{i_1})+c_2(A_{i_2}-\lambda\cdot B_{i_2})+\ldots+c_n(A_{i_n}-\lambda\cdot B_{i_n})=0 \qquad (i=1,2,\ldots n) \tag{19}$$

where:

$$\left.\begin{aligned}A_{ij}&=\int_0^l[\rho_{I_y}(x)\cdot\Psi_j''''(x)+2\rho'_{I_y}(x)\cdot\Psi_j'''(x)+\rho''_{I_y}(x)\cdot\Psi_j''(x)]\Psi_i(x)dx\\B_{ij}&=\int_0^l\rho_m(x)\cdot\Psi_i(x)\cdot\Psi_j(x)dx-\sum_{i=1}^{\kappa}\frac{J_{yi}}{m_o}[\Psi_j''\cdot\Psi(a_i)+\Psi_j'(a_i)\cdot\Psi_i'(a_i)]\end{aligned}\right\} \tag{20}$$

In order for the system to have non-trivial solutions, the determinant of the coefficients must be zero:

$$|\Gamma_{ij}|=0 \qquad \text{with: } i, j=1, 2,\ldots n \qquad \text{and} \qquad \Gamma_{ij}=A_{ij}-\lambda B_{ij} \tag{21}$$

From Eqs. (21) we determine the exact values for λ and, from Eqs. (17) the spectrum of the flexural eigenfrequencies ω_{wi}.

Assuming that the natural frequencies are dominated by the diagonal terms in the square matrice Γ_{ij},

$$\omega_{wi}=\sqrt{\frac{EI_{y0}\cdot A_{ii}}{m_0\cdot B_{ii}}} \tag{21a}$$

From the first (n–1) equation of Eqs. (21), we can finally find:

$$\left.\begin{aligned}
&\frac{c_j}{c_1}=\frac{\begin{vmatrix}\Gamma_{12}\ldots\ldots\ldots\Gamma_{11}\ldots\ldots\ldots\Gamma_{1n}\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \Gamma_{(n-1),2}\ldots\Gamma_{(n-1)1}\ldots\Gamma_{(n-1)n}\end{vmatrix}}{|\Gamma_{ij}|}\\
&\text{with:}\quad i=1, 2, \ldots (n-1) \qquad j=2, 3, \ldots(n)\\
&\text{and therefore: } X_n(x)=c_1\sum_{j=2}^{n}\left(\Psi_1+\frac{c_j}{c_1}\Psi_j\right)
\end{aligned}\right\} \tag{22}$$

2.3.2. Free torsional vibration

The equation of the free torsional vibration, if the damping $c_\varphi = 0$, is:

$$[EC_T(x)\cdot\varphi''(x, t)]''-[GI_D(x)\cdot\varphi'(x, t)]'+\Theta(x)\cdot\ddot{\varphi}(x, t)=0 \tag{23}$$

Following the same analysis as in *Free flexural vibration* , we obtain:

$$[EC_T(x)\cdot\varphi''(x)]''-[GI_D(x)\cdot\varphi'(x)]'-\omega_\varphi^2\Theta(x)\cdot\varphi(x)=0 \tag{24}$$

Then Eq. (24), because of Eq. (1), gives:

$$\left.\begin{aligned}
&[\rho_c(x)\varphi''''(x)+2\rho_c'(x)\,\varphi'''(x)+\rho_c''(x)\,\varphi''(x)]-\mu_0\,[\rho_{I_D}(x)\varphi''(x)+\rho'_{I_D}(x)\varphi'(x)]-\nu_0\rho_\theta(x)\varphi(x)=0\\
&\text{with:}\qquad \mu_0=\frac{GI_{D_0}}{EC_{T_0}},\quad \nu_0=\frac{\omega_\varphi^2\Theta_0}{EC_{T_0}}
\end{aligned}\right\} \tag{25}$$

Applying Galerkin's method, we finally find:

$$|\Delta_{ij}|=0,\quad \varphi_n(x)=d_1\sum_{j=2}^{n}\left(\Phi_1+\frac{d_j}{d_1}\Phi_j\right) \tag{26}$$

which, gives the spectrum of torsional eigenfrequencies ω_φ and the shape functions. Where:

$$\left.\begin{aligned}
&\Delta_{ij}=Z_{ij}-\mu_0\cdot H_{ij}-\nu_0\cdot\Lambda_{ij},\\
&Z_{ij}=\int_0^l[\rho_c(x)\Phi_j''''(x)+2\rho_c'(x)\Phi_j'''(x)+\rho_c''(x)\Phi_j''(x)]\Phi_i(x)dx\\
&H_{ij}=\int_0^l[\rho_{I_D}(x)\Phi_j''(x)+\rho_{I_D}'(x)\Phi_j'(x)]\Phi_i(x)dx,\\
&\Lambda_{ij}=\int_0^l\rho_\theta(x)\cdot\Phi_i(x)\cdot\Phi_j(x)dx
\end{aligned}\right\}\tag{27}$$

The coefficients $\frac{d_j}{d_1}$ are found by solving the (n–1) first equations of the above system.

2.4. The forced vibration

2.4.1. Flexural forced vibration

The equation of motion for the flexural forced vibration, is given by Eqs. (14), which, because of Eq. (1), becomes:

$$EI_{y_0}[\rho_{I_y}(x)w''''(x,t)+2\rho_{I_y}'(x)w'''(x,t)+\rho_{I_y}''(x)w''(x,t)]+c_w\cdot\dot{w}(x,t)+m_0\rho_m(x)\cdot\ddot{w}(x,t)+$$
$$+\sum_{i=1}^{\kappa}J_{y_i}\ddot{w}'(x,t)\delta'(x-a_i)=p_z(x,t)-m_y'(x,t)\tag{28}$$

We search for a solution with the form:

$$w(x,t)=\sum_n X_n(x)\cdot P_{w_n}(t)$$

where $P_{w_n}(t)$ are unknown functions of the time, which will be determined and $X_n(x)$ are functions of x, arbitrarily chosen, which satisfy the boundary conditions. As such functions, we choose those of Eqs. (22), which still satisfy Eq. (18). Then Eq. (28), because of Eqs. (16), becomes:

$$m_0\sum_n\omega_{kn}^2\rho_mX_nP_n+c_w\sum_nX_n\dot{P}_n+m_0\sum_n\rho_mX_n\ddot{P}_n+\sum_{i=1}^{\kappa}\left[J_{y_i}\delta'(x-a_i)\cdot\sum_nX_n'\ddot{P}_n\right]=p_z-m_y'$$

or after multiplication with X_σ and integration of the outcome from 0 to l, becomes:

$$\left.\begin{aligned}
&m_0\sum_n\omega_{w_n}^2D_{\sigma,n}^1\cdot P_{w_n}+c_w\sum_nD_{\sigma,n}^2\cdot\dot{P}_{w_n}+m_0\sum_nD_{\sigma,n}^1\cdot\ddot{P}_{w_n}-\sum_{i=1}^{\kappa}\left[J_{y_i}\cdot\sum_nD_{\sigma,n}^3\cdot\ddot{P}_{w_n}\right]=D_\sigma^4\\
&\text{and:}\quad D_{\sigma,n}^1=\int_0^l\rho_m(x)X_nX_\sigma dx\\
&\qquad\quad D_{\sigma,n}^2=\int_0^l\rho_mX_nX_\sigma dx\\
&\qquad\quad D_{\sigma,n}^3=X_n''(a_i)X_\sigma(a_i)+X_n'(a_i)X_\sigma'(a_i)\\
&\qquad\quad D_\sigma^4=\int_0^l[p_z(x,t)-m_y'(x,t)]X_\sigma dx
\end{aligned}\right\}\tag{29}$$

The above system of Eqs. (29) is a differential system with unknowns the P_{w_1}, P_{w_2}, $\ldots P_{w_n}$.

2.4.2. Torsional forced vibration

The equation of motion for the torsional forced vibration is given by Eqs. (14). Following the same methodology as in flexural forced vibration, we reach the system:

$$\left.\begin{aligned}&\Theta_0 \sum_n \omega^2_{\varphi_n} \Xi^1_{\sigma,n} P_{\varphi_n} + 2 \sum_{i=1}^{\kappa} D_i(1-\mu) F_{\delta i} \Xi^2_{\sigma,n} P_{\varphi_n} + c_\varphi \sum_n \Xi^3_{\sigma,n} \dot{P}_{\varphi_n} + \Theta_0 \sum_n \Xi^1_{\sigma,n} \ddot{P}_{\varphi_n} = \Xi^4_\sigma \\ &\text{where:}\quad \Xi^1_{\sigma,n} = \int_0^l \rho_\theta(x) \varphi_n \varphi_\sigma dx,\ \Xi^2_{\sigma,n} = \varphi_n(a_i)\varphi_\sigma(a_i),\ \Xi^3_{\sigma,n} = \int_0^l \varphi_n \varphi_\sigma dx \\ &\qquad \Xi^4_\sigma = \int_0^l m_x(x,t)\varphi_\sigma dx \quad \text{and:}\quad \sigma = 1, 2, \dots n\end{aligned}\right\} \tag{30}$$

The above system of Eqs. (30) is a differential system, with unknown factors $P_{\varphi_1}, P_{\varphi_2}, \ldots P_{\varphi_n}$.

2.5. Solution of the systems (29) and (30)

The systems (29) and (30), with use of Carlson-Laplace's transformation, and initial conditions: $w(x,0) = \dot{w}(x,0) = \varphi(x,0) = \dot{\varphi}(x,0) = 0$ takes the following form:

$$a_{i1}g_1(p) + a_{i2}g_2(p) + \ldots + a_{in}g_n(p) = \beta_i \cdot F(p) \tag{31}$$

where:

$$\left.\begin{aligned}&g_u(p) = LP_u(t) \quad F(p) = L \cdot f(t) \quad p_z(x,t) = \bar{p}_z(x) \cdot f(t) \quad m_y(x,t) = \bar{m}_y(x) \cdot f(t) \\ &\text{For the flexural vibration:} \\ &\alpha_{ij} = m_0 \omega^2_{wi} D^1_{ij} + c_w D^2_{ij} \cdot p + \left[m_0 D^1_{ij} + \left(\sum_{\lambda=1}^{\kappa} J_{y_\lambda} \right) D^3_{ij} \right] p^2 = A_{ij} + p \cdot B_{ij} + p^2 \cdot \Gamma_{ij} \\ &\beta_i = \int_0^l [\bar{p}_z(x) - \bar{m}'_y(x)] x_i dx \\ &\text{and for the torsional vibration:} \\ &\alpha_{ij} = \left[\Theta_0 \omega^2_{\varphi i} \Xi^1_{ij} + 2 \cdot \left(\sum_{i=1}^{\kappa} D_i(1-\mu) F_{\delta i} \right) \Xi^2_{\sigma,n} \right] + c_w \Xi^3_{ij} \cdot p + \Theta_0 \Xi^1_{ij} \cdot p^2 = A_{ij} + p \cdot B_{ij} + p^2 \cdot \Gamma_{ij}, \\ &\beta_i = \int_0^l \bar{m}_x(x) \cdot \varphi_i \cdot dx\end{aligned}\right\} \tag{32}$$

The usual forms of functions $f(t)$, $F(p)$ are rational functions of p. Then g_u takes the form:

$$g_u(p) = \frac{N_u(p)}{M_u(p)} \quad u = 1, 2, \ldots n \tag{33}$$

where N_u, M_u are polynomials with respect to p with $M_u(p)$ of higher order than $N_u(p)$.

Heaviside's rule can thus be applied, leading finally to Eq. (22):

$$P_u(t) = L^{-1} g_u(p) = L^{-1} \frac{N_u(p)}{M_u(p)} = \frac{N_u(0)}{M_u(0)} + \sum_{i=1}^{n} \frac{N_u(p_i) \cdot e^{p_i t}}{p_i \cdot M'_u(p_i)} \tag{34}$$

in which p_i, are the roots of polynomial $M_u(p)$.

3. Numerical results and discussion

We consider three towers which have height L=100 m, 70 m and 40 m.

Table 1 Data of the towers

				Masses					
L	SET	Tower	a_i	I_i	J_i	M_i	F_i	I_{Di}	Θ_i
		R_o=6 I_{Do}=5.55	75	50	5	5.3	27	15	13.3
	1	I_o=10 Θ_o=30.5	90	330	250	293	300	60	1950
		m_o=6.72 C_T=2.50	95	330	250	293	300	60	1950
		$R(x)$=0.00005*x^2–0.01083*x+1	100	50	5	5.3	27	15	13.3
		R_o=6 I_{Do}=5.55	75	25	5	5.3	27	15	13.3
100	2	I_o=10 Θ_o=30.5	90	25	5	5.3	90	20	130
		m_o=6.72 C_T=2.50	95	25	5	5.3	90	20	130
		$R(x)$=0.00005*x^2–0.01083*x+1	100	25	5	5.3	27	15	13.3
		R_o=2.5 I_{Do}=2.80	75	50	5	5.3	27	15	13.3
	3	I_o=4.5 Θ_o=12	90	330	250	293	300	60	1950
		m_o=3 C_T=1.50	95	330	250	293	300	60	1950
		$R(x)$=0.0001*x^2–0.025*x+2.5	100	50	5	5.3	27	15	13.3
		R_o=5 I_{Do}=3	50	50	5	5.3	25	15	13.3
	1	I_o=3.60 Θ_o=25	57	330	250	193	300	60	1950
		m_o=3 C_T=1.40	62	330	250	193	300	60	1950
		$R(x)$=0.0000326*x^2–0.01086*x+1	70	50	5	5.3	25	15	13.3
		R_o=5 I_{Do}=3	50	15	5	5.3	25	15	13.3
70	2	I_o=3.60 Θ_o=25	57	15	5	5.3	90	20	130
		m_o=3 C_T=1.40	62	15	5	5.3	90	20	130
		$R(x)$=0.0000326*x^2–0.01086*x+1	70	15	5	5.3	25	15	13.3
		R_o=1.50 I_{Do}=1.70	50	50	5	5.3	25	15	13.3
	3	I_o=0.80 Θ_o=6	57	330	250	193	300	60	1950
		m_o=1.40 C_T=0.60	62	330	250	193	300	60	1950
		$R(x)$=0.000816*x^2–0.0157*x+1.50	70	50	5	5.3	25	15	13.3
		R_o=2.50 m_o=0.80 Θ_o=12000	30	10	3	3	6	15	13.3
	1	I_o=1 I_{Do}=0.80 C_T=0.60	35	150	120	100	300	60	1950
		$R(x)$=–0.015*x+1	40	10	3	3	6	15	13.3
		R_o=2.50 m_o=0.80 Θ_o=12	30	10	3	3	6	15	13,3
40	2	I_o=1 I_{Do}=0.80 C_T=0.60	35	10	3	3	90	20	130
		$R(x)$=–0.015*x+1	40	10	3	3	6	15	13.3
		R_o=0.60 m_o=0.35 Θ_o=2	30	10	3	3	6	15	13.3
	3	I_o=0.10 I_{Do}=0.15 C_T=0.20	35	150	120	100	300	60	1950
		$R(x)$=–0.625*x+0.60	40	10	3	3	6	15	13.3

Table 2 Influence (in percentage) of the rotary inertia and diaphragmatic operation

		Flexural			Torsional		
Length	SET	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
100	1	0.06	0.37	1.05	2.90	2.40	2.38
	2	0.01	0.01	0.02	0.13	0.10	0.05
	3	4.08	5.11	5.14	1.30	1.21	1.08
70	1	0.05	1.25	2.17	0.6	0.08	0.012
	2	0.01	0.09	0.14	0.03	0.01	0
	3	5.36	7.22	6.43	1.57	1.45	0.97
40	1	3.15	6.40	5.88	0.08	0	0
	2	0.01	0.68	0.83	0	0	0
	3	7.30	14.20	11.45	2.70	1.89	2.03

Each of these towers has sets of concentrated masses, at the points a_i, which are symbolized as SET 1, SET 2, and SET 3. Generally SET 1 and SET 2 have the same characteristics for the pylons, but different ones for the concentrated masses while SET 1 and SET 3 have the same masses but different pylons.

As for the conical surface of the tower, it is a parabola of second order (for the towers with $L = 100$ m and 70 m) and a straight line (for the tower with $L = 40$ m). Table 1 shows clearly the above data: For the above cases, we have found the first three eigenfrequencies.

In Table 2 the influence (on percentage) of the rotary inertia on the flexural eigenfrequencies, and of the diaphragmatic operation on the torsional ones is shown.

From the above table the following conclusions can be obtained:

1. The influence of the rotary inertia of the masses is serious for slender towers, which have heavy masses. This is obvious from SETS 3 for all the tower heights (the maximum influence is increased from 4% to 11%).
2. The eigenfrequencies of a tower are least affected by the change of masses (heavy or light masses). This is clear from the comparison between SETS 1 and SETS 2 for all the tower heights. That influence is increased from 0.01% to 2.71%.
3. The most affected eigenfrequencies are the second and third ones.
4. The effect of the diaphragmatic operation is very small. The diaphragmatic operation, on the contrary to the rotary inertia of the masses, has influence on the towers with big diameter and heavy masses. Generally the differences are very small and the influence is increased from 0% to 2.90%.

References

Bouchet, A.V. and Biswas. (1977), “Non-linear analysis of towers and stacks”. *Journal of Structures Div. ASCE,* **103**(8), 1631-1641.

Gaines, J.H. and Voltera, E. (1976), “Upper and lower frequencies of tapered beams”. *Journal of Engineering Mech Div, ASCE,* **94**(2), 465-488.

Kounadis, A. (1976), “Dynamic response of cantilever beam-columns with attached masses supported on a flexible foundation”, *Proceedings of International Symposium on Earthquake Structure Engineering* , St. Louis Missouri, U.S.A. 473-482.

Michaltsos, G.T and Konstantakopoulos, T.G. (1998), "Non-linear dynamic analysis of thin-walled tapered towers with concentrated masses", *Proceedings of the Second International Conference on Thin-Walled Structures, Singapore,* 509-516.
Prathap, G. and Varadan, T.K. (1976), "Finite deflection of tapered cantilevers", *Journal of Engineering Mech Div, ASCE,* **102**(3), 549-552.
Rohde, F.V. (1953), "Large deflections of cantilever beam with uniformly distributed load", *Quarnt Appl Math,* **11**, 337.
Takabatake, H. (1990), "Cantilever and linearly tapered thin-walled members", *Journal of Engineering Mech, ASCE,* **116**(4), 733-750.
Takabatake, H., Mukai, H. and Hirano, T. (1993), "Doubly symmetric tube structures, I: Static analysis", *Journal of Structure Engineering, ASCE*, **119**(7), 981-2001.
Takabatake, H., Mukai, H. and Hirano, T. (1993), "Doubly symmetric tube structures, II: Dynamic analysis", *Journal of Structure Engineering, ASCE,* **119**(7), 2002-2006.
Takabatake, H. and Mizuki, A. (1995), "Simplified dynamic analysis of slender tapered thin-walled towers with additional mass and rigidity", *Structures Engineering and Mechanics*, **3**(1), 61-74.
Wang, T.M. and Lee, C.H. (1973), "Tapered cantilevers with varying distributed loads", *Journal of Engineering Mechanics Div, ASCE*, **99**(4), 919-925.

Appendix

1. The shape functions of the free flexural vibrating cantilever (without varying cross-section) are:

$$\left.\begin{array}{c}\Psi_n(x)=c_{2n}\left[\dfrac{c_{1n}}{c_{2n}}(\sin\lambda_n x-\sinh\lambda_n x)-(\cos\lambda_n x-\cosh\lambda_n x)\right]\\ \text{with: } \dfrac{c_{1n}}{c_{2n}}=\dfrac{\cos\lambda_n l+\cosh\lambda_n l}{\sin\lambda_n l+\sinh\lambda_n l}\end{array}\right\} \tag{a}$$

2. The shape functions of the free tortional vibrating simple cantilever (without varying cross-section) are:

$$\left.\begin{array}{l}\Phi_n(x)=c_{1n}\left[\sin k_{1n}x-\dfrac{k_{1n}}{k_{2n}}\cdot\sinh k_{2n}x-\dfrac{c_{4n}}{c_{1n}}(\cos k_{1n}x-\cos k_{2n}x)\right],\ \text{with:}\\ k_{1n},\ k_{2n}=\sqrt{\mp\dfrac{\mu}{2}+\sqrt{\left(\dfrac{\mu}{2}\right)^2+\dfrac{\omega_{n\sigma}^2\cdot\Theta}{EC_T}}},\quad \mu=\dfrac{GI_D}{EC_T},\quad \dfrac{c_{4n}}{c_{1n}}=\dfrac{k_{1n}^2\sin k_{1n}l+k_{1n}k_{2n}\sinh k_{2n}l}{k_{1n}^2\cos k_{1n}l+k_{2n}^2\cosh k_{2n}l}\end{array}\right\} \tag{b}$$